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# On some fan-tightness type properties

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*Abstract.* Properties similar to countable fan-tightness are introduced and compared to countable tightness and countable fan-tightness. These properties are also investigated with respect to function spaces and certain classes of continuous mappings.

*Keywords:* $\omega$-cover, $vet^*(X)$, tightness, fan-tightness, biquotient mapping, $C_p(X)$
*Classification:* 54A25, 54C35

In this paper, all spaces are assumed to be Hausdorff. We denote by $\mathbb{R}$ the set of real numbers; $\beta X$ denotes the Stone-Čech compactification of a Tychonoff space $X$ and $C_p(X)$ stands for the space of all real-valued continuous functions on $X$ with the topology of pointwise convergence. A basic open neighborhood of a function $f \in C_p(X)$ is of the form $W(x_1, \dots, x_k;\ f;\ \epsilon) = \{g \in C_p(X) : |f(x_i) - g(x_i)| < \epsilon,\ i = 1, 2, \dots, k\}$, where $k \in \omega$, $x_i \in X$ and $\epsilon > 0$. We denote by $C_p^0(X)$ the set of all bounded continuous functions on $X$ equipped with the topology of pointwise convergence. A cover $\gamma$ of $X$ is said to be an *$\omega$-cover* if for any finite subset $F$ of $X$ there is a $G \in \gamma$ such that $F \subseteq G$. The notion of countable fan-tightness was introduced in [1]: a space $X$ is said to have *countable fan-tightness* (denoted $vet(X) \le \omega$) if for each point $x$ in $X$ and any countable family $\{A_n\}_{n\in\omega}$ of subsets of $X$ satisfying $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist finite sets $H_n \subseteq A_n$ such that $x \in \overline{\bigcup\{H_n : n \in \omega\}}$. A space $X$ is said to have *countable strong fan-tightness* if for each $x \in X$ and for each countable family $\{A_n : n \in \omega\}$ of subsets of $X$ such that $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist $a_i \in A_i$ such that $x \in \overline{\{a_i : i \in \omega\}}$.

A space $X$ is said to *have property* $vet^*(X) \le \omega$ if for each point $x$ in $X$ and any countable family $\{A_n\}_{n\in\omega}$ of subsets of $X$ satisfying $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist sets $H_n \subseteq A_n$ with $|H_n| \le n$ such that $x \in \overline{\bigcup\{H_n : n \in \omega\}}$. Clearly, every space $X$ of countable strong fan-tightness has $vet^*(X) \le \omega$, and $vet^*(X) \le \omega$ in turn implies that the fan-tightness of $X$ is countable.

The following theorems were proved in [1] and [5], respectively:

**Theorem 1** (Arhangelskii). *For a Tychonoff space $X$, the following are equivalent:*

(a) *$vet\ C_p(X) \le \omega$;*
(b) *for each $n \in \omega$, $X^n$ is a Hurewicz space.*

**Theorem 2** (Sakai). *For a Tychonoff space $X$, the following are equivalent:*

(a) *$C_p(X)$ has countable strong fan-tightness;*

(b) *for each sequence $\{\gamma_n : n \in \omega\}$ of open $\omega$-covers of $X$ there exist $U_n \in \gamma_n$ such that $\{U_n : n \in \omega\}$ is an $\omega$-cover of $X$.*

**Lemma 3.** *For a topological space $X$, the following are equivalent:*

(a) *$vet^*(X) \leq \omega$;*

(b) *for each mapping $\phi : \omega \to \omega$ such that $\phi(n) \geq n$ for each $n \in \omega$, for each point $x \in X$ and for each (decreasing) family $\{A_n\}_{n\in\omega}$ of subsets of $X$ satisfying $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist $H_i \subseteq A_i$ such that $x \in \overline{\{H_n : n \in \omega\}}$ and $|H_n| \leq \phi(n)$;*

(c) *for each point $x \in X$ and for each decreasing family $\{A_n\}_{n\in\omega}$ of subsets of $X$ satisfying $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist $a_i \in A_i$ such that $x \in \overline{\{a_n : n \in \omega\}}$.*

Proof: (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c). Assume (b) and fix $x \in X$ and a decreasing family $\{A_n\}_{n\in\omega}$ of subsets of $X$ such that $x \in \bigcap\{\overline{A_n} : n \in \omega\}$. Consider the subset $\{n_k : k \in \omega\}$ of $\omega$ defined as follows: $n_1 = 1$ and $n_k = n_{k-1} + \phi(k)$. Since $x \in \overline{A_{n_k}}$ for each $k$, select $H_k \subseteq A_{n_k}$ with $|H_k| \leq \phi(k)$ and $x \in \overline{\bigcup\{H_n : n \in \omega\}}$. Without loss of generality it may be assumed that $H_k = \{x^k_1, x^k_2, \dots, x^k_{\phi(k)}\}$. For $i \in \omega$ such that $n_{k-1} < i \leq n_k$, put $a_i = x^k_{i-n_{k-1}}$. Clearly, $a_i \in A_i$ and $x \in \overline{\{a_i : i \in \omega\}} = \overline{\bigcup\{H_n : n \in \omega\}}$.

(c) $\Rightarrow$ (a). Assume (c) and fix $\{B_n\}_{n\in\omega}$ with $x \in \bigcap\{\overline{B_n} : n \in \omega\}$. Put $A_i = \bigcup\{B_k : k \geq i\}$. The family $\{A_i : i \in \omega\}$ satisfies (c); select $a_i \in A_i$ with $x \in \overline{\{a_n : n \in \omega\}}$ and put $H_i = B_i \cap \{a_n : 1 \leq n \leq i\}$. Clearly, $|H_i| \leq i$ and $x \in \overline{\bigcup\{H_n : n \in \omega\}}$. $\square$

**Proposition 4.** *Let $X$ be a Fréchet space of countable fan-tightness. Then $vet^*(X) \leq \omega$.*

Proof: Fix a decreasing sequence $\{A_n : n \in \omega\}$ of subsets of $X$ and a point $x \in X$ such that $x \in \bigcap\{\overline{A_n} \setminus A_n : n \in \omega\}$. There exist finite sets $H_n \subseteq A_n$ with $x \in \overline{\bigcup\{H_n : n \in \omega\}}$. Choose a sequence $\{a_n : n \in \omega\} \subseteq \bigcup\{H_n : n \in \omega\}$ converging to $x$ and define a countable subset of $A_1$ as follows: for each $i \in \omega$, put $k_i = \max\{n : a_n \in H_i\}$ if $\{a_n\}_{n\in\omega} \cap H_i \neq \emptyset$ and put $b_i = a_{k_i}$, if $\{a_n\}_{n\in\omega} \cap H_i \neq \emptyset$ and $b_i = b_{i+1}$ otherwise. Since each $H_i$ is finite, the sequence $\{b_i\}_{i\in\omega}$ is well-defined and $b_i \in A_i$ for each $i$. Since $\{b_i\}_{i\in\omega}$ contains a subsequence of $\{a_n\}_{n\in\omega}$, we have $x \in \overline{\{b_i : i \in \omega\}}$ and therefore $vet^*(X) \leq \omega$. $\square$

**Theorem 5.** *Let $X$ be a Tychonoff space. Then the following are equivalent:*

(a) *$vet^*C_p(X) \leq \omega$;*

(b) *for every sequence $\{\gamma_n : n \in \omega\}$ of open $\omega$-covers of $X$ there exist $\lambda_n \subseteq \gamma_n$ such that $|\lambda_n| \leq n$ and $\bigcup\{\lambda_n : n \in \omega\}$ is an $\omega$-cover of $X$.*

PROOF: Assume (a). Fix a sequence $\{\gamma_n : n \in \omega\}$ of open $\omega$-covers of $X$ and for each natural number $n$ put $A_n = \{f \in C_p(X) : \exists U \in \gamma_n$ such that $f(X \setminus U) = \{0\}\}$. Put $f^*(x) = 1$ for each $x \in X$. Clearly, $f^* \in \overline{A_n}$ for each $n$. Choose $H_n \subseteq A_n$ such that $f^* \in \overline{\bigcup\{H_n : n \in \omega\}}$ and $|H_n| \leq n$. For each $n$ and for each $f \in H_n$ fix $U_f \in \gamma_n$ such that $f(X \setminus U_f) = \{0\}$ and put $\lambda_n = \{U_f : f \in H_n\}$. To show that $\bigcup\{\lambda_n : n \in \omega\}$ is an $\omega$-cover of $X$, fix $x_1, \dots, x_k \in X$. There exist $n \in \omega$ and $f \in H_n$ such that $f \in W(x_1, \dots, x_k;\ f^*;\ 1/2)$. Thus for each $i = 1, \dots, k$ we have $f(x_i) > \frac{1}{2}$ and $x_i \in U_f$.

Assume (b) and fix $f \in C_p(X)$ and a sequence $\{A_n\}_{n\in\omega}$ of subsets of $X$ such that $f \in \bigcap\{\overline{A_n} : n \in \omega\}$. Put $\gamma_n = \{(g - f)^{-1}(-\frac{1}{n},\ \frac{1}{n}) : g \in A_n\}$. To show that $\gamma_n$ is an $\omega$-cover of $X$, fix $x_1, \dots, x_k \in X$. Since $W(x_1, \dots, x_k;\ f;\ \frac{1}{n}) \cap A_n \neq \emptyset$, there exists $g \in A_n$ such that $x_i \in (g - f)^{-1}(-\frac{1}{n},\ \frac{1}{n})$ for each $i = 1, \dots, k$.
Case 1. There exists a subsequence $\{n_k\}_{k\in\omega}$ such that $X \in \gamma_{n_k}$ for each $k$. Fix $g_{n_k} \in A_{n_k}$ such that $X = (g_{n_k} - f)^{-1}(-\frac{1}{n_k}, \frac{1}{n_k})$. It is easy to see that $f \in \overline{\{g_{n_k} : k \in \omega\}}$.
Case 2. $X$ is an element of finitely many members of $\{\gamma_n\}$. Without loss of generality we may assume that $X \notin \gamma_n$ for each $n$. Choose $\lambda_n \subseteq \gamma_n$ with $|\lambda_n| \leq n$ and for each $U \in \lambda_n$, fix $g_U \in A_n$ with $U = (g_U - f)^{-1}(-\frac{1}{n}, \frac{1}{n})$. Put $H_n = \{g_U : U \in \lambda_n\}$. Fix a basic open neighborhood $W(x_1, \dots, x_k;\ f;\ \frac{1}{n})$ of $f$. Since $X \notin \bigcup\{\lambda_n : n \in \omega\}$, $|\{U \in \bigcup\{\lambda_n : n \in \omega\} : x_i \in U$, for each $i = 1, \dots, k\}| = \omega$ and there exists $N \geq n$ such that for some $U \in \lambda_N$, $x_i \in (g_U - f)^{-1}(-\frac{1}{N}, \frac{1}{N})$ for all $i$. Hence $g_U \in W(x_1, \dots, x_k;\ f;\ \frac{1}{n}) \cap H_N$ and $f \in \overline{\bigcup\{H_n : n \in \omega\}}$. This completes the proof. □

*Question* 1. Does $vet^*(X) \leq \omega$ imply that $X$ has countable strong fan-tightness? In particular, are these two properties equivalent for function spaces (equivalently, are condition (b) of Theorem 2 and condition (b) of Theorem 5 equivalent)?

**Corollary 6.** *Condition (b) of Theorem 5 is preserved by t-equivalence.*

*Remark.* It can be shown that a space $X$ satisfies condition (b) of Theorem 5 if and only if for each finite power $X^k$ of $X$ and for each sequence $\{\gamma_n : n \in \omega\}$ of open covers of $X^k$ there exist $\lambda_n \subseteq \gamma_n$ such that $|\lambda_n| \leq n$ and $\bigcup\{\lambda_n : n \in \omega\}$ is a cover of $X^k$. It can also be shown that every Tychonoff space $X$ satisfying condition (b) of Theorem 5 is zero-dimensional.

**Example 7.** Countable fan-tightness does not imply that $vet^*(X) \leq \omega$: Consider $X = C_p(0,1)$, where $(0,1)$ is the open unit interval. By Arhangelskii's theorem, $vet\ C_p(0,1) \leq \omega$. It is easy to see, however, that the sequence $\{\gamma_n\}$ of open covers of $(0,1)$, where $\gamma_n = \{\bigcup\{(a_i,\ b_i) : 1 \leq i \leq k\} : k \in \omega,\ a_i, b_i \in (0,1)$, and $\sum_{i=1}^{k}(b_i - a_i) < \frac{1}{n3^n}\}$, does not admit the choice of $\lambda_n \subseteq \gamma_n$ satisfying condition (b) of Theorem 5 and, therefore, $vet^* C_p(0,1) \not\leq \omega$.

Denote by $S_c$ the space obtained by identifying the limit points of continuum many convergent sequences.

**Theorem 8.** *Let $X$ be a topological space such that $t(S_c \times X) \le \omega$. Then $vet^*(X) \le \omega$.*

Proof: Enumerate the convergent sequences of $S_c$ by the elements of $\mathbb{R}$: $S_c = \{C_\alpha : \alpha \in \mathbb{R}\} \cup \{O\}$, where $C_\alpha = \{\frac{\alpha}{n} : n \in \omega\}$ and $O$ is the only non-isolated point of $S_c$.

Fix $x^* \in X$ and a countable family $\{A_n : n \in \omega\}$ of subsets of $X$ such that $x^* \in \bigcap\{\overline{A_n} : n \in \omega\}$. Since $t(X) \le t(S_c \times X) = \omega$, we may assume without loss of generality that $|A_n| = \omega$.

Consider $K = \{(a_i)_{i\in\omega} : a_i \in A_i \ \forall\, i \in \omega\}$. Since $|K| = 2^\omega$, $K = \{\xi_\alpha : \alpha \in \mathbb{R}\}$, where each $\xi_\alpha = (a_i^\alpha)_{i\in\omega}$ and $\xi_\alpha \ne \xi_{\alpha'}$ whenever $\alpha \ne \alpha'$.

For each $\alpha \in \mathbb{R}$, put $\zeta_\alpha = \{(\frac{\alpha}{n}, a_n^\alpha) : n \in \omega\}$. Let $B = \bigcup\{\zeta_\alpha : \alpha \in \mathbb{R}\} \subseteq S_c \times X$.

Claim 1: $\overline{B} \ni (O, x^*)$. Fix a neighborhood $O_{x^*}$ of $x^*$ in $X$ and a neighborhood $V$ of $O$ in $S_c$. For each $n \in \omega$ there exists an $a_n^* \in O_{x^*} \cap A_n$. Also, there is a real number $\alpha^*$ such that $\xi_{\alpha^*} = (a_i^*)_{i\in\omega}$. Since $V$ contains all but finitely many points of $C_{\alpha^*}$, $\zeta_{\alpha^*} \cap (V \times O_{x^*}) \ne \emptyset$.

Choose a countable subset $M$ of $B$ such that $\overline{M} \ni (O, x^*)$. Without loss of generality, it may be assumed that $M = \bigcup\{\zeta_{\alpha_k} : k \in \omega\}$. Put $H_i = \{a_i^{\alpha_k} : 1 \le k \le i\}$. Clearly, each $H_i$ is a subset of $A_i$ and $|H_i| \le i$.

Claim 2: $x^* \in \overline{\bigcup\{H_i : i \in \omega\}}$. Fix a neighborhood $O_{x^*}$ of $x^*$ in $X$. Put $V = S_c \setminus \{\frac{\alpha_k}{n} : k \in \omega, n \in \omega, n < k\}$. Clearly, $V$ is an open neighborhood of $O$, and, consequently, $M \cap (V \times O_{x^*}) \ne \emptyset$. Fix natural numbers $k$ and $n$ such that $(\frac{\alpha_k}{n},\ a_n^{\alpha_k}) \in (V \times O_{x^*}) \cap M$. Since $\frac{\alpha_k}{n} \in V$, we have $n \ge k$ and, therefore, $a_n^{\alpha_k} \in H_n \cap O_{x^*}$. This completes the proof. $\square$

The following two corollaries provide answers to questions posed in [3]:

**Corollary 9.** *Let $X$ be a Hausdorff space such that $t(S_c \times X) \le \omega$. Then $vet\ (X) \le \omega$.*

*Remark.* Example 7 shows that the last corollary cannot be reversed: It follows from theorem (on product) that $t(C_p(0,1) \times S_c) > \omega$. In fact, it follows from Example 2 in [6] that $t(C_p(0,1) \times S_c) \ge 2^\omega$.

*Question* 2. Can Theorem 8 be reversed? Also, is it true that for a space $X$ of countable strong fan-tightness we have $t(S_c \times X) \le \omega$?

**Corollary 10.** *Let $X$ be a Tychonoff space such that $t(C_p(X) \times S_c) \le \omega$. Then $X$ is a Hurewicz space.*

It was shown in [3] that for a regular countably compact space $X$ of countable tightness, the tightness of the product space $S_c \times X$ is countable. It was also proved that for a regular countably compact space, countable fan-tightness and countable tightness are equivalent. The following corollary improves the last result:

**Corollary 11.** *Let $X$ be a regular countably compact space of countable tightness. Then $vet^*(X) \leq \omega$.*

*Question* 3. Let $X$ be a regular pseudocompact space of countable tightness. Is it true that $t(S_c \times X) \leq \omega$? In particular, is it true that $vet^*(X) \leq \omega$?

**Definition.** A mapping $f : X \to Y$ is said to be *countably biquotient*, if for each point $y \in Y$ and for each increasing open cover $\{U_n : n \in \omega\}$ of $f^{-1}(y)$ there is a number $n$ such that $y \in Int(f(U_n))$.

**Proposition 12.** *Let $X$ be a space such that $vet^*(X) \leq \omega$ and let $f : X \to Y$ be a continuous countably biquotient mapping onto $Y$. Then $vet^*(Y) \leq \omega$.*

Proof: Use condition (c) of Lemma 3. Fix $y \in Y$ and a countable decreasing family $\{A_n\}_{n\in\omega}$ of subsets of $Y$ such that $y \in \bigcap \overline{A_n}$. Put $B_n = f^{-1}(A_n)$. There is a point $x \in f^{-1}(y)$ such that $x \in \bigcap\{\overline{B_n} : n \in \omega\}$. Indeed, otherwise $\{X \setminus \overline{B_n} : n \in \omega\}$ be an increasing cover of $f^{-1}(y)$ and for some $n$, we would have $y = f(x) \in Int(f(X \setminus \overline{f^{-1}(A_n)})) \subseteq Y \setminus A_n$, a contradiction to $y \in \overline{A_n}$.

Fix $x \in f^{-1}(y)$ such that $x \in \overline{B_n}$ for each $n$ and choose $b_i \in B_i$ with the property $x \in \overline{\{b_i : i \in \omega\}}$. It is easy to see that $f(b_i) \in A_i$ and $y \in \overline{\{f(b_i) : i \in \omega\}}$. By Lemma 3, $vet^*(Y) \leq \omega$. □

**Corollary 13.** *If $X$ is a topological space such that $vet^*(X) \leq \omega$ and $Y$ is an image of $X$ under a continuous open mapping, then $vet^*(Y) \leq \omega$.*

We shall say that a space $X$ has *countable omega-fan-tightness* ($vet_\omega(X) \leq \omega$) if for each point $x \in X$ and any countable family $\{A_n\}_{n\in\omega}$ of countable subsets of $X$ satisfying $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist finite sets $H_n \subseteq A_n$ such that $x \in \overline{\bigcup\{H_n : n \in \omega\}}$.

**Theorem 14.** *Let $X$ be a Tychonoff space such that for every finite $k$ and for every sequence $\{\gamma_n\}_{n\in\omega}$ of countable open covers of $X^k$ there exist finite subfamilies $\lambda_n \subseteq \gamma_n$ such that $\bigcup\{\lambda_n : n \in \omega\}$ is a cover of $X^k$. Then $vet_\omega C_p(X) \leq \omega$.*

Proof: Fix a family $\{A_n : n \in \omega\}$ of countable subsets of $C_p(X)$ and a function $f \in C_p(X)$ such that $f \in \bigcap \overline{A_n}$. Fix natural numbers $n$ and $k$ and for each $g \in A_k$, put $V_n(g) = (g - f)^{-1}(-\frac{1}{n}, \frac{1}{n})$. Put $\gamma_k^n = \{V_n(g)^n : g \in A_k\}$. The family $\gamma_k^n$ is an open cover of $X^n$. Indeed, for $(x_1, x_2, \dots, x_n) \in X^n$ there is $h \in A_k$ such that $h \in W(x_1, x_2, \dots, x_n; f; \frac{1}{n})$ and hence $(x_1, x_2, \dots, x_n) \in V_n(h)^n$.

Consider a sequence $\{\gamma_k^n : k \geq n\}$ of open countable covers of $X^n$ and select finite families $\lambda_k^n = \{V_n(g) : g \in H_k^n\} \subseteq \gamma_k^n$, where $H_k^n$ is a finite subset of $A_n$ and $\bigcup\{\lambda_k^n : k \geq n\} \supseteq X$. Put $H_i = \bigcup\{H_i^n : n \leq i\}$. Clearly, $H_i$ is a finite subset of $A_i$.

Claim: $f \in \overline{\bigcup\{H_i : i \in \omega\}}$. Fix $x_1, x_2, \dots, x_n \in X$ and $\epsilon > 0$. It may be assumed without loss of generality that $\frac{1}{n} < \epsilon$. For some natural number $k \geq n$, we have $\bigcup \lambda_k^n \ni (x_1, x_2, \dots, x_n)$ and there is an $h \in H_k^n \subseteq H_k$ such that

$(x_1, x_2, \ldots, x_n) \in V_n(h)^n$, i.e. $h \in W(x_1, x_2, \ldots, x_n; f; \epsilon) \cap H_k$. The proof is complete. $\square$

The following two theorems were proved by Professor Arhangelskii, who kindly permitted me to include them in this paper.

**Theorem 15.** *Let $X$ be a Tychonoff pseudocompact space. Then $vet_\omega C_p(X) \leq \omega$.*

PROOF: The restriction mapping $r : C_p(\beta X) \to C_p(X)$ is a continuous bijection. Fix a countable subset $A \subseteq C_p(X)$. Then the restriction of the inverse mapping $r^{-1}|_A : A \to C_p(\beta X)$ is continuous. Indeed, for each $z \in \beta X \setminus X$ the set $F = \bigcap\{\tilde{g}^{-1}(g(z)) : g \in A\}$, where $\tilde{g}$ is an extension of function $g$ to a continuous function on $\beta X$, is a $G_\delta$-set in $\beta X$ and, therefore, there exists a $y_z \in X \cap F$, i.e. $g(z) = g(y_z)$ for each $g \in A$. From here, $r(W(z, r^{-1}(f), \epsilon) \cap r^{-1}(A)) = W(y_z, f, \epsilon) \cap A$ for any $f \in A$. Clearly, for each $x \in X$ and each $f \in A$ we have $r(W(x, r^{-1}(f), \epsilon) \cap r^{-1}(A)) = W(x, f, \epsilon) \cap A$.

Fix $f \in C_p(X)$ and a sequence $\{A_n\}_{n \in \omega}$ of subsets of $C_p(X)$ such that $f \in \bigcup \overline{A_n}$. Then $r^{-1}(f) \in \bigcap \overline{r^{-1}(A_n)}$ and by Arhangelskii's Theorem, $vet(C_p(\beta X)) \leq \omega$. Fix finite $H_n \subseteq r^{-1}(A_n)$ such that $r^{-1}(f) \in \overline{\bigcup\{H_n : n \in \omega\}}$. It follows that $f \in \overline{\bigcup\{r(H_n) : n \in \omega\}}$ and each $r(H_n)$ is a finite subset of $A_n$. $\square$

*Remark.* The last theorem shows that Theorem 14 cannot be reversed: a pseudocompact Tychonoff space not satisfying the assumptions of Theorem 14 would be a counterexample.

It is known that countable fan-tightness is preserved by continuous open surjective mappings. The next theorem shows that it is not true for countable omega-fan-tightness.

**Theorem 16.** *Let $Y$ be a Tychonoff space. Then there exist a Tychonoff space $X$ with $vet_\omega(X) \leq \omega$ and a continuous open surjection $f : X \to Y$.*

PROOF: Consider the space $Z = \big((\omega_1 + 1) \times \beta(C_p(Y))\big) \setminus \big(\{\omega_1\} \times (\beta(C_p(Y)) \setminus C_p(Y))\big)$. Since $Z$ contains a dense countably compact space $\omega_1 \times \beta(C_p(Y))$, the space $Z$ is pseudocompact, and therefore $vet_\omega(C_p(Z)) \leq \omega$. It is easy to see that $\{\omega_1\} \times C_p(Y)$ is closed in $Z$ and every bounded continuous function on $\{\omega_1\} \times C_p(Y)$ can be extended to a continuous function on $Z$. Thus the restriction mapping $r : C_p(Z) \to C_p(\{\omega_1\} \times C_p(Y)) = C_p(C_p(Y))$ is an open mapping; a topological copy of $Y$ is contained in $C_p^0(C_p(Y)) \subseteq r(C_p(Z))$. Put $X = r^{-1}(Y)$ and put $f = r|_X$. It is easy to see that $f$ is a continuous open mapping onto $Y$ and $vet_\omega(X) \leq \omega$. $\square$

**Remark.** After this paper had been submitted, the author proved independently from S. Garcia-Ferreira and A. Tamariz-Mascarua that for a Tychonoff space $X$, $vet^* C_p(X) \leq \omega$ implies that $C_p(X)$ has countable strong fan-tightness. In the article “Some generalizations of rapid ultrafilters in topology and id-fan tightness”, Tsukuba J. Math, 19 (1) (1995), 173–185, the two authors also showed that $vet^*(X) \leq \omega$ does not imply in general that $X$ has countable strong fan-tightness. This provides a complete answer to Question 1. It is not clear, however, whether the two properties coincide for topological groups.

Also, A. Bella noticed that the answer to Question 2 is negative.

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