Function spaces have essential sets

Jan Čerych

Abstract. It is well known that any function algebra has an essential set. In this note we define an essential set for an arbitrary function space (not necessarily algebra) and prove that any function space has an essential set.

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Definition 1. Let X be a compact Hausdorff topological space. Denote by $C(X)$ the commutative Banach algebra consisting of all continuous complex-valued functions on X (with respect to usual point-wise algebraic operations) endowed with the sup-norm.

By a *function algebra* on X we mean any closed subalgebra of $C(X)$ which contains constant functions on X and which separates points of X. (The last property, more precisely, means: whenever $x,\ y$ is a couple of distinct points in X, then there exists a function $f \in A$ such that $f(x) \neq f(y)$.)

By a *function space* on X we mean any closed subspace of $C(X)$ which separates points of X.

Remark 1. Any function algebra on X is a function space on X; of course, not conversely.

Definition 2. A function algebra A on X is said to be *maximal* if it is a proper subalgebra of $C(X)$ and has the following property: whenever B is a function algebra on X, $B \supset A$, then either $B = A$ or $B = C(X)$.

Definition 3. A being a function space on X, a closed subset E is said to be an *essential set of* A if the following conditions are fulfilled:

(i) A consists of all continuous extensions of functions from A/E (the space of all restrictions of functions in A to E);

(ii) whenever a closed subset F of X has the same property as E in (i), then $E \subset F$ (or, E is a unique minimal closed subset of X satisfying the condition (i) above).

The notion of an "essential set" in the case of function algebras is due to H.S. Bear, who proved in [1] that any maximal algebra on X has an essential set.

Hoffman and Singer [2] found an essential set of any, not necessarily maximal, function algebra on X.

Our aim here is to find an essential set for any function space, not necessarily algebra, and to prove that an essential set of a function space has the same properties as in the case of function algebras.

I. Glicksberg has shown in [3] that the rather useful tool for the research into function algebras is so-called "annihilator". Let us define an annihilator in our more general case:

Definition 4. Denote by $M(X)$ the space of all complex Borel regular measures on X, i.e., by the Riesz Representation Theorem, the dual space of $C(X)$.

The *annihilator* $A^{\perp}$ of a function space A is defined to be the set of all measures $m \in M(X)$ such that $\int f\,dm = 0$ for any $f \in A$, or the set of all measures *orthogonal* to A.

Remark 2. The dual space A' of A is then canonically isomorphic to the quotient space $M(X)/A^{\perp}$.

Let us endow $M(X)$ with the weak-star topology: it is well known that $M(X)$ becomes a locally convex topological linear space with the dual space $C(X)$.

We shall now characterize the essential set of a function space A by means of the properties of measures in $A^{\perp}$.

Theorem 1. *Let A be a function space on X. Denote by E the closure of the union of all closed supports of measures in $A^{\perp}$. Then E is the essential set of A.*

PROOF: Let $f \in C(X)$, $g \in A$ and let $f/E = g/E$, where f/E denotes the restriction of the function f from X to E. Let us denote $M = \operatorname{spt}(m)$ for $m \in A^{\perp}$; then

$$\int f\,dm = \int_M f\,dm = \int_M g\,dm = \int g\,dm = 0,$$

hence f is orthogonal to $A^{\perp}$ and, by Banach's Bipolar Theorem, $f \in A$. It means that E has the property (i) from Definition 3.

Let a closed subset K have the property (i); we shall prove that $K \supset E$. Suppose that $K \not\supset E$. Then there is a measure $m \in A^{\perp}$ whose closed support is not a subset of K. Take $x \in \operatorname{spt}(m) \setminus K$. Let V be an open neighbourhood of x in X such that its closure $\overline{V}$ is disjoint with K. We shall find a function $f \in C(\overline{V})$ which fulfills the following two conditions:

$$\operatorname{spt}(f) \subset V, \ \int_V f\,dm \neq 0,$$

where $\operatorname{spt}(f)$ means the closed support of f. Denote by g a function in $C(X)$, which is equal to f on $\overline{V}$ and to 0 outside of $\overline{V}$. Then $g/K = 0 \in A/K$, but

$$\int g\,dm = \int_V g\,dm = \int_V f\,dm \neq 0$$

and then g is not orthogonal to A, so $g \notin A$. It follows that K has not the property (i). $\square$

Remark 3. Now, the following question arises: whether the word "closure" in Theorem 1 may be omitted, or whether the union of closed supports of all measures in $A^{\perp}$ is automatically a closed set. We shall show that it is true if X is a metric space (Theorem 2), but in general it is not the case (Remark 4).

Theorem 2. *Let X be a compact metric space and A a function space on X. Then the essential set E of A is equal to the union of closed supports of all measures in $A^{\perp}$. (Especially, the union of closed supports of all orthogonal measures is a closed set.)*

Proof: Let $x \in E$. We shall find the measure $m \in A^{\perp}$ such that $\mathrm{spt}(m) \ni x$. Denote by U_n, $n = 1, 2, \dots$ the open balls in X with centres at x and radii $\frac{1}{n}$. We shall construct a finite or infinite sequence of measures $m_n \in A^{\perp}$ such that

$$|m_n|(X) \leq 1, \tag{1}$$

$$(\mathrm{spt}(m_n) \setminus \bigcup_{k=1}^{n-1} \mathrm{spt}(m_k)) \cap U_n \stackrel{def}{=} M_n \neq \emptyset \text{ and then } |m_n|(M_n) > 0, \tag{2}$$

$$|m_n|(X) < \min_{1 \leq k \leq n-1} |m_k|(M_k), \tag{3}$$

where $|m|$ means the total variation of a measure m.

By Theorem 1, we can find a measure $m_1 \in A^{\perp}$ such that $|m_1|(X) = 1$ for which $\mathrm{spt}(m_1) \cap U_1 \neq \emptyset$. If $x \in \mathrm{spt}(m_1)$, the proof is finished. If it is not the case, then, by Theorem 1, there exists a measure $m_2 \in A^{\perp}$ such that $(\mathrm{spt}(m_2) \setminus \mathrm{spt}(m_1)) \cap U_2 \neq \emptyset$; then (2) follows. Multiplying m_2 by an appropriate nonzero constant, we can get (1) and (3). If $x \in \mathrm{spt}(m_2)$, we are done. In the opposite case, we shall continue the construction.

In the case the sequence $\{m_n\}$ is finite, the proof is finished. If it is not the case, put

$$m = \sum_{n=1}^{\infty} \frac{1}{2^n} m_n.$$

By (1), $m \in M(X)$. Also $m \in A^{\perp}$ because $m_n \perp A$.

Take an arbitrary n. By (2), $|m_n|(M_n) > 0$, while $|m_k|(M_n) = 0$ for $1 \leq k \leq n-1$. By (3), we have

$$|m|(M_n) = |\sum_{k=n}^{\infty} \frac{1}{2^k} m_k(M_n)| \geq \frac{1}{2^n}|m_n|(M_n) - \sum_{k=n+1}^{\infty} \frac{1}{2^k}|m_k|(X) \geq$$

$$\geq \frac{1}{2^n}|m_n|(M_n) - \sum_{k=n+1}^{\infty} \frac{1}{2^k}|m_k|(X) > \frac{1}{2^n}|m_n|(M_n) - \frac{1}{2^n}|m_n|(M_n) = 0$$

and then $\mathrm{spt}(m) \cap U_n \neq \emptyset$. Since n was arbitrary, Theorem 2 follows. $\square$

Remark 4. In [4] we have constructed a function algebra A on X such that there exists a point $x \in E$ which is not contained in the closed support of any measure in $A^{\perp}$. It follows that the word “closure” in Theorem 1 cannot be omitted in the general (non-metric) case.

References

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Department of Mathematical Analysis, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail: cerych@karlin.mff.cuni.cz

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