Uniformly μ-continuous topologies on Köthe-Bochner spaces and Orlicz-Bochner spaces

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Abstract. Some class of locally solid topologies (called uniformly μ-continuous) on Köthe-Bochner spaces that are continuous with respect to some natural two-norm convergence are introduced and studied. A characterization of uniformly μ-continuous topologies in terms of some family of pseudonorms is given. The finest uniformly μ-continuous topology $\mathcal{T}_I^{\varphi}(X)$ on the Orlicz-Bochner space $L^{\varphi}(X)$ is a generalized mixed topology in the sense of P. Turpin (see [11, Chapter I]).

Keywords: Orlicz spaces, Orlicz-Bochner spaces, Köthe-Bochner spaces, locally solid topologies, generalized mixed topologies, uniformly μ-continuous topologies, inductive limit topologies

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1. Preliminaries.

For notation and terminology concerning locally solid Riesz spaces we refer to [1].

Throughout the paper let (Ω, Σ, μ) be a complete σ-finite measure space and let L^0 denote the corresponding space of equivalence classes of all Σ-measurable real valued functions. Then L^0 is a super Dedekind complete Riesz space under the ordering $u_1 \le u_2$ whenever $u_1(\omega) \le u_2(\omega)$ μ-a.e. on Ω.

For $u \in L^0$ let us put

$$\|u\|_\mu = \inf\{\lambda > 0 : \mu(\{\omega \in \Omega : |u(\omega)| > \lambda\}) \le \lambda\}.$$

It is easy to see that a sequence (u_n) in L^0 is convergent to $u \in L^0$ in measure on Ω (in symbols $u_n \to u$ $(\mu - \Omega)$) iff $\|u_n - u\|_\mu \to 0$. We will denote by $\mathcal{T}_\mu$ the topology on L^0 of $\|\cdot\|_\mu$.

For a subset A of Ω let χ_A stand for its characteristic function.

Let $[x]$ denote the greatest integer which is less or equal to a real number x.

Let $(E, \|\cdot\|_E)$ be an F-normed function space, that is E is an ideal of L^0 with $\operatorname{supp} E = \Omega$ and $\|\cdot\|_E$ is a complete Riesz F-norm. The Köthe dual E' of E is defined by

$$E' = \{v \in L^0 : \int_\Omega |u(\omega)v(\omega)|\, d\mu < \infty \text{ for all } u \in E\}.$$

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In case $(E, \|\cdot\|_E)$ is a Banach function space the associated norm $\|\cdot\|_{E'}$ on E' can be defined for $v \in E'$ by

$$\|v\|_{E'} = \sup\Big\{\Big|\int_\Omega u(\omega)v(\omega)\,d\mu\Big| : u \in E,\ \|u\|_E \le 1\Big\}.$$

We will write $A_n \searrow \emptyset$ when (A_n) is a decreasing sequence in Σ such that $\mu(A_n \cap A) \to 0$ for every $A \in \Sigma$ with $\mu(A) < \infty$.

We denote by E_a the ideal of elements of absolutely continuous norm in E, i.e. $E_a = \{u \in E : \|\chi_{A_n} u\|_E \to 0 \text{ as } A_n \searrow \emptyset\}$.

Let $(X, \|\cdot\|_X)$ be a real Banach space, and let S_X and B_X denote the unit sphere and the closed unit ball in X, respectively.

By $L^0(X)$ we will denote the linear space of equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$.

For $f \in L^0(X)$ let us put

$$\|f\|_\mu^X = \inf\{\lambda > 0 : \mu(\{\omega \in \Omega : \|f(\omega)\|_X > \lambda\}) \le \lambda\}.$$

We say that a sequence (f_n) in $L^0(X)$ is convergent to $f \in L^0(X)$ in measure on Ω (in symbols $f_n \to f\ (\mu - \Omega)$) whenever $\mu(\{\omega \in \Omega : \|f_n(\omega) - f(\omega)\|_X > \varepsilon\}) \to 0$ for every $\varepsilon > 0$. It can be seen that a sequence (f_n) in $L^0(X)$ is convergent to $f \in L^0(X)$ in measure on Ω iff $\|f_n - f\|_\mu^X \to 0$. The topology on $L^0(X)$ of $\|\cdot\|_\mu^X$ will be denoted by $\mathcal{T}_\mu(X)$.

For $f \in L^0(X)$ let

$$\widetilde{f}(\omega) := \|f(\omega)\|_X \quad \text{for} \quad \omega \in \Omega.$$

The linear space $E(X) = \{f \in L^0(X) : \widetilde{f} \in E\}$ provided with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is called a Köthe-Bochner space (see [2], [3]).

Now we recall some concepts and terminology concerning locally solid topologies on vector-valued function spaces as set out in [3].

A subset H of $E(X)$ is said to be *solid* whenever $\|f_1(\omega)\|_X \le \|f_2(\omega)\|_X$ μ-a.e. and $f_1 \in E(X)$, $f_2 \in H$ imply $f_1 \in H$.

A pseudonorm ρ on $E(X)$ is said to be *solid* whenever for $f_1, f_2 \in E(X)$, $\|f_1(\omega)\|_X \le \|f_2(\omega)\|_X$ μ-a.e. imply $\rho(f_1) \le \rho(f_2)$.

A linear topology τ on $E(X)$ is said to be *locally solid* if it has a basis for neighbourhoods of zero consisting of solid sets.

A linear topology τ on $E(X)$ that is at the same time locally solid and locally convex will be called a *locally convex-solid topology* on $E(X)$.

Theorem 1.1 (see [3, Theorem 2.2, Theorem 2.3]). *For a linear topology τ on $E(X)$ the following statements are equivalent:*

(i) *τ is a locally solid topology (respectively τ is a locally convex-solid topology);*

(ii) *τ is generated by some family of solid pseudonorms (respectively seminorms).*

Now we are going to explain the relationship between locally solid topologies on E and $E(X)$ (see [3]).

Let p be a Riesz pseudonorm (respectively seminorm) on E, and let

$$\overline{p}(f) := p(\widetilde{f}) \quad \text{for} \quad f \in E(X).$$

Then $\overline{p}$ is a solid pseudonorm (respectively seminorm) on $E(X)$.

Next, fix $x \in S_X$. Given $u \in E$ let us put $\overline{u}(\omega) := u(\omega) \cdot x$ for $\omega \in \Omega$. Then $\overline{u} \in L^0(X)$ and $\|\overline{u}(\omega)\|_X = |u(\omega)|$ for $\omega \in \Omega$, so $\overline{u} \in E(X)$.

Let ρ be a solid pseudonorm (respectively seminorm) on $E(X)$, and let

$$\widetilde{\rho}(u) := \rho(\overline{u}) \quad \text{for} \quad u \in E.$$

Then $\widetilde{\rho}$ is a Riesz pseudonorm (respectively seminorm) on E.

Theorem 1.2 (see [3, Lemma 3.1]). (i) *If ρ is a solid pseudonorm on $E(X)$, then $\overline{\widetilde{\rho}}(f) = \rho(f)$ for $f \in E(X)$.*

(ii) *If p is a Riesz pseudonorm on E, then*

$$\widetilde{\overline{p}}(u) = p(u) \quad \textit{for} \quad u \in E.$$

Let τ be a locally solid topology on $E(X)$ generated by some family $\{\rho_\alpha : \alpha \in \{\alpha\}\}$ of solid pseudonorms defined on $E(X)$. By $\widetilde{\tau}$ we will denote the locally solid topology on E generated by the family $\{\widetilde{\rho}_\alpha : \alpha \in \{\alpha\}\}$ of Riesz pseudonorms on E. If τ is a Hausdorff topology, then so is $\widetilde{\tau}$.

In turn, let ξ be a locally solid topology on E generated by some family $\{p_\alpha : \alpha \in \{\alpha\}\}$ of Riesz pseudonorms on E. By $\overline{\xi}$ we will denote the locally solid topology on $E(X)$ generated by the family $\{\overline{p}_\alpha : \alpha \in \{\alpha\}\}$ of solid pseudonorms on $E(X)$. Then $\overline{\xi}$ is a Hausdorff topology, whenever ξ is Hausdorff.

Theorem 1.3 (see [3, Theorem 3.2]). (i) *For a locally solid topology τ on $E(X)$ we have: $\overline{\widetilde{\tau}} = \tau$.*

(ii) *For a locally solid topology ξ on E we have: $\widetilde{\overline{\xi}} = \xi$.*

Now we recall some notation and terminology concerning Orlicz spaces (see [5], [6], [11] for more details).

By an Orlicz function we mean a function $\varphi : [0, \infty) \to [0, \infty]$ which is non-decreasing, left continuous, continuous at 0 with $\varphi(0) = 0$ and not identically equal to 0.

A convex Orlicz function is usually called a Young function. For a Young function φ we denote by φ^* the function complementary to φ in the sense of Young, i.e.

$$\varphi^*(s) = \sup\{ts - \varphi(t) : t \geq 0\} \quad \text{for} \quad s \geq 0.$$

Let φ and ψ be a pair of Orlicz functions vanishing only at zero (respectively taking only finite values). We say that φ *increases essentially more rapidly than ψ*

for small t (respectively for large t) denoted $\psi \overset{\mathrm{s}}{\prec\prec} \varphi$ (respectively $\psi \overset{\mathrm{l}}{\prec\prec} \varphi$), whenever for any $c > 0$, $\psi(ct)/\varphi(t) \to 0$ as $t \to 0$ (respectively $t \to \infty$). We will write $\psi \prec\prec \varphi$ when $\psi \overset{\mathrm{s}}{\prec\prec} \varphi$ and $\psi \overset{\mathrm{l}}{\prec\prec} \varphi$ hold. For φ and ψ being Young functions the condition $\psi \overset{\mathrm{s}}{\prec\prec} \varphi$ (respectively $\psi \overset{\mathrm{l}}{\prec\prec} \varphi$) implies $\varphi^* \overset{\mathrm{s}}{\prec\prec} \psi^*$ (respectively $\varphi^* \overset{\mathrm{l}}{\prec\prec} \psi^*$) (see [5, Lemma 13.1]).

An Orlicz function φ determines a functional $m_\varphi : L^0 \to [0, \infty]$ by

$$m_\varphi(u) = \int_\Omega \varphi(|u(\omega)|)\, d\mu.$$

The Orlicz space generated by φ is the ideal of L^0 defined by

$$L^\varphi = \{u \in L^0 : m_\varphi(\lambda u) < \infty \ \text{ for some } \ \lambda > 0\}.$$

L^φ can be equipped with the complete metrizable topology $\mathcal{T}_\varphi$ of the F-norm

$$\|u\|_\varphi = \inf\Big\{\lambda > 0 : m_\varphi\big(\frac{u}{\lambda}\big) \le \lambda\Big\}.$$

Let

$$\varphi_0(t) = \begin{cases} 0 & \text{for } \ 0 \le t \le 1 \\ 1 & \text{for } \ t > 1. \end{cases}$$

It is known that L^{φ_0} is the largest Orlicz space and consists of all those $u \in L^0$ that are bounded outside of some set of finite measure and $\|u\|_{\varphi_0} = \|u\|_\mu$ for all $u \in L^{\varphi_0}$. (see [11, 0.3.4]).

Moreover one can check that L^{φ_0} is the largest linear subspace of L^0 such that the functional $\|\cdot\|_\mu$ restricted to L^{φ_0} is an F-norm.

We will write $\|\cdot\|_\mu$ and $\mathcal{T}_\mu$ instead of $\|\cdot\|_{\varphi_0}$ and $\mathcal{T}_{\varphi_0}$, respectively.

Moreover, if φ is a Young function, then the topology $\mathcal{T}_\varphi$ can be generated by the Luxemburg norm:

$$|||u|||_\varphi = \inf\Big\{\lambda > 0 : m_\varphi\big(\frac{u}{\lambda}\big) \le 1\Big\}.$$

For an Orlicz function φ let

$$E^\varphi = \{u \in L^0 : m_\varphi(\lambda u) < \infty \ \text{ for all } \ \lambda > 0\}$$

and

$$L^\varphi_a = \{u \in L^\varphi : \|u_{A_n}\|_\varphi \to 0 \ \text{ as } \ A_n \searrow \emptyset\}.$$

It is well known that $E^\varphi = L^\varphi_a$ whenever φ takes only finite values. Moreover, for every Young function φ the identity $(L^\varphi)' = L^{\varphi^*}$ holds.

Let $M_\varphi : L^0(X) \to [0,\infty]$ be defined by

$$M_\varphi(f) = \int_\Omega \varphi(\|f(\omega)\|_X)\, d\mu.$$

Thus $M_\varphi(f) = m_\varphi(\widetilde{f})$. The Köthe-Bochner space

$$\begin{aligned} L^\varphi(X) &= \{f \in L^0(X) : \widetilde{f} \in L^\varphi\} \\ &= \{f \in L^0(X) : M_\varphi(\lambda f) < \infty \ \text{ for some } \ \lambda > 0\} \end{aligned}$$

is usually called *an Orlicz-Bochner space* and is equipped with the F-norm

$$\|f\|_{L^\varphi(X)} = \|\widetilde{f}\|_\varphi \ \text{ for } \ f \in L^\varphi(X).$$

We will denote by $\mathcal{T}_\varphi(X)$ the topology on $L^\varphi(X)$ generated by the F-norm $\|\cdot\|_{L^\varphi(X)}$. Moreover, if φ is a Young function, then $\mathcal{T}_\varphi(X)$ is generated by the Luxemburg norm: $|||f|||_{L^\varphi(X)} = |||\widetilde{f}|||_\varphi$ for $f \in L^\varphi(X)$. We will write $\|\cdot\|_\mu^X$ and $\mathcal{T}_\mu(X)$ instead of $\|\cdot\|_{L^{\varphi_0}(X)}$ and $\mathcal{T}_{\varphi_0}(X)$, respectively.

2. Uniformly μ-continuous topologies on Köthe-Bochner spaces

Definition 2.1. (i) A solid pseudonorm ρ on $E(X)$ is said to be *uniformly μ-continuous*, whenever $f_n \in E(X)$, $f_n \to 0$ $(\mu - \Omega)$ with $\sup_n \|f_n\|_{E(X)} < \infty$ imply $\rho(f_n) \to 0$.
(ii) A locally solid topology τ on $E(X)$ is said to be *uniformly μ-continuous* whenever $f_n \in E(X)$, $f_n \to 0$ $(\mu - \Omega)$ with $\sup_n \|f_n\|_{E(X)} < \infty$ imply $f_n \xrightarrow{\tau} 0$.

In view of [3, Theorem 2.3] a locally solid topology τ on $E(X)$ is uniformly μ-continuous iff it is generated by some family $\{\rho_\alpha : \alpha \in \{\alpha\}\}$ of uniformly μ-continuous pseudonorms defined on $E(X)$.

It is easy to prove the following lemma.

Lemma 2.1. (i) *If ρ is a uniformly μ-continuous pseudonorm on $E(X)$, then $\widetilde{\rho}$ is a uniformly μ-continuous pseudonorm on E (i.e. $u_n \in E$ $u_n \to 0$ $(\mu - \Omega)$ with $\sup_n \|u_n\|_E < \infty$ imply $\widetilde{\rho}(u_n) \to 0$).*
(ii) *If p is a uniformly μ-continuous pseudonorm on E, then $\overline{p}$ is a uniformly μ-continuous pseudonorm on $E(X)$.*

From Lemma 2.1 we easily get the following theorem that explains the relationship between uniformly μ-continuous topologies on E and $E(X)$.

Theorem 2.2. (i) *If τ is a uniformly μ-continuous topology on $E(X)$, then $\widetilde{\tau}$ is a uniformly μ-continuous topology on E.*

(ii) *If ξ is a uniformly μ-continuous topology on E, then $\overline{\xi}$ is a uniformly μ-continuous topology on $E(X)$.*

We shall need the following result.

Theorem 2.3. (i) *If τ is the finest uniformly μ-continuous topology on $E(X)$, then $\widetilde{\tau}$ is the finest uniformly μ-continuous topology on E.*

(ii) *If ξ is the finest uniformly μ-continuous topology on E, then $\overline{\xi}$ is the finest uniformly μ-continuous topology on $E(X)$.*

PROOF: (i) Let ξ be a uniformly μ-continuous topology on E. By Theorem 2.2 $\overline{\xi}$ is a uniformly μ-continuous topology on $E(X)$, so $\overline{\xi} \subset \tau$. By [3, Theorem 3.3] and Theorem 1.3 $\xi = \widetilde{\overline{\xi}} \subset \widetilde{\tau}$, as desired.

(ii) Let τ be a uniformly μ-continuous topology on $E(X)$. By Theorem 2.2 $\widetilde{\tau}$ is a uniformly μ-continuous topology on E, so $\widetilde{\tau} \subset \xi$. By [3, Theorem 3.3] and Theorem 1.3 $\tau = \overline{\widetilde{\tau}} \subset \overline{\xi}$, as desired. □

Now we are going to give a description of uniformly μ-continuous topologies on Orlicz-Bochner spaces. We start with the following definition.

Definition 2.2. A solid pseudonorm ρ on $E(X)$ is said to be *uniformly summable* whenever the following conditions hold:

For every $r > 0$

$$(*) \qquad \sup\{\rho(\chi_{A(f,\lambda)}f) : f \in E(X), \|f\|_{E(X)} \leq r\} \to 0 \ \text{ as } \ \lambda \to 0_+,$$

where $A(f,\lambda) = \{\omega \in \Omega : \|f(\omega)\|_X \leq \lambda \text{ or } \|f(\omega)\|_X > \frac{1}{\lambda}\}$ for $0 < \lambda < 1$ and

$$(**) \qquad \rho(\overline{\chi}_A) \to 0 \ \text{ as } \ \mu(A) \to 0.$$

Theorem 2.4. *Let φ be an arbitrary Orlicz function and ψ be a finite valued Orlicz function such that $\psi \prec\prec \varphi$. Then the F-norm $\|\cdot\|_{L^\psi(X)}$ (restricted to $L^\varphi(X)$) is uniformly summable on $L^\varphi(X)$.*

PROOF: Since $\psi \prec\prec \varphi$, so $L^\varphi \subset L^\psi$ (see [11, 0.2.5, 0.3.5]). Hence $L^\varphi(X) \subset L^\psi(X)$. Let $r > 0$, $\varepsilon > 0$ be given. Choose $\eta > 0$ such that $\eta(r+1) < \varepsilon$ and let $c = \frac{\varepsilon}{r+1}$. Then there exist $0 < t_1 < t_2$ such that $\psi(t) \leq \eta\varphi(ct)$ for $0 \leq t < t_1$ or $t > t_2$, and choose $\lambda_0 \in (0,1)$ such that $\lambda_0 \leq \varepsilon t_1$ and $\frac{1}{\lambda_0} > \varepsilon t_2$. Hence for $f \in L^\varphi(X)$ and $\|f\|_{L^\varphi(X)} \leq r$ we have:

$$M_\psi\Big(\frac{\chi_{A(f,\lambda)}f}{\varepsilon}\Big) = \int_{A(f,\lambda)} \psi\Big(\frac{\|f(\omega)\|_X}{\varepsilon}\Big)\,d\mu \leq \int_{A(f,\lambda)} \eta\varphi\Big(c\frac{\|f(\omega)\|_X}{\varepsilon}\Big)\,d\mu$$
$$\leq \int_\Omega \eta\varphi\Big(\frac{\|f(\omega)\|_X}{r+1}\Big)\,d\mu \leq \eta(r+1) < \varepsilon$$

for every $0 < \lambda \leq \lambda_0$. It follows that $\|\chi_{A(f,\lambda)}f\|_{L^\psi(X)} \leq \varepsilon$ for every $f \in L^\varphi(X)$, $\|f\|_{L^\varphi(X)} \leq r$ and $0 < \lambda \leq \lambda_0$. This means that for $r > 0$

$$\sup\{\|\chi_{A(f,\lambda)}f\|_{L^\psi(X)} : f \in L^\varphi(X),\ \|f\|_{L^\varphi(X)} \leq r\} \to 0 \ \text{ as } \ \lambda \to 0_+.$$

Now, choose $\delta > 0$ such that $0 < \delta < \frac{\varepsilon}{\psi\left(\frac{1}{\varepsilon}\right)}$. Then $M_\psi\left(\frac{\overline{\chi}_A}{\varepsilon}\right) = \int_A \psi\left(\frac{1}{\varepsilon}\right) d\mu = \mu(A)\cdot\psi\left(\frac{1}{\varepsilon}\right) \leq \delta\cdot\psi\left(\frac{1}{\varepsilon}\right) < \varepsilon$ for every $A \in \Sigma$ with $\mu(A) \leq \delta$. Hence $\|\overline{\chi}_A\|_{L^\psi(X)} \to 0$ as $\mu(A) \to 0$, and the proof is finished. $\square$

Remark 2.1. Let φ be an Orlicz function such that $\varphi(u) \to \infty$ as $u \to \infty$. Then $\varphi_0 \prec\prec \varphi$ and it follows that the F-norm $\|\cdot\|_\mu^X$ is uniformly summable on $L^\varphi(X)$.

Theorem 2.5. *Let φ be an Orlicz function such that $\varphi(u) \to \infty$ as $u \to \infty$. For a solid pseudonorm ρ on $L^\varphi(X)$ the following statements are equivalent:*

(i) *ρ is uniformly summable;*

(ii) *ρ is uniformly μ-continuous.*

Proof: (i) $\Rightarrow$ (ii) Take a sequence (f_n) in $L^\varphi(X)$ such that $f_n \to 0$ $(\mu - \Omega)$ and $\sup_n \|f_n\|_{L^\varphi(X)} \leq r$ for some $r > 0$. Fix $\varepsilon > 0$. There exists $\lambda_0 \in (0,1)$ such that $\sup_n \rho(\chi_{A(f_n,\lambda_0)} f_n) < \frac{\varepsilon}{2}$. Moreover, there exists $\delta > 0$ such that

$$\rho(\overline{\chi}_A) < \frac{\varepsilon}{2\left(\left[\frac{1}{\lambda_0}\right] + 1\right)} \quad \text{whenever} \quad A \in \Sigma \quad \text{with} \quad \mu(A) \leq \delta.$$

Since $f_n \to 0$ $(\mu - \Omega)$, we can find a natural number k such that for all $n \geq k$

$$\mu(\Omega \setminus A(f_n,\lambda_0)) \leq \mu(\{\omega \in \Omega : \|f_n(\omega)\|_X > \lambda_0\}) \leq \delta.$$

Hence for $n \geq k$

$$\begin{aligned} \rho(f_n) &= \rho(\chi_{A(f_n,\lambda_0)} f_n + \chi_{\Omega\setminus A(f_n,\lambda_0)} f_n) \leq \rho(\chi_{A(f_n,\lambda_0)} f_n) \\ &\qquad + \rho(\chi_{\Omega\setminus A(f_n,\lambda_0)} f_n) \\ &\leq \frac{\varepsilon}{2} + \rho\Big(\big(\big[\frac{1}{\chi_0}\big] + 1\big)\overline{\chi}_{\Omega\setminus A(f_n,\lambda_0)}\Big) \leq \frac{\varepsilon}{2} + \big(\big[\frac{1}{\lambda_0}\big] + 1\big)\rho(\overline{\chi}_{\Omega\setminus A(f_n,\lambda_0)}) \\ &\leq \frac{\varepsilon}{2} + \big(\big[\frac{1}{\lambda_0}\big] + 1\big)\frac{\varepsilon}{2\left(\left[\frac{1}{\lambda_0}\right] + 1\right)} \leq \varepsilon. \end{aligned}$$

Thus $\rho(f_n) \to 0$.

(ii) $\Rightarrow$ (i) For $r > 0$ let $B_X^\varphi(r) = \{f \in L^\varphi(X) : \|f\|_{L^\varphi(X)} \leq r\}$, $B_X^\rho(r) = \{f \in L^\varphi(X) : \rho(f) \leq r\}$, $B_X^\mu(r) = \{f \in L^{\varphi_0}(X) : \|f\|_\mu^X \leq r\}$. By (ii) the identity map

$$id : (B_X^\varphi(r), \mathcal{T}_\mu(X)|_{B_X^\varphi(r)}) \to (B_X^\varphi(r), \tau(\rho)|_{B_X^\varphi(r)})$$

is continuous at zero for any $r > 0$, where $\tau(\rho)$ denotes the topology on $L^\varphi(X)$ generated by ρ. Let $\varepsilon > 0$, $r > 0$ be given. There exists $\eta > 0$ such that $B_X^\mu(\eta) \cap B_X^\varphi(r) \subset B_X^\rho(\varepsilon)$. Since $\|\cdot\|_\mu^X$ is uniformly summable on $L^\varphi(X)$ (see Remark 2.1) there exists $\lambda_0 \in (0,1)$ such that

$$\sup\{\|\chi_{A(f,\lambda)} f\|_\mu^X : f \in L^\varphi(X),\ \|f\|_{L^\varphi(X)} \leq r\} \leq \eta \quad \text{whenever} \quad 0 < \lambda \leq \lambda_0.$$

Then $\sup\{\rho(\chi_{A(f,\lambda)}f) : f \in L^\varphi(X),\ \|f\|_{L^\varphi(X)} \leq r\} \leq \varepsilon$ whenever $0 < \lambda \leq \lambda_0$. Hence $\sup\{\rho(\chi_{A(f,\lambda)}f) : f \in L^\varphi(X),\ \|f\|_{L^\varphi(X)} \leq r\} \to 0$ as $\lambda \to 0_+$.

Moreover, there exists $\delta > 0$ such that $\|\overline{\chi}_A\|_\mu^X \leq \eta$ for $A \in \Sigma$ with $\mu(A) \leq \delta$. Then $\rho(\overline{\chi}_A) \leq \varepsilon$ whenever $A \in \Sigma$ with $\mu(A) \leq \delta$. It follows that $\rho(\overline{\chi}_A) \to 0$ as $\mu(A) \to 0$.

Thus ρ is a uniformly summable pseudonorm on $L^\varphi(X)$. □

Theorem 2.6. *Let φ be an Orlicz function such that $\varphi(u) \to \infty$ as $u \to \infty$. For a locally solid topology τ on $L^\varphi(X)$ the following statements are equivalent:*

(i) *τ is uniformly μ-continuous;*
(ii) *$\tau|_{B_X^\varphi(r)} \subset \mathcal{T}_\mu(X)|_{B_X^\varphi(r)}$ for every $r > 0$;*
(iii) *τ is generated by some family of uniformly summable pseudonorms.*

PROOF: (i) ⇒ (ii) Since $\mathcal{T}_\mu(X)$ is a linear metrizable topology, it follows from Definition 2.1 (ii).

(ii) ⇒ (i) Obvious.

(i) ⇒ (iii) Let τ be defined by the family $\{\rho_\alpha : \alpha \in \{\alpha\}\}$ of solid pseudonorms. Then by Definition 2.1 and Theorem 2.5 τ is generated by the family $\{\rho_\alpha : \alpha \in \{\alpha\}\}$ of uniformly summable pseudonorms.

(iii) ⇒ (i) It follows from Theorem 2.5. □

3. Generalized mixed topologies on Orlicz-Bochner spaces

In this section we consider some kind of inductive limit topology on Orlicz-Bochner space $L^\varphi(X)$.

Let φ be an arbitrary Orlicz function, and let

$$F_n^X = B_X^\varphi(2^n) \quad\text{and}\quad \mathcal{T}_n(X) = \mathcal{T}_\mu(X)|_{F_n^X} \quad\text{for}\quad n \geq 0.$$

It can be seen that the metric bounded sets F_n^X $(n \geq 0)$ are balanced subsets of $L^\varphi(X)$. Moreover, the sequence $(F_n^X, \mathcal{T}_n(X))$ $(n \geq 0)$ of balanced topological spaces satisfies the following conditions:

(i) $L^\varphi(X) = \bigcup_{n\geq 0} F_n^X$;
(ii) $F_n^X + F_n^X \subset F_{n+1}^X$, and the function

$$F_n^X \times F_n^X \ni (f,g) \mapsto f + g \in F_{n+1}^X$$

is continuous $(n \geq 0)$;
(iii) the function $[-1,1] \times F_n^X \ni (\lambda, f) \mapsto \lambda \cdot f \in F_n^X$ is continuous $(n \geq 0)$;
(iv) $\mathcal{T}_{n+1}(X)|_{F_n^X} = \mathcal{T}_n(X)$ for $n \geq 0$.

Thus the space $L^\varphi(X)$ with the system $\{(F_n^X, \mathcal{T}_n(X)) : n \geq 0\}$ comes under the definition of the strict inductive limit of balanced topological spaces (in the sense of Turpin; see [11, Definition 1.1.1]).

Definition 3.1. Let φ be an Orlicz function and let (ε_n) be a sequence of positive numbers. The family of all sets of the form:

$$(*)\qquad \bigcup_{N=0}^{\infty}\Big(\sum_{n=0}^{N} B_X^{\varphi}(2^n)\cap B_X^{\mu}(\varepsilon_n)\Big)$$

forms a base of neighbourhoods of zero for a linear topology $\mathcal{T}_I^{\varphi}(X)$ on $L^{\varphi}(X)$ that will be called *generalized mixed topology.* $\mathcal{T}_I^{\varphi}(X)$ is exactly the strict inductive limit topology of balanced topological spaces $\{(B_X^{\varphi}(2^n),\mathcal{T}_{\mu}(X)|_{B_X^{\varphi}(2^n)}) : n\geq 0\}$ in the sense of Turpin [11, Chapter I].

Using the solid decomposition property (see [3, Lemma 1.1]) it is easy to verify that the sets of the form $(*)$ are solid, so $\mathcal{T}_I^{\varphi}(X)$ is locally solid.

According to [11, Theorem 1.1.6] $\mathcal{T}_I^{\varphi}(X)$ is the finest of all linear topologies τ on $L^{\varphi}(X)$, which satisfy the condition

$$(1)\qquad \tau|_{B_X^{\varphi}(2^n)}\subset \mathcal{T}_{\mu}(X)|_{B_X^{\varphi}(2^n)}\quad\text{for}\quad n\geq 0.$$

Moreover, in view of [11, Theorem 1.1.8] we have

$$(2)\qquad \mathcal{T}_I^{\varphi}(X)|_{B_X^{\varphi}(2^n)} = \mathcal{T}_{\mu}(X)|_{B_X^{\varphi}(2^n)}\quad\text{for}\quad n\geq 0.$$

Since $\mathcal{T}_{\mu}(X)|_{L^{\varphi}(X)}\subset\mathcal{T}_{\varphi}(X)$ we have $\mathcal{T}_I^{\varphi}(X)\subset\mathcal{T}_{\varphi}(X)$; hence $\mathcal{T}_{\mu}(X)|_{L^{\varphi}(X)}\subset \mathcal{T}_I^{\varphi}(X)\subset\mathcal{T}_{\varphi}(X)$.

Henceforth, we assume in this section that $\varphi(u)\to\infty$ as $u\to\infty$.

Theorem 3.1. *The topology $\mathcal{T}_I^{\varphi}(X)$ is the finest uniformly μ-continuous topology on $L^{\varphi}(X)$.*

PROOF: It follows from (1) and Theorem 2.6. □

The generalized mixed topology $\mathcal{T}_I^{\varphi}$ on Orlicz spaces L^{φ} has been studied in [11], [8], [9], [10]. Now we will extend the study of the generalized mixed topology to the Orlicz-Bochner spaces.

Theorem 3.2. *The space $(L^{\varphi}(X),\mathcal{T}_I^{\varphi}(X))$ is complete.*

PROOF: First we show that the balls $B_X^{\varphi}(2^n)$ are closed subsets of $(L^{\varphi_0}(X),\mathcal{T}_{\mu}(X))$. Indeed, let (f_k) be a sequence in $B_X^{\varphi}(2^n)$ and let $f\in L^{\varphi_0}(X)$ be such that $f_k\to f$ for $\mathcal{T}_{\mu}(X)$. This means that $\mu(\{\omega\in\Omega : \|f_k(\omega)-f(\omega)\|_X > \varepsilon\})\to 0$ for any $\varepsilon>0$. Hence $\mu(\{\omega\in\Omega : \big|\|f_k(\omega)\|_X-\|f(\omega)\|_X\big| > \varepsilon\})\to 0$ for every $\varepsilon>0$. Thus $\widetilde{f_k}\to\widetilde{f}$ for $\mathcal{T}_{\mu}$ in L^{φ_0}. It is known that the balls $B_{\varphi}(2^n)$ are closed subsets of $(L^{\varphi_0},\mathcal{T}_{\mu})$ (see [11, 0.3.6]). But $\widetilde{f_k}\in B_{\varphi}(2^n)$ $(k=1,2,\dots)$, $\widetilde{f}\in L^{\varphi_0}$, so we get $\widetilde{f}\in B_{\varphi}(2^n)$. It follows that $f\in B_X^{\varphi}(2^n)$.

Since the spaces $(B_X^{\varphi}(2^n),\mathcal{T}_{\mu}(X)|_{B_X^{\varphi}(2^n)})$ $(n\geq 0)$ are complete, by [11, Theorem 1.1.10] the space $(L^{\varphi}(X),\mathcal{T}_I^{\varphi}(X))$ is complete.

□

Theorem 3.3. *For a subset $Z \subset L^{\varphi}(X)$ the following statements are equivalent:*

(i) $\sup\{\|f\|_{L^{\varphi}(X)} : f \in Z\} < \infty$;

(ii) *Z is bounded for $\mathcal{T}_I^{\varphi}(X)$.*

Proof: Observe that the balls $B_X^{\varphi}(2^n)$ are bounded subsets of $(L^{\varphi}(X), \mathcal{T}_{\mu}(X)|_{L^{\varphi}(X)})$. In fact, fix an $r > 0$, let $f_n \in B_X^{\varphi}(r)$ $(n = 1, 2, \dots)$ and let $\lambda_n \to 0$. For $\varepsilon > 0$ let $\Omega_n(\varepsilon) = \{\omega \in \Omega : \|\lambda_n f_n(\omega)\|_X > \varepsilon\}$. Then we have

$$\mu(\Omega_n(\varepsilon)) \cdot \varphi\Big(\frac{\varepsilon}{r|\lambda_n|}\Big) \le \int_{\Omega_n(\varepsilon)} \varphi\Big(\frac{\|f_n(\omega)\|_X}{r}\Big)\, d\mu \le M_{\varphi}\Big(\frac{f_n}{r}\Big) \le r.$$

Since $\varphi(u) \to \infty$ as $u \to \infty$ we get $\mu(\Omega_n(\varepsilon)) \to 0$ and this means that $\lambda_n f_n \to 0$ for $\mathcal{T}_{\mu}(X)$.

Moreover the balls $B_X^{\varphi}(2^n)$ are also closed in $(L^{\varphi}(X), \mathcal{T}_{\mu}(X)|_{L^{\varphi}(X)})$. In view of (1) and (2) $\mathcal{T}_I^{\varphi}(X)$ is the finest of all linear topologies τ on $L^{\varphi}(X)$ such that $\tau|_{B_X^{\varphi}(2^n)} = \mathcal{T}_{\mu}(X)|_{B_X^{\varphi}(2^n)}$ $(n = 0, 1, 2, \dots)$. Hence by [11, Corollary 1.1.12] the equivalence (i) $\Leftrightarrow$ (ii) holds. □

Theorem 3.4. *For a subset $Z \subset L^{\varphi}(X)$ the following statements are equivalent:*

(i) *Z is relatively compact for $\mathcal{T}_I^{\varphi}(X)$;*

(ii) *Z is relatively compact for $\mathcal{T}_{\mu}(X)|_{L^{\varphi}(X)}$ and* $\sup\{\|f\|_{L^{\varphi}(X)} : f \in Z\} < \infty$.

Proof: follows from Theorem 3.3 and (2). □

Definition 3.2. A sequence (f_n) in $L^{\varphi}(X)$ is said to be γ_{φ}^X-convergent to $f \in L^{\varphi}(X)$, in symbols $f_n \xrightarrow{\gamma_{\varphi}} f$, whenever

$$f_n \to f \ (\mu - \Omega) \ \text{ and } \ \sup_n \|f_n\|_{L^{\varphi}(X)} < \infty.$$

Theorem 3.5. *For a sequence (f_n) in $L^{\varphi}(X)$ the following statements are equivalent:*

(i) $f_n \to 0$ *for* $\mathcal{T}_I^{\varphi}(X)$;

(ii) $f_n \xrightarrow{\gamma_{\varphi}} 0$.

Moreover, $\mathcal{T}_I^{\varphi}(X)$ is the finest of all linear topologies τ on $L^{\varphi}(X)$ which satisfy the condition:

$$f_n \xrightarrow{\gamma_{\varphi}} 0 \ \textit{ implies } \ f_n \to 0 \ \textit{ for } \ \tau. \tag{+}$$

Proof: The equivalence (i) $\Leftrightarrow$ (ii) follows immediately from Theorem 3.3 and (2). Now let τ be a linear topology on $L^{\varphi}(X)$ for which the condition (+) holds. Then $\tau|_{B_X^{\varphi}(r)} \subset \mathcal{T}_{\mu}(X)|_{B_X^{\varphi}(r)}$ for $r > 0$, because $\mathcal{T}_{\mu}(X)$ is a linear metrizable topology. Hence by (1) we get $\tau \subset \mathcal{T}_I^{\varphi}(X)$. □

Definition 3.3. Let (Y,η) be a linear topological space. A linear mapping $T: L^\varphi(X)\to Y$ is said to be γ_φ-linear, whenever

$$f_n \xrightarrow{\gamma_\varphi} 0 \ \text{ implies } \ T(f_n)\to 0 \ \text{ for } \ \eta.$$

Then following theorem gives a characterization of γ_φ-linear operators on $L^\varphi(X)$.

Theorem 3.6. *For a linear topological space (Y,η) and a linear mapping $T: L^\varphi(X)\to Y$ the following statements are equivalent:*

(i) *T is $(\mathcal{T}_I^\varphi(X),\eta)$-continuous;*
(ii) *T is γ_φ-linear;*
(iii) *for every $r>0$, the restriction $T|_{B_X^\varphi(r)}$ is $(\mathcal{T}_\mu(X)|_{B_X^\varphi(r)},\eta)$-continuous.*

Proof: (i) $\Rightarrow$ (ii) It follows from Theorem 3.5.
(ii) $\Rightarrow$ (iii) Obvious.
(iii) $\Rightarrow$ (i) Let W be a neighbourhood of zero in Y for η. Since η is a linear topology, there exists a sequence $(W_n : n\ge 0)$ of neighbourhoods of zero for η such that $\sum_{n=0}^N W_n\subset W$ for every $N\ge 0$. By (iii) we can find a sequence $(\varepsilon_n : n\ge 0)$ of positive numbers such that $T(B_X^\varphi(2^n)\cap B_X^\mu(\varepsilon_n))\subset W_n$ for $n\ge 0$. Thus for $N\ge 0$ we have

$$T\Big(\sum_{n=0}^N (B_X^\varphi(2^n)\cap B_X^\mu(\varepsilon_n))\Big)\subset \sum_{n=0}^N W_n\subset W,$$

so

$$T\Big(\bigcup_{N=0}^\infty\Big(\sum_{n=0}^N (B_X^\varphi(2^n)\cap B_X^\mu(\varepsilon_n))\Big)\Big)\subset \bigcup_{N=0}^\infty T\Big(\sum_{n=0}^N (B_X^\varphi(2^n)\cap B_X^\mu(\varepsilon_n))\Big)\subset W.$$

It follows that T is $(\mathcal{T}_I^\varphi(X),\eta)$-continuous. □

Theorem 3.7. *Assume that (Ω,Σ,μ) is an atomless measure space or that μ is the counting measure on $\mathbb{N}$. If $(L^\varphi(X),\mathcal{T}_\varphi(X))$ is a locally bounded space then for a subset Z of $L^\varphi(X)$ the following statements are equivalent:*

(i) *Z is bounded for $\mathcal{T}_I^\varphi(X)$;*
(ii) $\sup\{\|f\|_{L^\varphi(X)} : f\in Z\}<\infty$;
(iii) *Z is bounded for $\mathcal{T}_\varphi(X)$.*

Proof: (i) $\Leftrightarrow$ (ii) See Theorem 3.3.
(ii) $\Rightarrow$ (iii) In view of [11, 0.3.10.2] $\sup\{\|f\|_{L^\varphi(X)} : f\in Z\}<\infty$ iff Z is additively bounded (see [11, 0.3.10.1]), so arguing as in the proof of [9, Lemma 2.5] we obtain that Z is bounded for $\mathcal{T}_\varphi(X)$.
(iii) $\Rightarrow$ (i) Obvious. □

The next theorem compares the topology $\mathcal{T}_I^\varphi(X)$ with the mixed topology $\gamma[\mathcal{T}_\varphi(X),\mathcal{T}_\mu(X)|_{L^\varphi(X)}]$ in the sense of Wiweger (see [12]).

Theorem 3.8. *Assume that* (Ω, Σ, μ) *is an atomless measure space or that* μ *is the counting measure on* $\mathbb{N}$. *If* $(L^\varphi(X), \mathcal{T}_\varphi(X))$ *is a locally bounded space, then the generalized mixed topology* $\mathcal{T}_I^\varphi(X)$ *coincides with the mixed topology* $\gamma[\mathcal{T}_\varphi(X), \mathcal{T}_\mu(X)|_{L^\varphi(X)}]$.

Proof: In view of Theorem 3.7 it follows from [12, 2.2.1, 2.2.2]. □

An Orlicz function φ continuous for all $u \geq 0$, taking only finite values, vanishing only at zero and not bounded is usually called a φ-function. By Φ we will denote the collection of all φ-functions.

A Young function φ vanishing only at zero and taking only finite values is called an N-function whenever $\frac{\varphi(t)}{t} \to 0$ as $t \to 0$ and $\frac{\varphi(t)}{t} \to \infty$ as $t \to \infty$. By Φ_N we will denote the collection of all N-functions.

Let Φ_1 be the set of all Orlicz functions φ vanishing only at zero and such that $\varphi(t) \to \infty$ as $t \to \infty$. Denote by

$\Phi_{11} = \{\varphi \in \Phi_1 : \varphi(t) < \infty \ \text{ for } \ t \geq 0\}$,

$\Phi_{12} = \{\varphi \in \Phi_1 : \varphi \ \text{ jumps to } \ \infty\}$.

Then $\Phi_1 = \Phi_{11} \cup \Phi_{12}$.

Theorem 3.9. *Let* $\varphi \in \Phi_{1i}$ $(i = 1, 2)$. *Then the topology* $\mathcal{T}_I^\varphi(X)$ *is generated by the family of solid* F*-norms:*

$$\{\|\cdot\|_{L^\psi(X)} : \psi \in \Psi_{1i}^\varphi\},$$

where $\Psi_{11}^\varphi = \{\psi \in \Phi : \psi \prec\prec \varphi\}$, $\Psi_{12}^\varphi = \{\psi \in \Phi : \psi \overset{\mathrm{s}}{\prec\prec} \varphi\}$.

Moreover, the following identities hold:

$$L^\varphi(X) = \bigcap\{L^\psi(X) : \psi \in \Psi_{1i}^\varphi\} = \bigcap\{E^\psi(X) : \psi \in \Psi_{1i}^\varphi\}. \tag{3}$$

Proof: Let $\varphi \in \Phi_{1i}$ $(i = 1, 2)$. Then $\mathcal{T}_I^\varphi$ is the finest uniformly μ-continuous topology on L^φ (see [10, Theorem 2.4]) and is generated by the family $\{\|\cdot\|_\psi : \psi \in \Psi_{1i}^\varphi\}$ (see [10, Theorem 4.5, Theorem 3.8]). Then the topology $\overline{\mathcal{T}_I^\varphi}$ on $L^\varphi(X)$ is generated by the family $\{\|\cdot\|_{L^\psi(X)} : \psi \in \Psi_{1i}^\varphi\}$ of solid F-norms and by Theorem 2.3 $\overline{\mathcal{T}_I^\varphi}$ is the finest uniformly μ-continuous topology on $L^\varphi(X)$. By Theorem 3.1 $\overline{\mathcal{T}_I^\varphi} = \mathcal{T}_I^\varphi(X)$, and we are done.

The identities (3) follow from [10, Theorem 3.1]. □

Let Φ_1^c be the set of all Young functions φ vanishing only at zero and such that $\frac{\varphi(t)}{t} \to \infty$ as $t \to \infty$. Denote by

$\Phi_{11}^c = \{\varphi \in \Phi_1^c : \varphi(t) < \infty \ \text{ for } \ t \geq 0 \ \text{ and } \ \frac{\varphi(t)}{t} \to 0 \ \text{ as } \ t \to 0\}$,

$\Phi_{12}^c = \{\varphi \in \Phi_1^c : \varphi \ \text{ jumps to } \ \infty \ \text{ and } \ \frac{\varphi(t)}{t} \to 0 \ \text{ as } \ t \to 0\}$,

$\Phi^c_{13} = \{\varphi \in \Phi^c_1 : \varphi(t) < \infty$ for $t \geq 0$ and $\frac{\varphi(t)}{t} \to a$ as $t \to 0$ for some $a > 0\}$,

$\Phi^c_{14} = \{\varphi \in \Phi^c_1 : \varphi$ jumps to ∞ and $\frac{\varphi(t)}{t} \to a$ as $t \to 0$ for some $a > 0\}$.

Then $\Phi^c_1 = \bigcup_{i=1}^4 \Phi^c_{1i}$ and the sets Φ^c_{1i} $(i = 1,2,3,4)$ are pairwise disjoint. It can be seen that $\Phi^c_{11} = \Phi_N$.

Theorem 3.10. *Let $\varphi \in \Phi^c_{1i}$ $(i = 1,2,3,4)$. Then the topology $\mathcal{T}^\varphi_I(X)$ is generated by the family of solid norms*

$$\{\|\!| \cdot \|\!|_{L^\psi(X)} : \psi \in \Psi^\varphi_{1i}(N)\},$$

where $\Psi^\varphi_{11}(N) = \{\psi \in \Phi_N : \psi \prec\!\prec \varphi\}$, $\Psi^\varphi_{12}(N) = \{\psi \in \Phi_N : \psi \overset{s}{\prec\!\prec} \varphi\}$, $\Psi^\varphi_{13}(N) = \{\psi \in \Phi_N : \psi \overset{1}{\prec\!\prec} \varphi\}$, $\Psi^\varphi_{14}(N) = \Phi_N$.
Moreover, the following identities hold:

$$L^\varphi(X) = \bigcap\{L^\psi(X) : \psi \in \Psi^\varphi_{1i}(N)\} = \bigcap\{E^\psi(X) : \psi \in \Psi^\varphi_{1i}(N)\}. \tag{4}$$

Proof: Let $\varphi \in \Phi^c_{1i}$ $(i = 1,2,3,4)$. Then $\mathcal{T}^\varphi_I$ is the finest uniformly μ-continuous topology on L^φ (see [10, Theorem 2.4]) and is generated by the family $\{\|\!| \cdot \|\!|_\psi : \psi \in \Psi^\varphi_{1i}(N)\}$ (see [10, Theorem 3.12 and Theorem 4.5]). Then the topology $\overline{\mathcal{T}^\varphi_I}$ on $L^\varphi(X)$ is generated by the family $\{\|\!| \cdot \|\!|_{L^\psi(X)} : \psi \in \Psi^\varphi_{1i}(N)\}$ of solid norms, and by Theorem 2.3 $\overline{\mathcal{T}^\varphi_I}$ is the finest uniformly μ-continuous topology on $L^\varphi(X)$. By Theorem 3.1 $\overline{\mathcal{T}^\varphi_I} = \mathcal{T}^\varphi_I(X)$, as desired.

The identities (4) follow from [10, Theorem 3.2]. □

As an application of Theorem 3.10 we get a characterization of uniformly μ-continuous seminorms on $L^\varphi(X)$.

Theorem 3.11. *Let $\varphi \in \Phi^c_{1i}$ $(i = 1,2,3,4)$. Then for a solid seminorm ρ on $L^\varphi(X)$ the following statements are equivalent:*

(i) *ρ is uniformly μ-continuous;*
(ii) *there exist $\psi \in \Psi^\varphi_{1i}(N)$ and a number $a > 0$ such that*

$$\rho(f) \leq a\|\!|f\|\!|_{L^\psi(X)} \quad \textit{for all} \quad f \in L^\varphi(X).$$

Proof: (i) $\Rightarrow$ (ii) Since $\mathcal{T}^\varphi_I(X)$ is the finest uniformly μ-continuous topology on $L^\varphi(X)$ (see Theorem 3.1), in view of Theorem 3.10 and [4, Chapter 4, § 18(4)] there exist $\psi_1, \dots, \psi_n \in \Psi^\varphi_{1i}(N)$ and a number $a > 0$ such that

$$\rho(f) \leq a \max(\|\!|f\|\!|_{L^{\psi_1}(X)}, \dots, \|\!|f\|\!|_{L^{\psi_n}(X)}) \quad \text{for all} \quad f \in L^\varphi(X).$$

Let $\psi(u) = \max(\psi_1(u), \dots, \psi_n(u))$ for $u \geq 0$. Then $\psi \in \Psi^{\varphi}_{1i}(N)$ and $|||f|||_{L^{\psi_i}(X)} \leq |||f|||_{L^{\psi}(X)}$ for $i = 1, \dots, n$ and all $f \in L^{\varphi}(X)$, so

$$\rho(f) \leq a|||f|||_{L^{\psi}(X)} \quad \text{for all} \quad f \in L^{\varphi}(X).$$

(ii) $\Rightarrow$ (i) It is obvious, because for each $\psi \in \Psi^{\varphi}_{1i}(X)$, $|||\cdot|||_{L^{\psi}(X)}$ is a uniformly μ-continuous norm on $L^{\varphi}(X)$. $\square$

To present the general form of $\mathcal{T}^{\varphi}_I(X)$-continuous linear functionals on $L^{\varphi}(X)$ we recall the terminology concerning some spaces of X-weak measurable functions (see [2]).

Given a function $g : \Omega \to X^*$ and $x \in X$ we denote by g_x the real function on Ω defined by $g_x(\omega) = g(\omega)(x)$. A function g is said to be X-weak measurable if the functions g_x are measurable for each $x \in X$. We say that two X-weak measurable functions g_1, g_2 are equivalent whenever $g_1(\omega)(x) = g_2(\omega)(x)$ μ-a.e. for all $x \in X$.

By $L^0(X^*, X)$ we denote the linear space of equivalence classes of all X-weak measurable functions $g : \Omega \to X^*$. It is known that the set $\{|g_x| : x \in B_X\}$ is order bounded in L^0 for every $g \in L^0(X^*, X)$.

The function $\vartheta : L^0(X^*, X) \to L^0$ defined by

$$\vartheta(g) = \sup\{|g_x| : x \in B_X\} \quad \text{for} \quad g \in L^0(X^*, X)$$

is called an abstract norm.

It is known that for $f \in L^0(X)$, $g \in L^0(X^*, X)$ the function $\langle f, g\rangle : \Omega \to R$ defined by $\langle f, g\rangle(\omega) = \langle f(\omega), g(\omega)\rangle = g(\omega)(f(\omega))$ is measurable and

$$|\langle f, g\rangle(\omega)| \leq \|f(\omega)\|_X \cdot \vartheta(g)(\omega) \quad \mu\text{-a.e.}$$

For an ideal I of L^0 let

$$I(X^*, X) = \{g \in L^0(X^*, X) : \vartheta(g) \in I\}.$$

Theorem 3.12. *Let $\varphi \in \Phi^c_{1i}$ $(i = 1, 2, 3, 4)$. Then for a linear functional F on $L^{\varphi}(X)$ the following statements are equivalent:*

(i) *F is continuous for $\mathcal{T}^{\varphi}_I(X)$;*
(ii) *F is γ_{φ}-linear;*
(iii) *there exists a unique $g \in E^{\varphi^*}(X^*, X)$ such that*

$$F(f) = F_g(f) = \int_{\Omega} \langle f(\omega), g(\omega)\rangle \, d\mu \quad \text{for} \quad f \in L^{\varphi}(X).$$

Proof: (i) $\Leftrightarrow$ (ii) The equivalence follows from Theorem 3.6.

(i) $\Rightarrow$ (iii) Let $\varphi \in \Phi_{1i}^c$ $(i = 1,2,3,4)$. In view of Theorem 3.10 (see also the proof of Theorem 3.11) there exist $\psi \in \Psi_{1i}^{\varphi}(N)$ and $r > 0$ such that F is bounded on $B_X^{(\psi)}(r) \cap L^{\varphi}(X)$, where $B_X^{(\psi)}(r) = \{f \in L^{\psi}(X) : |||f|||_{L^{\psi}(X)} \leq r\}$. This means that F is continuous on the linear subspace $(L^{\varphi}(X), \mathcal{T}_{\psi}(X)|_{L^{\varphi}(X)})$ of the normed space $(E^{\psi}(X), \mathcal{T}_{\psi}(X)|_{E^{\psi}(X)})$. Hence by the Hahn-Banach extension theorem there exists a $\mathcal{T}_{\psi}(X)|_{E^{\psi}(X)}$-continuous linear functional $\overline{F}$ on $E^{\psi}(X)$ such that $\overline{F}(f) = F(f)$ for $f \in L^{\varphi}(X)$. Since $E^{\psi} = L_a^{\psi}$, we get $E^{\psi}(X) = L_a^{\psi}(X)$. By [2, Corollary 4.1] there exists a unique $g \in (L_a^{\psi})'(X^*, X)$ such that

$$\overline{F}(f) = \int_{\Omega} \langle f(\omega), g(\omega) \rangle \, d\mu \quad \text{for} \quad f \in L_a^{\psi}(X).$$

But $(L_a^{\psi})' = L^{\psi^*}$ (see [6, p.56]), so by [10, Corollary 3.5] we get $L^{\psi^*} \subset E^{\varphi^*}$. Finally, there exists a unique $g \in E^{\varphi^*}(X^*, X)$ such that

$$\overline{F}(f) = \int_{\Omega} \langle f(\omega), g(\omega) \rangle \, d\mu \quad \text{for} \quad f \in L_a^{\psi}(X).$$

Hence

$$F(f) = F_g(f) = \int_{\Omega} \langle f(\omega), g(\omega) \rangle \, d\mu \quad \text{for} \quad f \in L^{\varphi}(X).$$

(iii) $\Rightarrow$ (i) Let $\varphi \in \Phi_{1i}^c$ $(i = 1,2,3,4)$. According to [10, Corollary 3.5] there exists $\psi \in \Psi_{1i}^{\varphi}(N)$ such that $g \in L^{\psi^*}(X^*, X)$. Then $L^{\varphi}(X) \subset E^{\psi}(X) \subset L^{\psi}(X)$. Moreover, by [2, Theorem 1.1] using the Hölder's inequality we get for $f \in L^{\varphi}(X)$

$$\begin{aligned} |F_g(f)| &\leq \int_{\Omega} |\langle f(\omega), g(\omega) \rangle| \, d\mu \leq \int_{\Omega} \|f(\omega)\|_X \cdot \vartheta(g)(\omega) \, d\mu \\ &\leq 2|||\widetilde{f}|||_{\psi} \cdot |||\vartheta(g)|||_{\psi^*} = 2|||f|||_{L^{\psi}(X)} \cdot |||\vartheta(g)|||_{\psi^*}. \end{aligned}$$

This means that F_g is $\mathcal{T}_{\psi}(X)|_{L^{\varphi}(X)}$-continuous, so F_g is $\mathcal{T}_I^{\varphi}(X)$-continuous, because $\mathcal{T}_{\psi}(X)|_{L^{\varphi}(X)} \subset \mathcal{T}_I^{\varphi}(X)$ by Theorem 3.10.

Thus the proof is complete. $\square$

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