Linking the closure and orthogonality properties of perfect morphisms in a category

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Abstract. We define perfect morphisms to be those which are the pullback of their image under a given endofunctor. The interplay of these morphisms with other generalisations of perfect maps is investigated. In particular, closure operator theory is used to link closure and orthogonality properties of such morphisms. A number of detailed examples are given.

Keywords: perfect morphism, (pullback) closure operator, factorisation theory, orthogonal morphisms

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1. Introduction

This paper continues the study of categorical generalisations of perfect morphisms begun in [20]. Such generalisations have been twofold — generalising the orthogonality properties of perfect maps on the one hand and their closure and compactness properties on the other. Central to our investigations is a notion of perfect morphism relative to a pointed endofunctor on a category $\mathbf{X}$. Using this notion as well as the pullback closure operator induced by the endofunctor, we explore the links between these two previously disjoint categorical studies of perfect maps.

The emphasis of [20] was on introducing the pullback closure operator and investigating its role in describing — in closure and compactness terms — perfect morphisms defined via a pointed endofunctor. We now look more closely at perfect morphisms defined via orthogonality properties, and build on the main result of [20], providing a theorem that summarises ties between various generalisations of perfect maps.

Our approach is to investigate what properties of the pointed endofunctor and underlying category enable links to be established between these notions of perfect morphism. The closure operator, when employed, is strictly a tool in the process. A number of examples are given that illustrate the theory and endeavour to provide an intuition for the assumptions upon which it is built.

I am grateful to my doctoral supervisor Guillaume Brümmer for his hand in this work. In particular it was on his instigation that I embarked on the study of perfect morphisms.

2. Categorical background

Categorical notation is taken from [1]. We work in a category $\mathbf{X}$. The pair (R, r) will denote a pointed endofunctor on $\mathbf{X}$ throughout. For $X \in Ob\mathbf{X}$, $r_X : X \to RX$ will denote the natural morphism induced by (R, r). A number of central definitions relate to this pointed endofunctor.

2.1 Definition. (1) $\Sigma_R = \{f \in Mor\mathbf{X} \mid Rf \text{ is an isomorphism}\}$;

(2) $\mathrm{Fix}(R, r) = \{X \in Ob\mathbf{X} \mid r_X : X \to RX \text{ is an isomorphism}\}$;

(3) (R, r) is *idempotent* if $RX \in \mathrm{Fix}(R, r)$ for every $X \in Ob\mathbf{X}$;

(4) (R, r) is *well pointed* if $r_{RX} = Rr_X$ for every $X \in Ob\mathbf{X}$;

(5) (R, r) is *direct* if for any $f : X \to Y$ in $\mathbf{X}$ the pullback P below can be formed, and the induced morphism $u \in \Sigma_R$.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, dashed, "u"] \arrow[dd, "r_X"'] & & Y \arrow[dd, "r_Y"] \\
& P \arrow[ur, "p"] \arrow[dl, "q"] & \\
RX \arrow[rr, "Rf"] & & RY
\end{tikzcd}$$

Two notions of orthogonal morphism will be used below — one relative to a morphism class, the other relative to an object class.

2.2 Definition. Let $\mathcal{A} \subseteq Mor\mathbf{X}$, $\mathcal{B} \subseteq Ob\mathbf{X}$ and $f : X \to Y$ be a morphism in $\mathbf{X}$.

(1) $f \in \mathcal{A}^{\downarrow}$ if any commutative square $vg = fu$ with $g \in \mathcal{A}$ has a unique diagonal.

(2) $f \in \mathcal{B}_{\perp_w}$ if for any morphism $g : X \to B$ with codomain $B \in \mathcal{B}$ there is an $h : Y \to B$ such that $hf = g$. We say $f \in \mathcal{B}_{\perp}$ if h is *unique.* Such f will be termed the (uniquely) $\mathcal{B}$-extendable morphisms.

We use categorical closure operators as introduced in [8]. A standard reference is now [10]. Suffice to say that $\mathbf{X}$ is an $(\mathbf{E}, \mathcal{M})$ category for sinks. The class $\mathcal{M}$ constitutes subobjects in $\mathbf{X}$ and closure operators act on these subobjects. The class $\mathcal{E} = \mathbf{E} \cap Mor\mathbf{X}$ and the class $\mathcal{M}$ will be fixed throughout.

The *pullback closure operator* was introduced and studied in [20] and [19]. Its construction is as follows for a subobject $m : M \to X$ in $\mathcal{M}$. Take the $(\mathcal{E}, \mathcal{M})$-factorisation $ne = Rm$ and then form the pullback $\overline{m}$ of n along r_X. The pullback closure of m is then $\Phi_{(R,r)}(m) := \overline{m}$.

$$
\begin{tikzcd}
M \arrow[rr, "m"] \arrow[dr, dashed, "j"] \arrow[dd, "r_M"'] & & X \arrow[dd, "r_X"] \\
& \overline{M} \arrow[ur, "\overline{m}"'] \arrow[dd, "\overline{r_X}"'] & \\
RM \arrow[rr, "Rm"] \arrow[dr, "e"] & & RX \\
& N \arrow[ur, "n"] &
\end{tikzcd}
$$

Closure operators and the two notions defined below have been used extensively to study epimorphisms in categories.

2.3 Definition. Let $f : X \to Y$ be a morphism in $\mathbf{X}$.

(1) f is said to be $\Phi_{(R,r)}$-*dense* if when we take the $(\mathcal{E}, \mathcal{M})$-factorisation $me = f$ we get that $\Phi_{(R,r)}(m) \cong 1_X$.

(2) f is $\mathcal{A}$-*cancellable* for a class $\mathcal{A}$ of $\mathbf{X}$-objects, if for any pair $g, h : Y \to A$ with $gf = hf$ and $A \in \mathcal{A}$ it follows that $g = h$.

3. Different notions of perfect morphism — a brief survey

Since their introduction, a number of characterisations — and indeed definitions — of perfect maps have been given. Thus when categorical topologists in the 1970's set about generalising the notion of a perfect map, a number of different generalisations were possible. A particularly good summary of these can be found in [17]. Below is an outline of five characterisations that will be used in our investigations.

For now, consider perfect continuous maps in <u>TYCH</u>. (R, r) is the pointed endofunctor induced by the Čech-Stone compactification. This is the paradigmatic example for our study. For a continuous $f : X \to Y$ in <u>TYCH</u>, the following are five different ways of characterising f as a perfect map.

(1) **f is a closed map and for any $y \in Y$, $f^{-1}(y)$ is compact.** This is usually considered to be the definition of a perfect map. Until recently no attempts had been made to generalise this definition. To our knowledge, [9] was the first endeavour to make a more general study of morphisms that preserve closure and have compact preimages of points.

(2) **For any space Z the map $f \times 1_Z : X \times Z \to Y \times Z$ is closed.** [3] uses this as the definition of a perfect map, and shows the equivalence of this definition with the one above. The first attempts to generalise this characterisation were made in [4] (for sequential closure) and [22] (in categories of "structured sets").

[18] takes these generalisations further in an hereditary construct. More recently [9] investigated the interrelation of this notion with the one in (1) above. They also restrict themselves to certain constructs. Some improvements on their joint results were made in [7].

(3) ***f* is orthogonal to every compact extendable epimorphism.** In our notation we could write that $f \in (\underline{\mathrm{HC}\text{OMP}}_{\perp_w} \cap \mathit{Epi})^{\downarrow} = (\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi})^{\downarrow} = \{\mathit{Dense}\ C^*\text{-}\mathit{embeddings}\}^{\downarrow}$. [16] made use of this characterisation of perfect maps to find a categorical generalisation. (An appendix to [23] introduces independently an equivalent notion.)

A string of papers [25], [17], [24], [26] and finally [27] exploited this line of study. The final paper introduced perfect sources. Collections of such sources occur as the second part of factorisation structures for sources in **X**, and so are in one-one correspondence with epireflective subcategories of **X**.

(4) $\boldsymbol{f \in \Sigma_{\mathbf{R}}^{\downarrow}}$**.** This is a result of the fact that in our setting, $\{\mathit{Dense}\ C^*\text{-}\mathit{embeddings}\} = \Sigma_R$. While this is obviously strongly related to (3) above, we have not found any author who has specifically generalised this fact by considering an arbitrary endofunctor or even reflector (R,r).

(5) ***f* is the pullback of its image under $\boldsymbol{(R,r)}$.** More precisely, the diagram below is a pullback square.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r_X"'] \arrow[dr, phantom, "(*)"] & Y \arrow[d, "r_Y"] \\
RX \arrow[r, "Rf"'] & RY
\end{tikzcd}$$

The fact that this characterises perfect maps was first proved in [15]. A number of authors ([2], [13], [28] and [14]) have taken this approach to generalising perfect maps in relatively restricted settings.

[27] calls this notion of perfectness R-strongly perfect and extends it to sources. He gives a few results that relate this notion to the one in (3) that was so widely studied. It seems that no-one took these ideas any further apart from the recent work in [5]. *This last notion of a perfect map is the central one that we use.*

We should point out here that more recently in [6] the definition of a perfect morphism $f : X \to Y$ as one which is compact in the comma category $\mathbf{X}/Y$ has been given. We do not consider that definition here.

4. (R,r)-perfect and weakly (R,r)-perfect morphisms

4.1 Definition. A morphism $f : X \to Y$ in $\mathbf{X}$ will be called *weakly* (R,r)*-perfect* if $f \in \Sigma_R^{\downarrow}$. We will call f (R,r)*-perfect* if the commutative square as shown in $(*)$ above is a pullback.

There are numerous results in topology regarding properties of perfect maps and their relation to compact spaces. Taking the class $\mathrm{Fix}(R,r)$ as the analogue of the compact Hausdorff spaces, a number of these are easily generalised for both weakly (R,r)-perfect and (R,r)-perfect morphisms. First an important observation which appears as Proposition 15 in [20].

4.2 Proposition. *If $f : X \to Y$ in $\mathbf{X}$ is (R,r)-perfect, then f is weakly (R,r)-perfect.*

4.3 Lemma. *Any morphism $h : X \to Y$ in Σ_R is* $\mathrm{Fix}(R,r)$*-cancellable.*

PROOF: Take $X \xrightarrow{h} Y \in \Sigma_R$ and $u, v : Y \to Z$ such that $uh = vh$ and Z is in $\mathrm{Fix}(R,r)$.

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "r_X"'] & Y \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, "r_Y"'] & Z \arrow[d, "r_Z"] \\
RX \arrow[r, "Rh"] & RY \arrow[r, shift left, "Ru"] \arrow[r, shift right, "Rv"'] & RZ
\end{tikzcd}$$

Since $uh = vh$, $RuRh = RvRh$, but Rh is an isomorphism, so $Ru = Rv$. Thus $r_Z u = r_Z v$, and since $Z \in \mathrm{Fix}(R,r)$, $u = v$. $\square$

4.4 Proposition. *The class of weakly (R,r)-perfect morphisms contains all $\mathbf{X}$ isomorphisms and is closed under composition, pullbacks, multiple pullbacks and products in $\mathbf{X}$.*

PROOF: True for any class $\mathcal{A}^{\downarrow}$, cf. for example [25, Proposition 1]. $\square$

4.5 Remark. It is easy to see that the class of (R,r)-perfect morphisms contains all isomorphisms and is closed under composition. We need to assume various properties for (R,r) before the other results follow.

4.6 Proposition. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathbf{X}$ such that their composition gf is (weakly) (R,r)-perfect.*

(a) *If g is a monomorphism, then f is (weakly) (R,r)-perfect.*
(b) *If g is (weakly) (R,r)-perfect, then f is (weakly) (R,r)-perfect.*
(c) *If f is a retraction, then g is weakly (R,r)-perfect.*

Proof: (a) Let gf be weakly (R,r)-perfect. Assume we have a commutative square $fu = vh$ with $A \xrightarrow{h} B \in \Sigma_R$.

There is a unique diagonal $d : B \to X$ for the square $(gf)u = (gv)h$. But since g is a monomorphism d is a unique diagonal for the square $fu = vh$ and f is weakly (R,r)-perfect.

Now let gf be (R,r)-perfect, we must show that the left hand square in the diagram below is a pullback.

$$\begin{tikzcd}
A \arrow[dr, dashed, "h"] \arrow[drr, "u"] \arrow[ddr, "v"'] & & & \\
& X \arrow[r, "f"'] \arrow[d, "r_X"] & Y \arrow[r, "g"] \arrow[d, "r_Y"'] & Z \arrow[d, "r_Z"] \\
& RX \arrow[r, "Rf"] & RY \arrow[r, "Rg"] & RZ
\end{tikzcd}$$

Say we have a source $(A,(u,v))$ such that $r_Y u = Rfv$ then $r_Z gu = Rg r_Y u = RgRfv = R(gf)v$, so there is a unique $h : A \to X$ such that $gfh = gu$ and $r_X h = v$. Since g is a monomorphism, h is also the unique morphism such that $r_X h = v$ and $fh = u$, so f is (R,r)-perfect.

(b) Let gf be weakly (R,r)-perfect and $A \xrightarrow{h} B \in \Sigma_R$ with morphisms u and v such that $fu = vh$.

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[d, "u"'] & & B \arrow[d, "gv"] \arrow[dll, dashed, "d"'] \arrow[dl, dashed, "d^*"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

There is a unique $d : B \to X$ such that $dh = u$ and $gfd = gv$. There is also a unique $d^* : B \to Y$ such that $d^*h = fu$ and $gd^* = gv$. But then since $fdh = fu$, $gfd = gv$, $vh = fu$ and $gv = gv$ the uniqueness condition on d^* gives that $fd = v = d^*$. Thus d is a unique diagonal for the square $fu = vh$ and f is weakly (R,r)-perfect.

The case for both gf and g (R,r)-perfect is a simple application of [1, Proposition 11.10(2)].

(c) Say gf is weakly (R,r)-perfect and we have $A \xrightarrow{h} B \in \Sigma_R$ and morphisms u and v such that $gu = vh$.

$$\begin{tikzcd}
& A \arrow[r, "h"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dll, dashed, "d"] \\
X \arrow[r, shift left, "f"] & Y \arrow[r, "g"] \arrow[l, shift left, "s"] & Z
\end{tikzcd}$$

f has a right inverse s, so $gfsu = gu = vh$ and thus there is a unique $d : B \to X$ such that $dh = su$ and $gfd = v$. Put $d^* := fd$ then $d^*h = fdh = fsu = u$ and $gd^* = gfd = v$. Then d^* is a unique diagonal for the square $gu = vh$, since any other d' such that $d'h = u$ and $gd' = v$ would give $sd'h = su$ and $gfsd' = v$ and so by the uniqueness condition on d, $sd' = d$ and then $d' = fsd' = fd = d^*$. $\square$

4.7 Proposition. *Let $f : X \to Y$ be a morphism in $\mathbf{X}$ with codomain Y in* $\mathrm{Fix}(R,r)$.

(a) *f is (R,r)-perfect iff X is in* $\mathrm{Fix}(R,r)$.
(b) *If (R,r) is idempotent and well-pointed then f is weakly (R,r)-perfect iff X is in* $\mathrm{Fix}(R,r)$.

Proof: (a) Clear since if r_Y is an isomorphism, then the commutative square $(*)$ is a pullback iff r_X is an isomorphism.

(b) The reverse implication is immediate since if X is in $\mathrm{Fix}(R,r)$, then by (a) f is (R,r)-perfect, hence weakly (R,r)-perfect. On the other hand assume that (R,r) is idempotent and well-pointed and that f is weakly (R,r)-perfect.

$$\begin{tikzcd}
X \arrow[r, "r_X"] \arrow[dd, "1_X"'] & RX \arrow[d, "Rf"] \arrow[ddl, dashed, "d"'] \\
& RY \\
X \arrow[r, "f"] & Y \arrow[u, "r_Y"']
\end{tikzcd}$$

$f1_X = r_Y^{-1}Rfr_X$ so since $r_X \in \Sigma_R$, there is a unique $d : RX \to X$ such that $dr_X = 1_X$ and $fd = r_Y^{-1}Rf$. But $r_X d r_X = r_X$ and so since $r_X \in \Sigma_R$ and $RX \in \mathrm{Fix}(R,r)$, Lemma 4.3 gives that $r_X d = 1_{RX}$, thus r_X is an isomorphism and X is in $\mathrm{Fix}(R,r)$. $\square$

4.8 Corollary. *Let* $\mathbf{X}$ *have a terminal object* T *such that* $T \cong RT$. *(If* (R,r) *is idempotent and well-pointed) an* $\mathbf{X}$*-object* X *is in* $\mathrm{Fix}(R,r)$ *iff the unique morphism* $X \xrightarrow{t_X} T$ *is (weakly)* (R,r)*-perfect.*

4.9 Remark. In most instances it is the case that $T \cong RT$ (for example if (R,r) is pointwise epimorphic). It is worth noting that this condition is not needed to prove that $X \in \mathrm{Fix}(R,r) \Rightarrow X \xrightarrow{t_X} T$ is (R,r)-perfect. (An alternative proof can be given.) Also the assumption of idempotence and well-pointedness is only needed for the one direction in the weakly (R,r)-perfect case.

4.10 Proposition. *Let* $\mathbf{X}$ *have products of pairs. If* $X \in \mathrm{Fix}(R,r)$*, then for any* $Y \in Ob\mathbf{X}$*, the projection* $\pi_2 : X \times Y \to Y$ *is weakly* (R,r)*-perfect.*

Proof: Let $X \in \mathrm{Fix}(R,r)$ and $Y \in Ob\mathbf{X}$. Say $\pi_2 u = vh$ with $A \xrightarrow{h} B \in \Sigma_R$

$$\begin{tikzcd}
 & & A \arrow[rr, "h"] \arrow[dll, "r_A"'] \arrow[dd, "u"] & & B \arrow[dd, "v"] \arrow[dr, "r_B"] & \\
RA \arrow[rrrrr, "Rh"] \arrow[dd, "Ru"'] & & & & & RB \arrow[dd, "Rv"] \\
 & & X \times Y \arrow[rr, "\pi_2"] \arrow[dll, "r_{X\times Y}"'] \arrow[ddrr, "\pi_1"] & & Y \arrow[dr, "r_Y"] & \\
R(X \times Y) \arrow[rrrrr, "R\pi_2"] \arrow[drr, "R\pi_1"'] & & & & & RY \\
 & & RX & & X \arrow[ll, "r_X"'] &
\end{tikzcd}$$

Put $d := \langle r_X^{-1} R\pi_1 Ru (Rh)^{-1} r_B, v\rangle$, then $\pi_1 dh = r_X^{-1} R\pi_1 Ru (Rh)^{-1} r_B h = r_X^{-1} R\pi_1 Ru r_A = r_X^{-1} R\pi_1 r_{X\times Y} u = r_X^{-1} r_X \pi_1 u = \pi_1 u$ and $\pi_2 dh = vh = \pi_2 u$, so $dh = u$ and d is a diagonal for the square $\pi_2 u = vh$.

Say we have a morphism d^* such that $d^* h = u$ and $\pi_2 d^* = v$ then since $\pi_1 d^* h = \pi_1 dh$ and $h \in \Sigma_R$ and $X \in \mathrm{Fix}(R,r)$ Lemma 4.3 gives that $\pi_1 d^* = \pi_1 d$. We also have that $\pi_2 d^* = v = \pi_2 d$, so $d^* = d$ and π_2 is weakly (R,r)-perfect. □

4.11 Remark. The extent of the work done on perfectness is such that some similar results in various guises have appeared in many publications considering different definitions of perfectness (cf. for example [2], [13], [28] and [14]). Good summaries of the basic results in topology that these results extend can be found in [3, § 10] and [11, § 3.7].

5. Perfect morphisms defined via orthogonality classes

As has been mentioned, early categorical investigations into perfectness generalised the fact that in $\underline{\mathrm{TYCH}}$ the perfect maps are exactly those in the class $(\underline{\mathrm{HCOMP}}_{\perp_w} \cap \mathit{Epi})^{\downarrow}$. For a class $\mathcal{X}$ of $\mathbf{X}$-objects, a morphism in $\mathbf{X}$ was called *$\mathcal{X}$-perfect* iff it was in the class $(\mathcal{X}_{\perp_w} \cap \mathit{Epi}\,\mathbf{X})^{\downarrow}$. This notion was introduced in [16] and in its final investigations was extended to sources in [27]. Theorem 4 of [27] touches on some of the links between this notion of $\mathcal{X}$-perfectness and our present notion of (R,r)-perfectness. We now explore these matters further. *For any class $\mathcal{X}$ of $\mathbf{X}$-objects we will use the term $\mathcal{X}$-perfect as above. Note that for $\mathcal{X} = \mathrm{Fix}(R,r)$ the term $\mathrm{Fix}(R,r)$-perfect should not be confused with the term (R,r)-perfect already being used.*

We explore the links between $\mathrm{Fix}(R,r)$-perfect morphisms and (weakly) (R,r)-perfect morphisms. Crucial to this is understanding how Σ_R relates to the class $\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}$.

5.1 Proposition. $\Sigma_R \subseteq \mathrm{Fix}(R,r)_{\perp} \subseteq \{\mathrm{Fix}(R,r)\text{-}\mathit{cancellable}\}$.

PROOF: Let $f : X \to Y$ be in Σ_R and let $g : X \to Z$ have codomain Z in $\mathrm{Fix}(R,r)$.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] \arrow[dd, "r_X"'] & & Y \arrow[dl, dashed, "h"] \arrow[dd, "r_Y"] \\
& Z \arrow[dd, "r_Z"] & \\
RX \arrow[rr, "Rf"] \arrow[dr, "Rg"'] & & RY \\
& RZ &
\end{tikzcd}$$

Put $h := r_Z^{-1} Rg (Rf)^{-1} r_Y$ then $hf = r_Z^{-1} Rg (Rf)^{-1} r_Y f = r_Z^{-1} Rg r_X = g$. By Lemma 4.3, h is a unique extension so $f \in \mathrm{Fix}(R,r)_{\perp}$.

If we have a morphism $f : X \to Y$ in $\mathrm{Fix}(R,r)_{\perp}$ and morphisms $u, v : Y \to Z$ with codomain in $\mathrm{Fix}(R,r)$ such that $uf = vf$, then since $uf = vf$ is a morphism from X to Z it follows immediately that $u = v$ is the unique extension of f to Z over $uf = vf$. Thus f is $\mathrm{Fix}(R,r)$-cancellable. $\square$

5.2 Proposition. *If $\mathcal{E} \subseteq \mathit{Epi}\,\mathbf{X}$, then $\{\Phi_{(R,r)}\text{-dense}\} \subseteq \{\mathrm{Fix}(R,r)\text{-cancellable}\}$.*

PROOF: Let $f : X \to Y$ be $\Phi_{(R,r)}$-dense, and $u, v : Y \to Z$ have codomain in $\mathrm{Fix}(R,r)$ with $uf = vf$. The diagram below shows the construction of $\Phi_{(R,r)}(m)$ where $me = f$ is the $(\mathcal{E},\mathcal{M})$ factorisation of f and ne' is the $(\mathcal{E},\mathcal{M})$ factorisation of Rm.

$$
\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[dr, "e"] \arrow[dd, "r_X"'] & & & Y \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[dd, "r_Y"] & Z \arrow[dd, "r_Z"] \\
& M \arrow[r, "j"'] \arrow[urr, "m"] \arrow[dd, "r_M"] & \overline{M} \arrow[ur, "\Phi(m)"'] \arrow[dd, "\overline{r_Y}"] & & \\
RX \arrow[rrr, "Rf"] \arrow[dr, "Re"'] & & & RY \arrow[r, shift left, "Ru"] \arrow[r, shift right, "Rv"'] & RZ \\
& RM \arrow[r, "e'"'] \arrow[urr, "Rm"] & N \arrow[ur, "n"'] & &
\end{tikzcd}
$$

Since $\mathcal{E} \subseteq Epi\,\mathbf{X}$, $RuRm = RvRm$ and $Run = Rvn$. Hence $Run\overline{r_Y} = Rvn\overline{r_Y} \Rightarrow Rur_Y\Phi_{(R,r)}(m) = Rvr_Y\Phi_{(R,r)}(m) \Rightarrow r_Zu\Phi_{(R,r)}(m) = r_Zv\Phi_{(R,r)}(m)$. But since Z is in $\mathrm{Fix}(R,r)$ and f is $\Phi_{(R,r)}$-dense, both r_Z and $\Phi_{(R,r)}(m)$ are isomorphisms, so $u = v$. $\square$

5.3 Proposition. *If* (R,r) *is idempotent, then* $\mathrm{Fix}(R,r)_\perp \subseteq \{\Phi_{(R,r)}$*-dense*$\}$.

PROOF: Let $me = f$ be the $(\mathcal{E},\mathcal{M})$ factorisation of a morphism $f : X \to Y$ in $\mathrm{Fix}(R,r)_\perp$, and construct $\Phi_{(R,r)}(m)$.

$$
\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[dr, "e"] \arrow[ddd, "r_X"'] & & & Y \arrow[ddd, "r_Y"] \arrow[dl, dashed, shift left, "k"'] \arrow[dddlll, dashed, "h"'] \\
& M \arrow[urr, "m"] \arrow[dr, "j"'] \arrow[ddd, "r_M"] & & \\
& & \overline{M} \arrow[uur, "\Phi(m)"'] \arrow[ddd, "\overline{r_Y}"] & \\
RX \arrow[rrr, "Rf"] \arrow[dr, "Re"'] & & & RY \\
& RM \arrow[urr, "Rm"] \arrow[dr, "e'"'] & & \\
& & N \arrow[uur, "n"'] &
\end{tikzcd}
$$

Since RX is in $\mathrm{Fix}(R,r)$ there is a (unique) $h : Y \to RX$ such that $hf = r_X$ which gives that $ne'Rehf = Rfhf = r_Yf$. By Proposition 5.1, f is $\mathrm{Fix}(R,r)$-cancellable, so $ne'Reh = r_Y$. This means that there is a unique $k : Y \to \overline{M}$ such that $\overline{r_Y}k = e'Reh$ and $\Phi_{(R,r)}(m)k = 1_Y$. Hence $\Phi_{(R,r)}(m)$ is an isomorphism and f is $\Phi_{(R,r)}$-dense. $\square$

5.4 Proposition. *If* (R, r) *is idempotent and well-pointed, then* $\mathrm{Fix}(R, r)_\perp \subseteq \Sigma_R$.

Proof: Construct the diagram below for $f : X \to Y$ in $\mathrm{Fix}(R, r)_\perp$. $RX \in \mathrm{Fix}(R, r)$, so there is a unique $h : Y \to RX$ such that $hf = r_X$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r_X"'] & Y \arrow[d, "r_Y"] \arrow[dl, dashed, "h"'] \\
RX \arrow[r, shift left, "Rf"] & RY \arrow[l, dashed, shift left, "h^*"]
\end{tikzcd}$$

By Proposition 5.1 f is $\mathrm{Fix}(R, r)$-cancellable, so $Rfhf = r_Y f \Rightarrow Rfh = r_Y$. But since (R, r) is both idempotent and well-pointed, $r_Y \in \Sigma_R \subseteq \mathrm{Fix}(R, r)_\perp$ so there is a unique $h^* : RY \to RX$ such that $h^* r_Y = h$. Then because r_Y is $\mathrm{Fix}(R, r)$-cancellable, we see that $Rfh^* = 1_{RY}$. Similarly $h^* Rfr_X = h^* r_Y f = hf = r_X$ implies that $h^* Rf = 1_{RX}$ and so Rf is an isomorphism and $f \in \Sigma_R$. $\square$

These results combine to give us the following valuable result which generalises Proposition 3.3 of [5] which is given for the case that (R, r) is a reflection.

5.5 Proposition. *If* (R, r) *is idempotent and well-pointed, then:*

$$\mathrm{Fix}(R, r)_{\perp_w} \cap \{\mathrm{Fix}(R, r)\text{-}\mathit{cancellable}\} = \mathrm{Fix}(R, r)_\perp = \Sigma_R.$$

If in addition $\mathcal{E} \subseteq \mathit{Epi}\,\mathbf{X}$, *these classes are also equal to* $\mathrm{Fix}(R, r)_{\perp_w} \cap \{\Phi_{(R,r)}\text{-}\mathit{dense}\}$.

Proof: Propositions 5.1 and 5.4 combine to give that $\Sigma_R = \mathrm{Fix}(R, r)_\perp$ and with the knowledge of Proposition 5.1 it is clear that $\mathrm{Fix}(R, r)_{\perp_w} \cap \{\mathrm{Fix}(R, r)\text{-}\mathit{cancellable}\} = \mathrm{Fix}(R, r)_\perp$. Furthermore Proposition 5.3 gives that $\mathrm{Fix}(R, r)_\perp \subseteq \mathrm{Fix}(R, r)_{\perp_w} \cap \{\Phi_{(R,r)}\text{-}\mathit{dense}\}$ and if $\mathcal{E} \subseteq \mathit{Epi}\,\mathbf{X}$, then Proposition 5.2 completes the argument by showing that $\mathrm{Fix}(R, r)_{\perp_w} \cap \{\Phi_{(R,r)}\text{-}\mathit{dense}\} \subseteq \mathrm{Fix}(R, r)_{\perp_w} \cap \{\mathrm{Fix}(R, r)\text{-}\mathit{cancellable}\}$. $\square$

5.6 Corollary. *If* (R, r) *is idempotent and well-pointed, then* $\{(R, r)\text{-}\mathit{perfect}\} \subseteq \{\mathit{Weakly}\ (R, r)\text{-}\mathit{perfect}\} \subseteq \{\mathrm{Fix}(R, r)\text{-}\mathit{perfect}\}$. *If in addition* $\Sigma_R \subseteq \mathit{Epi}\,\mathbf{X}$, *then* $\{\mathit{Weakly}\ (R, r)\text{-}\mathit{perfect}\} = \{\mathrm{Fix}(R, r)\text{-}\mathit{perfect}\}$.

Proof: The first inclusion is already known. By the above proposition, $\Sigma_R \cap \mathit{Epi}\,\mathbf{X} = \mathrm{Fix}(R, r)_\perp \cap \mathit{Epi}\,\mathbf{X} = \mathrm{Fix}(R, r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}$, from which it follows that

$\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X} \subseteq \Sigma_R$ and so $\Sigma_R^{\downarrow} \subseteq (\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X})^{\downarrow}$. This establishes the second inclusion.

If $\Sigma_R \subseteq \mathit{Epi}\,\mathbf{X}$, then obviously $\Sigma_R = \Sigma_R \cap \mathit{Epi}\,\mathbf{X} = \mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}$ and so $\{$*Weakly* (R,r)*-perfect*$\}$ $= \{\mathrm{Fix}(R,r)$*-perfect*$\}$. □

5.7 Theorem. *If (R,r) is idempotent, well-pointed and direct (and $\Sigma_R \subseteq \mathit{Epi}\,\mathbf{X}$), then:*

$$(R,r)\text{-}\mathit{perfect} = \mathit{Weakly}\ (R,r)\text{-}\mathit{perfect} \ \subseteq\ (=)\,\mathrm{Fix}(R,r)\text{-}\mathit{perfect}$$

If furthermore $\mathcal{E} \subseteq \mathit{Epi}\,\mathbf{X}$, then ($\Phi_{(R,r)}$-dense $\mathrm{Fix}(R,r)$-extendable, (R,r)-perfect) is a factorisation structure for morphisms in $\mathbf{X}$.

Proof: The second inclusion/equality follows from the above corollary. Under the assumption of directness, [20, Corollary 17] tells us that the first equality holds. That same result gives that $(\Sigma_R, \Sigma_R^{\downarrow})$ is a factorisation structure for morphisms in $\mathbf{X}$, and by Proposition 5.5 Σ_R is just the class of $\Phi_{(R,r)}$-dense $\mathrm{Fix}(R,r)$-extendable morphisms.

□

5.8 Remarks. (1) It is not generally the case that $\Sigma_R \subseteq \mathit{Epi}\,\mathbf{X}$. For example if (R,r) is the $\underline{\mathrm{Top}_0}$ reflection in $\underline{\mathrm{Top}}$, then any embedding of a point into any indiscrete space with more than 1 point is in Σ_R while obviously it is not an epimorphism in $\underline{\mathrm{Top}}$. If (R,r) is pointwise monomorphic, however, then we do have that $\Sigma_R \subseteq \mathit{Epi}\,\mathbf{X}$.

(2) It is also notable that in general $\{\Phi_{(R,r)}$*-dense*$\} \neq \{\mathrm{Fix}(R,r)$*-cancellable*$\}$, this is something typical of the regular closure (cf. [8, Remark (2), p. 137]). As an example, let (R,r) be the $\underline{\mathrm{Top}_0}$ reflection again. Take the space $\mathbf{N} \cup \{\infty\}$ which has the topology generated by basic opens of the form $U_n = \{m \in \mathbf{N} \mid m \geq n\} \cup \{\infty\}$ for $n \in \mathbf{N}$. The topological embedding of $\mathbf{N}$ into $\mathbf{N} \cup \{\infty\}$ is b-dense but not $\Phi_{(R,r)}$-dense and it is well known that in $\underline{\mathrm{Top}}$ the b-dense maps are $\underline{\mathrm{Top}_0}$-cancellable.

It is of course theoretically possible to have $\Sigma_R^{\downarrow} = (\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X})^{\downarrow}$ without necessarily having that $\Sigma_R = \mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}$. In most of the examples we consider, this cannot happen.

The class $\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}$ contains all isomorphisms and is closed under composition, pushouts and cointersections (cf. [25, Proposition 1 (viii)]). Thus if $\mathbf{X}$ has pushouts and cointersections, $\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}$ is the first component of a factorisation structure for sources in $\mathbf{X}$ ([1, Theorem 15.14]). In such cases, the source factorisation structure induces the factorisation structure $((\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}), (\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X})^{\downarrow})$ for morphisms in $\mathbf{X}$. Thus if $\Sigma_R^{\downarrow} = (\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X})^{\downarrow}$ it would mean that $\Sigma_R \subseteq \Sigma_R^{\downarrow\uparrow} = \mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}$. If furthermore the conditions of Proposition 5.4 hold, then $\mathrm{Fix}(R,r)_{\perp_w} \cap \mathit{Epi}\,\mathbf{X} \subseteq \Sigma_R$, and equality would follow.

So far we have only considered possible links between these notions from one perspective, namely given an endofunctor (R, r) how $\mathrm{Fix}(R, r)$-perfect and (weakly) (R, r)-perfect morphisms relate. What if we have an arbitrary class $\mathcal{X}$ of $\mathbf{X}$-objects and consider the $\mathcal{X}$-perfect morphisms?

If in our category $\mathbf{X}$ both pushouts of $(\mathcal{X}_{\perp_w} \cap \mathit{Epi}\,\mathbf{X})$-morphisms along any $\mathbf{X}$-morphism and cointersections of arbitrary families of $(\mathcal{X}_{\perp_w} \cap \mathit{Epi}\,\mathbf{X})$-morphisms exist, then there is a conglomerate $\mathbf{M}$ of sources in $\mathbf{X}$ such that $((\mathcal{X}_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}), \mathbf{M})$ is a factorisation structure for sources in $\mathbf{X}$ (cf. [25, Proposition 1 (vii)] and [1, Theorem 15.14]). This means that $\mathcal{X}$ has a $(\mathcal{X}_{\perp_w} \cap \mathit{Epi}\,\mathbf{X})$-reflective hull in $\mathbf{X}$. Denote the objects of this hull by $E(\mathcal{X})$ and let $(R_{\mathcal{X}}, r)$ be the reflector.

Since $\mathcal{X} \subseteq E(\mathcal{X})$ obviously $E(\mathcal{X})_{\perp_w} \subseteq \mathcal{X}_{\perp_w}$. Consider $X \xrightarrow{f} Y \in \mathcal{X}_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}$ and $g : X \to Z$ with codomain Z in $E(\mathcal{X})$. $Z \in E(\mathcal{X})$ means that there is a source $(m_i : Z \to A_i)_{i \in I} \in \mathbf{M}$ with each $A_i \in \mathcal{X}$, so we have the diagram below.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[dl, dashed, "d"'] \arrow[d, dashed, "h_i"] \\
Z \arrow[r, "m_i"] & A_i
\end{tikzcd}$$

For each $i \in I$ there is an $h_i : Y \to A_i$ such that $h_i f = m_i g$. Then by the $((\mathcal{X}_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}), \mathbf{M})$ diagonalisation property there is a morphism $d : Y \to Z$ such that in particular $df = g$, giving that $f \in E(\mathcal{X})_{\perp_w}$.

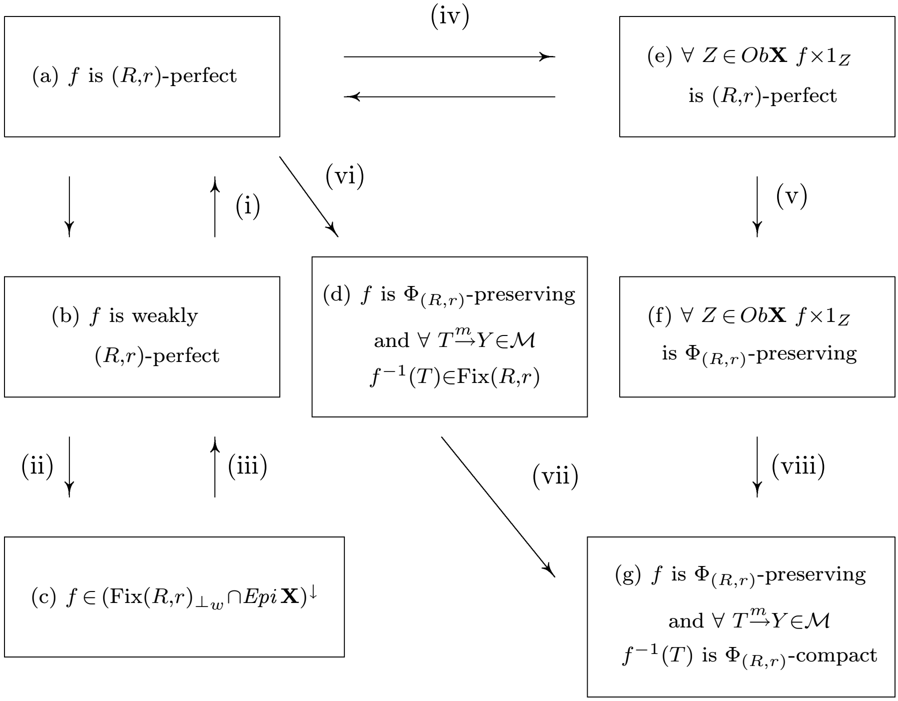
So we can conclude that $E(\mathcal{X})_{\perp_w} \cap \mathit{Epi}\,\mathbf{X} = \mathcal{X}_{\perp_w} \cap \mathit{Epi}\,\mathbf{X}$ which means (rewriting $E(\mathcal{X})$ as $\mathrm{Fix}(R_{\mathcal{X}}, r)$) that the $\mathcal{X}$-perfect morphisms, $(\mathcal{X}_{\perp_w} \cap \mathit{Epi}\,\mathbf{X})^{\downarrow} = (\mathrm{Fix}(R_{\mathcal{X}}, r) \cap \mathit{Epi}\,\mathbf{X})^{\downarrow}$, which are just the $\mathrm{Fix}(R_{\mathcal{X}}, r)$-perfect morphisms.

Since $(R_{\mathcal{X}}, r)$ is a reflection it fulfills the conditions of Corollary 5.6 above, so we can conclude that $\{(R_{\mathcal{X}}, r)$*-perfect*$\} \subseteq \{$*Weakly* $(R_{\mathcal{X}}, r)$*-perfect*$\} \subseteq \{\mathcal{X}$*-perfect*$\}$. Moreover if $\Sigma_{R_{\mathcal{X}}}$ is a class of epimorphisms, then $\{$*Weakly* $(R_{\mathcal{X}}, r)$*-perfect*$\} = \{\mathcal{X}$*-perfect*$\}$ and these in turn equal the $(R_{\mathcal{X}}, r)$-perfect morphisms if $(R_{\mathcal{X}}, r)$ is a direct reflection.

6. Summarising theorem and examples

The interrelation of the five notions of perfect morphism we have investigated can be presented in the following theorem, an improvement of [20, Theorem 23].

6.1 Theorem. *Let $f : X \to Y$ in $\mathbf{X}$. The properties of f in the boxes below imply others along the arrows drawn. The numerals alongside certain arrows represent conditions that are sufficient for the associated implication to hold.*



(i) (R, r) is direct and either $\Sigma_R \subseteq Epi\,\mathbf{X}$ or (R, r) is idempotent.
(ii) (R, r) is idempotent and well-pointed.
(iii) (ii) and $\Sigma_R \subseteq Epi\,\mathbf{X}$.
(iv) (i) or for any $A, B \in Ob\mathbf{X}$ the canonical morphism $k : R(A \times B) \to RA \times RB$ is a monomorphism.
(v) $\mathcal{E}$ is stable under pullback and $Re \in \mathcal{E}$ for every $e \in \mathcal{E}$.
(vi) (v) and (R, r) is pointwise epimorphic.
(vii) (i) and (v).
(viii) $\mathcal{E}$ is stable under pullback, (R, r) is direct and idempotent, $\mathbf{X}$ has a terminal object T and each $T \xrightarrow{m} Y \in \mathcal{M}$ is $\Phi_{(R,r)}$-closed.

For (e), (f) and (g) to be accessed we need to assume that $\mathbf{X}$ has products of pairs.

PROOF: (a) $\Rightarrow$ (b): Proposition 4.2.

(b) $\Rightarrow$ (a): This is proved for (R, r) direct and idempotent in [20, Theorem 23]. That $\Sigma_R \subseteq Epi\,\mathbf{X}$ can be substituted for idempotence is shown in [19, Theorem 2.3.5]. (This will appear in work on directness combining [5] and [19], presently in preparation.)

(b) $\Rightarrow$ (c): Corollary 5.6.

(c) $\Rightarrow$ (b): Corollary 5.6.

(b) $\Rightarrow$ (c): The proof for (R,r) direct and idempotent is given in [20]. As in the proof of (b) $\Rightarrow$ (a), $\Sigma_R \subseteq \mathit{Epi}\,\mathbf{X}$ can be substituted for idempotence of (R,r). On the other hand assume that for any $A, B \in \mathit{Ob}\mathbf{X}$ the canonical morphism $k : R(A\times B) \to RA \times RB$ is a monomorphism. Consider the diagram below where $u : A \to Y \times Z$ and $v : A \to R(X \times Z)$ are such that $r_{Y\times Z}u = R(f\times 1_Z)v$. The morphisms p_1, q_1 and π_1 are projections.

$$\begin{tikzcd}
A \arrow[drrr, "u"] \arrow[dr, dashed, "g"'] \arrow[dd, dashed, "h"'] \arrow[dddr, "v"] & & & & \\
& X\times Z \arrow[rr, "f\times 1_Z"] \arrow[dl, "p_1"] \arrow[dd, "r_{X\times Z}"] & & Y\times Z \arrow[dr, "q_1"] \arrow[dd, "r_{Y\times Z}"'] & \\
X \arrow[rrrr, "f"] \arrow[ddd, "r_X"'] & & & & Y \arrow[ddd, "r_Y"] \\
& R(X\times Z) \arrow[rr, "R(f\times 1_Z)"] \arrow[dr, "k"] \arrow[ddl, "Rp_1"] & & R(Y\times Z) \arrow[ddr, "Rq_1"'] & \\
& & RX\times RZ \arrow[dll, "\pi_1"] & & \\
RX \arrow[rrrr, "Rf"] & & & & RY
\end{tikzcd}$$

Since f is (R,r)-perfect and $r_Y q_1 u = Rq_1 r_{Y\times Z} u = Rq_1 R(f\times 1_Z)v = R(q_1(f\times 1_Z))v = R(fp_1)v = RfRp_1 v$, there is a unique $h : A \to X$ such that $fh = q_1 u$ and $r_X h = Rp_1 v$.

Put $g := \langle h, q_2 u\rangle : A \to X\times Z$ and note that $q_1(f\times 1_Z)g = fp_1 g = fh = q_1 u$ and $q_2(f\times 1_Z)g = 1_Z p_2 g = q_2 u$, so $(f\times 1_Z)g = u$. Also $\pi_1 k r_{X\times Z} g = Rp_1 r_{X\times Z} g = r_X p_1 g = r_X h = Rp_1 v = \pi_1 kv$ and $\pi_2 k r_{X\times Z} g = Rp_2 r_{X\times Z} g = r_Z p_2 g = r_Z q_2 u = Rq_2 r_{Y\times Z} u = Rq_2 R(f\times 1_Z)v = R1_Z Rp_2 v = Rp_2 v = \pi_2 kv$ so since k is a monomorphism we conclude that $r_{X\times Z} g = v$.

Say $g^* : A \to X\times Z$ is such that $(f\times 1_Z)g^* = u$ and $r_{X\times Z}g^* = v$. This gives $fp_1 g^* = q_1(f\times 1_Z)g^* = q_1 u$ and $r_X p_1 g^* = Rp_1 r_{X\times Z} g^* = Rp_1 v$ so by the uniqueness condition on h, $p_1 g^* = h$. On the other hand $p_2 g^* = 1_Z p_2 g^* = q_2(f\times 1_Z)g^* = q_2 u$, so in fact $g^* = \langle h, q_2 u\rangle = g$.

The implications (a) $\Rightarrow$ (d), (e) $\Rightarrow$ (a), (e) $\Rightarrow$ (f) and (f) $\Rightarrow$ (g) are proved in [20]. The proof of (d) $\Rightarrow$ (g) follows from [20] with the substitution of $\Sigma_R \subseteq \mathit{Epi}\,\mathbf{X}$ for idempotence as in (b) $\Rightarrow$ (a) above. $\square$

6.2 Remark. These many conditions may seem a little cluttered, but for certain (R, r) there are a number of conditions that are fulfilled simultaneously in which case the following clear deductions can be made.

(1) If (R, r) is a direct reflection, $\mathcal{E}$ is stable under pullback and for every $e \in \mathcal{E}$, $R(e) \in \mathcal{E}$, then all implications except (c) $\Rightarrow$ (b), (a) $\Rightarrow$ (d) and (f) $\Rightarrow$ (g) follow immediately from the theory.

(2) If in addition to (1) above, (R, r) is an epireflection, then only (c) $\Rightarrow$ (b) and (f) $\Rightarrow$ (g) cannot be concluded from the theorem.

(3) If moreover (R, r) is a bireflection, only (f) $\Rightarrow$(g) does not automatically hold. It is notable that condition (viii) is unnecessarily strong, in the examples below we show that the implication (f) $\Rightarrow$ (g) can in fact hold without (viii) being fulfilled.

6.3 Examples

We conclude with a number of examples. In each instance we have specifically investigated whether or not conditions (i) to (viii) of Theorem 6.1 are satisfied. Only (ii) and (iv) are satisfied by all examples. For the failure of each of the other conditions — with the exception of (viii) — we have been able to show in an example that the associated implication is not true. While this does not establish the necessity of the conditions given, it does give credence to the emphasis placed on them.

6.3.1 Čech-Stone compactification. Let (R, r) denote the Čech-Stone compactification in the category $\mathbf{X}$ of Tychonoff spaces and continuous maps. $(\mathcal{E}, \mathcal{M})$ is the (*Surjection, Embedding*) factorisation structure for morphisms in $\mathbf{X}$. $\Phi_{(R,r)}$ is the usual topological closure. All conditions (i) to (viii) in Theorem 6.1 are satisfied, hence all the implications are true.

Knowing what we do about (R, r) and $\mathcal{E}$, we see from Proposition 5.5 that $\Sigma_R = \mathrm{Fix}(R, r)_\perp = \mathrm{Fix}(R, r)_{\perp_w} \cap \{\Phi_{(R,r)}\text{-}dense\}$ which is the class of dense $\underline{\text{HComp}}$-extendable morphisms. Furthermore, Theorem 5.7 tells us that (*Dense* $\underline{\text{HComp}}$-*extendable,* (R, r)*-perfect*) is a factorisation structure for morphisms in $\mathbf{X}$. These facts are of course well known.

6.3.2 $\underline{\text{Top}_0}$ reflection. Let (R, r) be the $\underline{\text{Top}_0}$ reflector in $\mathbf{X} = \underline{\text{Top}}$. $\mathcal{E}$ is again the class of surjective continuous maps, and $\mathcal{M}$ the embeddings. In [5] it is shown that (R, r) is direct. Knowing this, that (R, r) is an $\mathcal{E}$-reflection and that surjective continuous maps are stable under pullback, it is immediately clear that conditions (i), (ii), (iv), (v), (vi) and (vii) of Theorem 6.1 are true.

Let $j : \{\bullet\} \to I_2$ be an embedding of a singleton space into a two point indiscrete space. Then $j \in \Sigma_R$, j is not $\Phi_{(R,r)}$-closed, j is not (R, r)-perfect and $j \in (\mathrm{Fix}(R, r)_{\perp_w} \cap Epi\,\mathbf{X})^{\downarrow}$. Thus conditions (iii) and (viii) in Theorem 6.1 do not hold, nor does the implication (c) $\Rightarrow$ (b).

The properties of (R, r) and $\mathcal{E}$ are such that we can conclude from Theorem 5.7 that we have a factorisation structure ($\Phi_{(R,r)}$-*dense* $\underline{\text{Top}_0}$-*extendable,* (R, r)-*perfect*) for morphisms in $\underline{\text{Top}}$.

6.3.3 Uniform completion. Let (R, r) be the completion reflector in $\underline{\mathrm{UNIF}_0}$. In this setting, $\mathcal{E}$ is the class of surjective uniformly continuous maps, and $\mathcal{M}$ is the class of uniform embeddings.

(R, r) is a bireflection and is shown in [5] to be direct. However, while $\mathcal{E}$ is stable under pullback in $\underline{\mathrm{UNIF}_0}$, (R, r) does not preserve $\mathcal{E}$-morphisms. (The uniformly continuous image of a complete space need not be complete.)

Thus conditions (i) to (vii) of Theorem 6.1 are satisfied, but not (v) to (vii). Since $\Phi_{(R,r)}$ is the underlying topological closure ([19]), (viii) holds.

Taking X to be the real line with discrete uniformity, and Y the real line with the usual uniformity, we see that both X and Y are complete, $1_{\mathbb{R}} : X \to Y$ is (R, r)-perfect (Proposition 4.7). However, $1_{\mathbb{R}}$ does not preserve the closure (e.g. $(0, 1)$ is closed in X but not in Y) so both implications (a) $\Rightarrow$ (d) and (e) $\Rightarrow$ (f) of Theorem 6.1 fail.

Also, since Y is in $\mathrm{Fix}(R, r)$ but not $\Phi_{(R,r)}$-compact, the surjection $f : Y \to \{\bullet\}$ to a singleton provides a counterexample to (d) $\Rightarrow$ (g).

We again conclude from Theorem 5.7 that (*Dense complete-extendable,* (R, r)*-perfect*) is a factorisation structure for morphisms in $\underline{\mathrm{UNIF}_0}$. (Dense means with respect to the underlying topology.)

In [14] it is observed that a uniformly continuous map $f : X \to Y$ is (R, r)-perfect iff for any Cauchy filter $\mathcal{U}$ in X, $\mathcal{U}$ converges in X if $f(\mathcal{U})$ converges in Y.

6.3.4 Sobrification. Let (S, s) be the sobrification reflector in $\underline{\mathrm{TOP}_0}$. As with the other examples thus far $(\mathcal{E}, \mathcal{M})$ is the (*Surjection, Embedding*) factorisation structure for morphisms restricted to $\underline{\mathrm{TOP}_0}$. Also in this setting $\mathcal{E}$ is stable under pullback, and again in [5] it is shown that (S, s) is direct.

Since in addition to the above (S, s) is a bireflection, conditions (i) to (iv) of Theorem 6.1 hold, and since $\Phi_{(S,s)}$ is the b-closure (cf. [20]) (viii) holds too. (S, s) does not, however, preserve surjective continuous maps as the following example shows.

Let X be the natural numbers $\mathbf{N}$ endowed with the discrete topology. Let Y be the natural numbers endowed with the co-finite topology. Both are $\underline{\mathrm{TOP}_0}$ spaces. X is clearly a sober space. Y however is not, since $\mathbf{N}$ is a closed irreducible subset of Y yet it cannot be expressed as the closure of a single point.

SY has underlying set $\mathbf{N} \cup \{\bullet\}$. U is an open set in SY iff $\{\bullet\} \subseteq U$ and $U \cap \mathbf{N}$ is open in Y.

The identity function on $\mathbf{N}$, $1_{\mathbf{N}} : X \to Y$ is a surjective $\underline{\mathrm{TOP}_0}$ morphism, yet $S1_{\mathbf{N}} : SX \to SY$ is not surjective. Thus (S, s) does not satisfy conditions (v) to (vii) of Theorem 6.1.

Consider now the spaces X and Y, both with underlying set $\mathbf{N} \cup \{\infty\}$ ($n \leq \infty\ \forall\, n \in \mathbf{N}$). Let X have the discrete topology and let Y have the upper topology, namely the topology with open sets of the form $U_n := \{m \in \mathbf{N} \mid n \leq m\} \cup \{\infty\}$ for $n \in \mathbf{N}$. Both X and Y are sober spaces.

Let $1_{\mathbf{N}\cup\{\infty\}}$ be the identity function on $\mathbf{N} \cup \{\infty\}$. Then $1_{\mathbf{N}\cup\{\infty\}} : X \to Y$ is a $\underline{\mathrm{SOB}}$-morphism and is thus (S, s)-perfect (cf. Proposition 4.7). Observe now that

$\mathbf{N}$ is a b-closed subset of X yet it is not b-closed in Y, thus since the b-closure is idempotent this means that $1_{\mathbf{N}\cup\{\infty\}} : X \to Y$ is not b-closure preserving. $\Phi_{(S,s)}$ is the b-closure, thus we have an example of an (S, s)-perfect map that is not $\Phi_{(S,s)}$-preserving. From this we conclude that in this example neither of the implications (a) $\Rightarrow$ (d) and (e) $\Rightarrow$ (f) of Theorem 6.1 holds.

It has been shown (cf. [12, Corollary 2] and [8, Example 3.2]) that in $\underline{\text{TOP}_0}$ the b-compact spaces are properly contained in the sober spaces. Let X be a sober space that is not b-compact. The map $f : X \to \{\bullet\}$ of X onto a singleton space then gives a simple example to show that the implication (d) $\Rightarrow$ (g) of Theorem 6.1 does not hold either.

The properties of (S, s) are such that we can conclude from Theorem 5.7 that we have a factorisation structure (*b-dense $\underline{\text{SOB}}$-extendable, (S, s)-perfect*) for morphisms in $\underline{\text{TOP}_0}$.

6.3.5 Endofunctors induced by congruence relations in varieties. In a number of algebraic settings, many pointed endofunctors are induced by congruence relations. We give these examples in Groups, but the material can be extended to an arbitrary variety (cf. [19]).

A congruence relation $\sim_G$ on a group G is an equivalence relation such that $\sim_G$ is a subgroup of $G \times G$. A family $(\sim_G)_{G\in\underline{\text{GRP}}}$ of congruence relations is termed a *natural family* if for any homomorphism $f : G \to H : x \sim_G y \Rightarrow f(x) \sim_H f(y)$. Such a family induces a pointed endofunctor (R, q) on $\underline{\text{GRP}}$, with $q_G : G \to RG$ being the quotient $G \to G/_{\sim_G}$.

If Φ is the pullback closure operator induced by (R, q), then it is not difficult to show that:

1. For a group homomorphism $f : G \to H$ the following are equivalent:
   (a) f is Φ preserving.
   (b) $f[[e_G]_{\sim_G}] = [e_H]_{\sim_H}$. ($e$ the identity element)
   (c) For every $g \in G$, $f[[g]_{\sim_G}] = [f(g)]_{\sim_H}$.
2. A group $G \in \text{Fix}(R, q) \Leftrightarrow [e_G]_{\sim_G} = \{e_G\}$.
3. A homomorphism $f : G \to H$ is (R, q)-perfect iff f is Φ-preserving and $f^{-1}(e_H) \cap [e_G]_{\sim_G} = \{e_G\}$.

If furthermore for each subgroup H of G, $\sim_H = \sim_G \cap H \times H$ ($(\sim_G)_{G\in\underline{\text{GRP}}}$ is hereditary) then:

4. $f : G \to H$ is (R, q)-perfect iff f is Φ-preserving and $f^{-1}(e_H) \in \text{Fix}(R, q)$.

Now we look at two specific examples in the categories of $\underline{\text{GRP}}$ and $\underline{\text{ABGRP}}$.

(1) Let (R, q) be the reflector from $\underline{\text{GRP}}$ to $\underline{\text{ABGRP}}$, induced by the family $(\sim_G)_{G\in\underline{\text{GRP}}}$, where $x \sim_G y \Leftrightarrow xC_G = yC_G$ for the commutator subgroup C_G of G.

(R, q) is not direct. (Consider the inclusion $\{e\} \xrightarrow{i} S_3$ for the group of permutations on 3 elements, noting that $RS_3 = \mathbf{Z}_2$.) So conditions (i), (vii) and (viii) of Theorem 6.1 do not hold. Condition (iii) of that theorem does not hold either.

Clearly conditions (ii), (v) and (vi) hold, and according to [21] (R,q) preserves products, so condition (iv) holds too.

[19, Theorem 2.3.7] tells us that in this setting (R,q)-perfect coincides with weakly (R,q)-perfect iff (R,q) is direct. From this we can conclude that the implication (b) $\Rightarrow$ (a) in Theorem 6.1 does not hold.

Consider the embedding $m : \mathbf{Z}_2 \to S_3$ where 0 is mapped to the identity permutation, and 1 is mapped to any one of the three transpositions. Since the domain of m is an abelian group, it is clear that $m \in (\mathrm{Fix}(R,q)_{\perp_w} \cap Epi)^{\downarrow}$. Considering the commutative square $1_{S_3}m = m1_{\mathbf{Z}_2}$ we see, however, that m is not in $\Sigma_R^{\downarrow}$ so the implication (c) $\Rightarrow$ (b) of Theorem 6.1 also does not hold in this example.

The results 1 & 3 above enable us to characterise the (R,q)-perfect homomorphisms as those $f : G \to H$ for which $f[C_G] = C_H$ and $f^{-1}(e_H) \cap C_G = \{e_G\}$. (Note that the family of congruence relations in this example is not hereditary, so we cannot apply 4. In fact the reflection map $q_{S_3} : S_3 \to \mathbf{Z}_2$ is Φ-preserving and $f^{-1}(0) = A_3 \in \mathrm{Fix}(R,q)$ yet it is not (R,q)-perfect.)

(2) Let (R,q) be the reflector from <u>AbGrp</u> to <u>TfAb</u>. The natural family of congruence relations that induces this reflection is defined by: $x \sim_G y \Leftrightarrow \exists$ nonzero integer n such that $nx = ny$. It is easy to see that this family of congruence relations is both hereditary and finitely productive (for G and H, $(x,y) \sim_{G\times H} (z,w) \Leftrightarrow x \sim_G z$ and $y \sim_H w$), from which it follows that the reflector (R,q) is direct. Hence all conditions in Theorem 6.1 except (iii) and (viii) hold.

Result 4 above tells us that a homomorphism $f : G \to H$ is (R,q)-perfect iff $f[tG] = tH$ and $f^{-1}(e_H)$ is torsion free. (Where $tG = [e_G]_{\sim_G}$ is the torsion subgroup.)

Since the congruence relations involved are productive, it is not difficult to see that every Abelian group is Φ-compact. Thus for a homomorphism $f : G \to H$, f is Φ-preserving and $f^{-1}(e_H)$ is Φ-compact iff f is Φ-preserving. Note that this tells us that the implication (f) $\Rightarrow$ (g) is true even though condition (viii) does not hold.

The inclusion map $i : \{0\} \to \mathbf{Z}_2 \in (\mathrm{Fix}(R,q)_{\perp_w} \cap Epi)^{\downarrow}$ yet it is not in $\Sigma_R^{\downarrow}$ (consider the square $1_{\mathbf{Z}_2}i = i1_{\{0\}}$). So yet again the implication (c) $\Rightarrow$ (b) does not hold.

Lastly, we conclude from Theorem 5.7 that (Φ-*dense* <u>TfAb</u>-*extendable,* (R,q)-*perfect*) is a factorisation structure for morphisms in <u>AbGrp</u>.

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