Weak Krull-Schmidt theorem

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Abstract. Recently, A. Facchini [3] showed that the classical Krull-Schmidt theorem fails for serial modules of finite Goldie dimension and he proved a weak version of this theorem within this class. In this remark we shall build this theory axiomatically and then we apply the results obtained to a class of some modules that are torsionfree with respect to a given hereditary torsion theory. As a special case we obtain that the weak Krull-Schmidt theorem holds for the class of modules that are both uniform and co-uniform. A simple example shows that this generalizes the result of [3] mentioned above.

Keywords: monogeny class, epigeny class, weak Krull-Schmidt theorem, hereditary torsion theory, uniform module, co-uniform module

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1. An axiomatical approach

By a ring R we shall mean an associative ring with the unit element $1 \neq 0$ and all modules are right unital R-modules elements of the category Mod-R.

Roughly speaking the weak Krull-Schmidt theorem characterizes the unicity of finite direct sums of modules up to the isomorphism in terms of monogeny and epigeny classes introduced by A. Facchini in [3]. In the first part we shall investigate a condition on the endomorphism ring of a module M under which this ring is semilocal with at most two maximal (one-sided) ideals. Further, we shall investigate two conditions working with "unusual cancellation" of monorphisms and epimorphisms. Following the ideas of [3], from these three properties together with a "direct summand property" we then derive a weak Krull-Schmidt theorem. In the second part we shall work with modules that are torsionfree with respect to a given hereditary torsion theory σ for Mod-R and that satisfies an injectivity-like condition. We show that these modules which are σ-uniform and σ-co-uniform satisfy the conditions from the first part and consequently the weak Krull-Schmidt theorem holds for classes of such modules. In the brief last item the results obtained are applied to the trivial torsion theory $\sigma = 0$ to prove the weak Krull-Schmidt theorem for the class of uniform co-uniform modules. A simple example (due to G. Baccella) shows that this is a proper generalization of the results of [3].

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1.1 Definition. For the modules A, B we shall use the notation $[A]_m \leq [B]_m$ whenever there is a monomorphism from A to B. Similarly, $[A]_e \leq [B]_e$ means that there exists an epimorphism from A to B. Following [3] we shall say that the modules A and B belong to the same *monogeny class*, $[A]_m = [B]_m$, if $[A]_m \leq [B]_m$ and $[B]_m \leq [A]_m$. Similarly we shall say that A and B belong to the same *epigeny class*, $[A]_e = [B]_e$, if $[A]_e \leq [B]_e$ and $[B]_e \leq [A]_e$.

1.2 Definition. We say that a module M satisfies the condition (2M) if the subset $I = \{\alpha \in E \mid \alpha \text{ is not injective}\}$ is a right ideal and the subset $J = \{\alpha \in E \mid \alpha \text{ is not surjective}\}$ is a left ideal of the endomorphism ring $E = \mathrm{End}_R(M)$ of the module M.

1.3 Definition. Let A and B be modules. We say, that B has the property (A-CI), if for the homomorphisms $\alpha : A \to B$ and $\beta : B \to A$ the homomorphism β is injective whenever the composition $\beta\alpha$ is. Further, we say that a class $\mathfrak{M}$ of modules has the property (CI) if for any two modules A, B from $\mathfrak{M}$ the module B has (A-CI) whenever $[A]_m \leq [B]_m$.

1.4 Definition. Let A and B be modules. We say, that B has the property (A-CS), if for the homomorphisms $\alpha : A \to B$ and $\beta : B \to A$ the homomorphism α is surjective whenever the composition $\beta\alpha$ is. Further, we say that a class $\mathfrak{M}$ of modules has the property (CS) if for any two modules A, B from $\mathfrak{M}$ the module B has (A-CS) whenever $[B]_e \leq [A]_e$.

1.5 Proposition. *Let M be a module satisfying the condition (2M). Then*

(a) *I and J are two-sided ideals of E;*
(b) *M is an indecomposable module;*
(c) *for every injective non-surjective $\alpha \in E$ and every surjective non-injective $\beta \in E$ the sum $\alpha + \beta$ is an automorphism of M;*
(d) *the ideals I and J are completely prime;*
(e) *every proper one-sided ideal K of E is contained either in I or in J;*
(f) *I and J are the only maximal left and right ideals of E.*

Proof: (a) Obvious.

(b) Assuming $M = U \oplus V$ non-trivial and denoting ι_U, ι_V and π_U, π_V the canonical injections and projections, respectively, we have $\iota_U\pi_U$, $\iota_V\pi_V \in I \cap J$, which is impossible since $\iota_U\pi_U + \iota_V\pi_V = 1_M$.

(c) Clearly, if $\alpha + \beta$ is not a unit, then either $\alpha + \beta \in I$, or $\alpha + \beta \in J$ and so either $\alpha \in I$ or $\beta \in J$, which is impossible.

(d) For $\alpha, \beta \notin I$ the composition $\alpha\beta$ is injective, i.e. $\alpha\beta \notin I$. Similarly for J.

(e) Let $K \leq E$ be a proper one-sided ideal. Then K contains no units and so

$$K \subseteq I \cup J = \{\alpha \in E \mid \alpha^{-1} \text{ does not exist}\}.$$

If neither $K \subseteq I$ nor $K \subseteq J$, then there are $\alpha \in K \setminus I$ and $\beta \in K \setminus J$, i.e. α injective non-surjective and β is surjective non-injective. Consequently, by (c), $\alpha + \beta \in K$ is a unit, which is impossible.

(f) It follows immediately from (e). □

1.6 Corollary. *Let A be a module satisfying the condition (2M) and B, C be arbitrary. If $A \oplus B \cong A \oplus C$, then $B \cong C$.*

PROOF: The endomorphism ring of A is semilocal by the preceding proposition and [5, Theorem 2] applies. □

1.7 Lemma. *If a module A satisfies the condition (2M) and $f_1, \dots, f_n : A \to B$ are non-isomorphisms such that $f = \sum_{i=1}^n f_i$ is an isomorphism, then there are $i \neq j$ such that f_i is injective non-surjective and f_j is surjective non-injective.*

PROOF: The homomorphisms $g_i = f^{-1} f_i \in E = \operatorname{End} A$ are non-isomorphisms. Moreover, $\sum_{i=1}^n g_i = 1_A$ and so E is not local. Thus the condition (2M) yields the existence of $i \neq j$ such that $g_i \in J \setminus I$ and $g_j \in I \setminus J$ and the assertion follows. □

1.8 Proposition. *Assume that the module A satisfies the condition (2M) and that $A \oplus B = C_1 \oplus \cdots \oplus C_n$ with $n \geq 2$. Then there are two indices $i \neq j$ such that $A' \oplus B' = C_i \oplus C_j$ where $A \cong A'$ and $B \cong B' \oplus (\oplus_{k \neq i,j} C_k)$.*

PROOF: For $A = 0$ it suffices to take $i = 1$, $j = 2$ and put $B' = C_1 \oplus C_2$. So assume that $A \neq 0$ and $E = \operatorname{End} A$ is local. Denoting ι_i, ι_A, ι_B, π_i, π_A, π_B the corresponding canonical mappings we have $1_A = \pi_A \iota_A = \sum_{i=1}^n \pi_A \iota_i \pi_i \iota_A$ and so there is an index i such that $\varrho = \pi_A \iota_i \pi_i \iota_A$ is a unit in E and consequently the composition $(\varrho^{-1} \pi_A \iota_i)(\pi_i \iota_A)$ is the identity map 1_A of A. Thus A is isomorphic to a direct summand of C_i, $C_i = A' \oplus X, A' \cong A$. Taking $j \neq i$ arbitrarily, we have $C_i \oplus C_j = A' \oplus X \oplus C_j = A' \oplus B'$, $A \oplus B \cong A \oplus B' \oplus (\oplus_{k \neq i,j} C_k)$ and Corollary 1.6 yields $B \cong B' \oplus (\oplus_{k \neq i,j} C_k) = B' \oplus (\oplus_{k \neq i,j} C_k)$.

Assuming finally that E is non-local Lemma 1.7 and Proposition 1.5 (c) yield the existence of $i \neq j$ such that $\pi_A \iota_i \pi_i \iota_A \in I \setminus J$, $\pi_A \iota_j \pi_j \iota_A \in J \setminus I$ and their sum α is an automorphism of A. Denoting ι', π' the canonical maps of the summand $C_i \oplus C_j$ of $A \oplus B$ we clearly have $\alpha = \pi_S \iota' \pi' \iota_A$. As above, the composition $(\alpha^{-1} \pi_A \iota')(\pi' \iota_A)$ is the identity map of A and A is isomorphic to a direct summand of $C_i \oplus C_j$, $C_i \oplus C_j = A' \oplus B'$, $A' \cong A$. Finally, $A \oplus B \cong A \oplus B' \oplus (\oplus_{k \neq i,j} C_k)$ and Corollary 1.6 applies. □

1.9 Proposition. *If the module A satisfies the condition (2M) and the module B has the properties (A-CI) and (A-CS), then the following are equivalent:*

(i) $A \cong B$;

(ii) $[A]_m = [B]_m$, $[A]_e = [B]_e$.

PROOF: It clearly suffices to prove that (ii) implies (i). By hypothesis there are monomorphisms $\alpha : A \to B$, $\beta : B \to A$ and epimorphisms $\gamma : A \to B$, $\delta : B \to A$. Proving indirectly we can assume that no of the mappings α, β, γ, δ is an isomorphism. Using the notations from Definition 1.2 we have: $\beta\alpha \in J \setminus I$ for otherwise we easily obtain that β is an isomorphism and $\delta\gamma \in I \setminus J$ for otherwise γ is an isomorphism. Thus $\sigma = \beta\alpha + \delta\gamma$ is a unit by Proposition 1.5 (c).

Further, $\beta\gamma \in I \cap J$ for otherwise it is either injective and so γ is an isomorphism, or it is surjective and so β is an isomorphism. Similarly, $\delta\alpha \in I \cap J$, for otherwise it is either injective and so δ is an isomorphism by the property (A-CI), or it is surjective and so α is an isomorphism by the property (A-CS).

Finally, the homomorphism $\varrho = (\beta + \delta)(\alpha + \gamma) = \beta\alpha + \beta\gamma + \delta\alpha + \delta\gamma$ is a unit in E, since $\varrho \in I$ yields $\sigma = \varrho - \beta\gamma - \delta\alpha \in I$ and $\varrho \in J$ yields $\sigma \in J$. Thus $\alpha + \gamma$ is injective, and assuming it is not an isomorphism we have $\alpha + \gamma \in J \setminus I$ and so $\gamma \in J$, which contradicts the choice of γ and completes the proof. $\square$

1.10 Proposition. *Let the module $A \neq 0$ satisfy the condition (2M) and let $U_1, \dots, U_n$, $n \geq 2$ be indecomposable modules such that $A \ncong U_i$ for every $i = 1, \dots, n$. If A is isomorphic to a direct summand of $U_1 \oplus \dots \oplus U_n$, then there are $i, j \in \{1, \dots, n\}$, $i \neq j$, such that $[A]_m \leq [U_i]_m$ and $[U_j]_e \leq [A]_e$. If, moreover, U_i has the property (A-CI), then $[A]_m = [U_i]_m$ and if U_j has the property (A-CS), then $[U_j]_e = [A]_e$.*

Proof: Applying Proposition 1.8 we see that for some $i \neq j$, $A' \oplus B' = U_1 \oplus \dots \oplus U_n$, $A' \cong A$ and $M = A'' \oplus B'' = U_i \oplus U_j$, $A' \cong A''$, $B' \cong B'' \oplus (\oplus_{k \neq i,j} U_k)$.

Consider the natural mappings and an isomorphism $\varphi : A \to A''$:

$$U_i \xrightarrow{\iota_i} M \xrightarrow{\pi_i} U_i, \quad U_j \xrightarrow{\iota_j} M \xrightarrow{\pi_j} U_j$$

$$A'' \xrightarrow{\iota} M \xrightarrow{\pi} A'', \quad B'' \xrightarrow{\lambda} M \xrightarrow{\sigma} B''$$

and denote $f = \pi_i \iota \varphi : A \to U_i$, $g = \pi_j \iota \varphi : A \to U_j$, $h = \varphi^{-1} \pi \iota_i : U_i \to A$, $l = \varphi^{-1} \pi \iota_j : U_j \to A$. We have $hf + lg = \varphi^{-1} \pi \iota_i \pi_i \iota \varphi + \varphi^{-1} \pi \iota_j \pi_j \iota \varphi = 1_A$.

First we show that $hf, lg : A \to A$ are not isomorphisms. Using the symmetry and proving indirectly we may suppose that $\varrho = hf$ is a unit in $E = \operatorname{End} A$. Then the composition map $A \xrightarrow{\pi_i \iota \varphi} U_i \xrightarrow{\varrho^{-1} \varphi^{-1} \pi \iota_i} A$ is the identity map on A which yields that A is isomorphic to a direct summand of U_i. However, the indecomposability of U_i gives $A \cong U_i$, which contradicts the hypothesis. Thus hf and lg are not units in E.

Now E is not local, for otherwise $hf + lg = 1_A \in I$, the maximal ideal of E. Thus, in view of Proposition 1.5 we have either $lg \in I \setminus J$ and $hf \in J \setminus I$, or conversely. With respect to the symmetry we may assume the first possibility. Thus lg is surjective non-injective, hf is injective non-surjective and so f is injective and l is surjective, which yields the first part. Under the respective additional hypotheses, h is injective and g is surjective and we are through. $\square$

1.11 Proposition. *Let the module A satisfy the condition (2M) and U, V be such that $[A]_m = [U]_m$, $A_e = [V]_e$. Then $A \oplus X \cong U \oplus V$ for some module X which is unique up to isomorphism. Moreover, X is either equal to U or to V or it is of the form $\beta^{-1}(\alpha(U))$ where $\alpha : U \to A$ is injective and $\beta : V \to A$ is surjective.*

Proof: The unicity follows at once from Corollary 1.6. Consider the monomorphisms $\alpha : U \to A$, $\gamma : A \to U$ and the epimorphisms $\beta : V \to A$, $\delta : A \to V$. If one of the modules A, U, V is zero, then all three are and so we may assume that they are all non-zero. If $\alpha\gamma$ is an isomorphism, then α is so and $X = V$ works. Similarly, if $\beta\delta$ is an isomorphism, then δ is so and $X = U$ works.

Thus it remains to investigate the case where neither $\alpha\gamma$ nor $\beta\delta$ is an isomorphism. Then $\alpha\gamma \in J \setminus I$, $\beta\delta \in I \setminus J$ (in the notation of Definition 1.2) and so $\varrho = \alpha\gamma+\beta\delta$ is a unit in $E = \operatorname{End} A$ by Proposition 1.5 (c). Denoting $\lambda : A \to U\oplus V$ the diagonal map induced by γ and δ, $\lambda(a) = \gamma(a) + \delta(a)$, and $\mu : U \oplus V \to A$ the codiagonal map induced by α and β, $\mu(u + v) = \alpha(u) + \beta(v)$, one can easily verify that the composition map $S \xrightarrow{\lambda} U \oplus V \xrightarrow{\varrho^{-1}\mu} A$ is the identity on A and consequently A is isomorphic to a proper direct summand of $U \oplus V$, A being indecomposable by Proposition 1.5 (b). Thus $U \oplus V \cong A \oplus \operatorname{Ker}\mu$ showing the first part. Now $\operatorname{Ker}\mu = \{u + v \in U \oplus V \mid \alpha(u) + \beta(v) = 0\} \cong \beta^{-1}(\alpha(U))$ which finishes the proof. □

1.12 Lemma. *Let U_1, U_2, V_1, V_2 be non-zero modules satisfying the condition (2M) and such that the set $\{U_1, U_2, V_1, V_2\}$ has the properties (CI) and (CS). If $U_1 \oplus U_2 \cong V_1 \oplus V_2$, then $\{[U_1]_m, [U_2]_m\} = \{[V_1]_m, [V_2]_m\}$ and $\{[U_1]_e, [U_2]_e\} = \{[V_1]_e, [V_2]_e\}$.*

Proof: Under suitable enumeration of modules we may assume first that $U_1 \cong V_1$. Then $U_1 \oplus U_2 \cong V_1 \oplus V_2 \cong U_1 \oplus V_2$, hence $U_2 \cong V_2$ by Corollary 1.6 and Proposition 1.9 applies.

Assume now that no U_i is isomorphic to any V_j. By Proposition 1.10 we have $[U_1]_m = [V_1]_m$, $[U_1]_e = [V_2]_e$ (for $A = U_1$ under a suitable enumeration of V_1, V_2). Using Proposition 1.10 for V_1, and V_2 we see that $[V_1]_e = [U_2]_e$ and $[V_2]_m = [U_2]_m$ and we are through. □

1.13 Definition. We say that the class $\mathfrak{M}$ of modules satisfies the condition (DSP) if for $A \oplus X = U \oplus V$ with A, U, V in $\mathfrak{M}$ the module X lies in $\mathfrak{M}$, too.

Following [3] we define the m-e collection of a finite family of module to be the collection of monogeny classes of its terms, each monogeny class being counted as often as it occurs, together with the collection of epigeny classes of the terms, again counting multiplicity.

Recall that a class $\mathfrak{M}$ of modules is called *abstract* if it is closed under isomorphisms and it is called *hereditary* if it is abstract and closed under submodules. For the sake of brevity we shall say that a class $\mathfrak{M}$ of modules satisfies the *weak Krull-Schmidt theorem* if, whenever $U_1, \dots, U_n, V_1, \dots, V_t$ are non-zero modules from $\mathfrak{M}$, then $U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots \oplus V_t$ if and only if $n = t$ and the families $\{U_1, \dots, U_n\}$ and $\{V_1, \dots, V_n\}$ have the same m-e collections, i.e. there are two permutations σ, τ of the set $\{1, \dots, n\}$ such that $[U_{\sigma(i)}]_m = [V_i]_m$ and $[U_{\tau(i)}]_e = [V_i]_e$ for every $i = 1, \dots, n$.

1.14 Theorem. *If $\mathfrak{M}$ is an abstract class of modules satisfying the conditions (DSP), having the properties (CI) and (CS) and such that each member of $\mathfrak{M}$ satisfies the condition (2M), then $\mathfrak{M}$ satisfies the weak Krull-Schmidt theorem.*

Proof: Assume first that $U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots \oplus V_t$ and continue by the induction on n, the case $n = 1$ being trivial by Propositions 1.5 (b) and 1.9. If $V_1 \cong U_1$ (under suitable enumerations of modules $U_1, \dots U_n, V_1, \dots, V_t$), then $U_2 \oplus \cdots \oplus U_n \cong V_2 \oplus \cdots \oplus V_t$ by Corollary 1.6 and the induction hypothesis together with Proposition 1.9 works. In the remaining case we have $[V_1]_m = [U_1]_m$, $[V_1]_e = [U_2]_e$ by Proposition 1.10, hence $V_1 \oplus X \cong U_1 \oplus U_2$ by Proposition 1.11, where $X \in \mathfrak{M}$ by the condition (DSP). Thus $V_1 \oplus X \oplus U_3 \oplus \cdots \oplus U_n \cong U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots \oplus V_t$, so $X \oplus U_3 \oplus \cdots \oplus U_n \cong V_2 \oplus \cdots \oplus V_t$ by Corollary 1.6 and $n = t$ by the induction hypothesis. Moreover, $\{X, U_3, \dots, U_n\}$ and $\{V_2, \dots, U_n\}$ have the same m-e collections and the same holds for the sets $\{V_1, X, U_3, \dots, U_n\}$ and $\{V_1, \dots, V_n\}$ by Proposition 1.9. Further, $\{V_1, X\}$ and $\{U_1, U_2\}$ have the same m-e collections by Lemma 1.12, the same is obviously true for $\{V_1, X, U_3, \dots, U_n\}$ and $\{U_1, \dots, U_n\}$ and the assertion follows easily.

Assume now that $\{U_1, \dots, U_n\}$ and $\{V_1, \dots, V_n\}$ have the same m-e collections. Again, we shall use the induction on n, the case $n = 1$ being trivial by Proposition 1.9. With respect to Proposition 1.10 we may assume that $[U_1]_m = [V_1]_m$ and $[U_1]_e = [V_i]_e$. For $i = 1$ Proposition 1.9 yields $U_1 \cong V_1$ and since $\{U_2, \dots, U_n\}$ and $\{V_2, \dots, V_n\}$ have the same m-e collections the induction hypothesis gives the desired result. Assuming now that $i = 2$, Proposition 1.11 gives $U_1 \oplus X \cong V_1 \oplus V_2$ for some $X \in \mathfrak{M}$ (by (DSP)). It follows from the first part of the proof that $\{U_1, X, V_3, \dots, V_n\}$ and $\{V_1, \dots, V_n\}$ have the same m-e collections as well as $\{U_1, \dots, U_n\}$ has by the hypothesis. Clearly, $\{X, V_3, \dots, V_n\}$ and $\{U_2, \dots, U_n\}$ have the same m-e collections, hence $X \oplus V_3 \oplus \cdots \oplus V_n \cong U_2 \oplus \cdots \oplus U_n$ by the induction hypothesis. Thus $U_1 \oplus U_2 \oplus \cdots \oplus U_n \cong U_1 \oplus X \oplus V_3 \oplus \cdots \oplus V_n \cong V_1 \oplus \cdots \oplus V_n$ and we are through. □

1.15 Corollary. *If $\mathfrak{M}$ is a hereditary class of modules having the properties (CI) and (CS) and such that each member of $\mathfrak{M}$ satisfies the condition (2M), then $\mathfrak{M}$ satisfies the weak Krull-Schmidt theorem.*

Proof: The class $\mathfrak{M}$, being hereditary, satisfies the condition (DSP) by Proposition 1.11 and Theorem 1.14 applies.

□

2. The relative case

Let σ be a hereditary torsion theory for the category Mod-R of right R-modules and $\mathcal{L}$ the Gabriel filter associated to σ. See [1], [4] and [6] for basic results on torsion theories.

Let M be a module and N be a submodule of M. The submodule

$$\mathrm{Cl}^M(N) = \{x \in M \mid (N : x) \in \mathcal{L}\}$$

of M is called *the σ-closure of N in M*. We say that N is *σ-closed in M* if $\mathrm{Cl}^M(N) = N$ and that N is *σ-dense in M* if $\mathrm{Cl}^M(N) = M$.

It is easy to see that a σ-torsionfree module M is uniform if and only if the intersection of any two non-zero σ-closed submodules of M is non-zero. However, the dual of this fact does not hold in general. To see it we shall consider the ordinary torsion theory σ on the category of abelian groups. Since every non-zero subgroup of Z is σ-dense in Z, the only σ-closed submodules of Z are 0 and Z and so the sum of any two σ-closed proper submodules of Z is a proper submodule of Z. On the other hand, the sum of proper submodules $2Z + 3Z = Z$ is not proper. Thus we are led to the following definition.

2.1 Definition. A σ-torsionfree module M is said to be *σ-co-uniform* (*σ-hollow*) if the σ-closure of the sum of any two σ-closed proper submodules is a proper submodule of M. Further, we say that a module M is *σ-cuniform* provided it is both uniform and σ-co-uniform.

2.2 Definition. We say that a σ-torsionfree module M satisfies the condition (I_e) if for all submodules A, B, K of M, with K σ-closed in M and A σ-dense in B every homomorphism $f : A \to M/K$ extends to $g : B \to M/K$.

2.3 Lemma. *If a σ-torsionfree module M satisfies the condition (I_e) and $\alpha : M \to M$ is any homomorphism, then $\alpha(M)$ is σ-closed in M.*

Proof: Let us suppose that $\alpha(M)$ is not σ-closed in M and denote $V = \mathrm{Cl}^M(\alpha(M))$. Taking $v \in V \setminus \alpha(M)$ we have $vR \cap \alpha(M) \neq 0$, for otherwise $vR \cong \frac{vR}{vR\cap\alpha(M)} \cong \frac{\alpha(M)+vR}{\alpha(M)} \le \frac{V}{\alpha(M)}$ is both σ-torsion and σ-torsionfree and hence zero. This also shows that $vR \cap \alpha(M)$ is σ-dense in vR. Let $K = \operatorname{Ker}\alpha$ and $\tilde{\alpha} : M/K \to \alpha(M)$ be the induced isomorphism. Consider the diagram

$$\begin{tikzcd} vR\cap\alpha(M) \arrow[r] \arrow[d, "\tilde{\alpha}^{-1}"'] & vR \arrow[d, "g"] \\ M/K \arrow[r, equal] & M/K \end{tikzcd}$$

where the existence of g is given by the condition (I_e). Setting $g(v) = y + K$ we can define the homomorphism $\psi : vR + \alpha(M) \to M/K$ via $\psi(vr + \alpha(u)) = yr + u + K$. Show first that ψ is well-defined. For $vr + \alpha(u) = v\tilde{r} + \alpha(\tilde{u})$ we have $v(r - \tilde{r}) = \alpha(\tilde{u} - u) \in vR \cap \alpha(M)$ and so $y(r - \tilde{r}) + K = \tilde{u} - u + K$, i.e. $yr + u + K = y\tilde{r} + \tilde{u} + K$, as desired.

Now we are going to show that ψ is injective. If not, we take $0 \neq x = vr+\alpha(u) \in \operatorname{Ker}\psi$. Since $\alpha(M) \cap vR$ is σ-dense in vR, M is σ-torsionfree and $vr \notin \alpha(M)$ (otherwise $vr = \alpha(u')$, $0 = \psi(x) = \psi(\alpha(u + u')) = u + u'$ and $x = \alpha(u + u') = 0$), there is $s \in (\alpha(M) : vr)\setminus(0 : x)$. Hence $0 \neq xs = vrs+\alpha(u)s \in \alpha(M)\cap\operatorname{Ker}\psi$ and so $\psi(xs) = 0 = u'$, where $\alpha(u') = vrs+\alpha(u)s$. This yields $xs = 0$, a contradiction showing that $\operatorname{Ker}\psi = 0$.

Finally, $y \in M$ and $v \notin \alpha(M)$ implies that $v - \alpha(y) \neq 0$, while $\psi(v - \alpha(y)) = y - y = 0$, which contradicts the monicity of ψ.

This finishes the proof of Lemma 2.3. □

2.4 Lemma. *Let $A \neq 0$ be a σ-torsionfree module satisfying the condition (I_e) and $E = \mathrm{End}_R(A)$. Then*

(i) *if A is uniform, then $I = \{\alpha \in E \mid \alpha \text{ is not injective}\}$ is a right ideal of E;*
(ii) *if A is σ-co-uniform, then the set $J = \{\alpha \in E \mid \alpha \text{ is not surjective}\}$ is a left ideal of E.*

PROOF: (i) For $\alpha, \beta \in I$ we have $\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta \subseteq \mathrm{Ker}(\alpha - \beta)$ showing that $\alpha - \beta \in I$. Further, for $\beta \in I$ and $\alpha \in E$ we have $\beta\alpha \in I$, for otherwise $\beta\alpha$ injective yields α injective, and $\mathrm{Im}\,\alpha \cap \mathrm{Ker}\,\beta = 0$ gives β injective, $\mathrm{Im}\,\alpha$ being σ-closed in A by the preceding lemma.

(ii) For $\alpha, \beta \in J$ we have $\mathrm{Im}(\alpha - \beta) \subseteq \mathrm{Im}\,\alpha + \mathrm{Im}\,\beta$ and so $\alpha - \beta \in J$. Further, for $\alpha \in J$ and $\beta \in E$ assume $\beta\alpha$ surjective. Then, taking $a \in A$ arbitrarily, $\beta(a) = \beta\alpha(a')$ for some $a' \in A$ and consequently $A = \alpha(A) + \mathrm{Ker}\,\beta$, $\mathrm{Ker}\,\beta \neq A$, β being surjective. By Lemma 2.3 the submodule $\alpha(A)$ is σ-closed in A and so $\alpha(A) = A$, which contradicts $\alpha \in J$. □

2.5 Lemma. *Let U, V be non-zero σ-torsionfree modules satisfying the condition (I_e). If $[U]_m = [V]_m$ and $\alpha : U \to V$ is a monomorphism, then the image $\alpha(U)$ is σ-closed in V.*

PROOF: By hypothesis there is a monomorphism $\beta : V \to U$. If v is an arbitrary element of $\mathrm{Cl}^V(\alpha(U))$, then $vI \subseteq \alpha(U)$ for some $I \in \mathcal{L}$ and so $(\beta(v))I = \beta(vI) \subseteq \beta\alpha(U)$. However, $\beta\alpha(U)$ is σ-closed in U by Lemma 2.3, hence $\beta(v) \in \beta\alpha(U)$ and consequently $v \in \alpha(U)$, β being injective. □

2.6 Lemma. *Let A, B be non-zero σ-torsionfree modules satisfying the condition (I_e). Then*

(i) *if B is uniform, then B has the property $(A\text{-}CI)$;*
(ii) *if B is σ-co-uniform, then B has the property $(A\text{-}CS)$.*

PROOF: Assume that $\alpha : A \to B$, $\beta : B \to A$ are such that the composition $\beta\alpha$ is injective and show that β is injective, too. So, the injectivity of $\beta\alpha$ yields $\mathrm{Im}\,\alpha \cap \mathrm{Ker}\,\beta = 0$. So $\mathrm{Ker}\,\beta = 0$ since $\alpha(A) \neq 0$, α being injective.

(ii) In the notation of the preceding part assume $\beta\alpha$ surjective and show that α is. As in the proof of Lemma 2.4 we have $B = \alpha(A) + \mathrm{Ker}\,\beta$ and $\mathrm{Ker}\,\beta \subseteq B$ is a proper σ-closed submodule, β being surjective. Thus it remains to show that $\alpha(A)$ is a σ-closed submodule of B. However, $\beta(B) = A$, and so $\alpha(A) = \alpha\beta(B)$ is σ-closed in B by Lemma 2.3. □

2.7 Lemma. *If V is a σ-closed submodule of a σ-torsionfree module M satisfying the condition (I_e), then V satisfies the condition (I_e), too.*

PROOF: Let A, B, K be submodules of V such that K is σ-closed in V and A is σ-dense in B. If $f : A \to V/K$ is an arbitrary homomorphism, then there is $g : B \to M/K$ extending f and it remains to show that $g(B) \subseteq V/K$. However, for $b \in B$ we have $bI \subseteq A$ for suitable $I \in \mathcal{L}$ and consequently $g(b)I = g(bI) = f(bI) \subseteq V/K$, which yields $g(b) \in V/K$, V/K being σ-closed in M/K. □

2.8 Theorem. *The weak Krull-Schmidt theorem holds for any class $\mathfrak{M}$ of σ-cuniform modules satisfying the condition (I_e).*

Proof: With respect to Theorem 1.9, Lemma 1.7 and Proposition 1.8 it suffices to verify that the class $\mathfrak{M}$ satisfies the condition (DSP). So, let $A \oplus X = U \oplus V$ with $A, U, V \in \mathfrak{M}$. Applying the ideas of Goldie's dimension to σ-closed submodules, we obtain easily that X should be of the dimension 1, i.e. uniform. Similarly, using the dual Goldie dimension we get that X is σ-co-uniform. It remains to verify that X satisfies the condition (I_e). By Proposition 1.11 we know that either $X = V$, or $X = U$, or X is isomorphic to a submodule $W = \beta^{-1}(\alpha(U))$ of V, where $\alpha : U \to A$ is injective and $\beta : V \to A$ is surjective. Moreover, by Lemma 2.6 the image $\alpha(U)$ is a σ-closed submodule of A. Taking $w \in \mathrm{Cl}^V W$, we have $wI \subseteq W$ for some $I \in \mathfrak{L}$. Hence $(\beta(w))I \subseteq \alpha(U)$, which yields $\beta(w) \in \alpha(U)$ and consequently $w \in W$ showing that W is σ-closed in V. An application of Lemma 2.7 completes the proof. □

2.9 Definition. We say that a σ-torsionfree module M is *σ-uniserial* if its σ-closed submodules form a chain under the inclusion.

2.10 Proposition. *The following are equivalent for a σ-torsionfree module M:*

(i) *M is σ-uniserial;*
(ii) *every σ-closed submodule of M is σ-co-uniform;*
(iii) *every σ-torsionfree factor-module of M is uniform.*

Proof: Since every σ-closed submodule and every σ-torsionfree homomorphic image of a σ-uniserial module is σ-uniserial, the condition (i) implies both (ii) and (iii). Conversely, if M is not σ-uniserial and K, L are two σ-closed submodules of M incomparable in the inclusion, then $\mathrm{Cl}^M(K + L)$ is not σ-co-uniform and $M/(K \cap L)$ is not uniform. □

2.11 Corollary. *The weak Krull-Schmidt theorem holds for the class of σ-uniserial modules satisfying the condition (I_e).*

Proof: By Proposition 2.10 and Theorem 2.8. □

2.12 Example. Let σ be the torsion theory on the category Ab of abelian groups such that the torsion class consists of all torsion groups with zero p-primary component, p a prime. The group $M = Z_p$ of all rationals with denominators prime to p is σ-torsionfree and the σ-closed submodules $p^k M$ form the chain under inclusion. Since σ-density of A in B means that B/A is torsion and $(B/A)_p = 0$, the module M satisfies the condition (I_e). We conclude that the weak Krull-Schmidt theorem holds for the σ-uniserial module M.

□

3. The absolute case

If $\sigma = 0$ is the trivial torsion theory for Mod-R, then we shall call the σ-cuniform modules simply cuniform. The next two consequences of Theorem 2.8

have been discovered independently by N.V. Dung and D. Herbera at the end of 1996 (the word biuniform is used instead of cuniform). This fact has been communicated to the author by Alberto Facchini together with the fact that these results have been already included in his Lecture Notes [8] "Module Theory. Endomorphism rings and direct decompositions in some classes of modules", Progress in Mathematics, Birkhäuser Verlag, which will appear in Summer 1998.

3.1 Theorem. *The weak Krull-Schmidt theorem holds for the class of cuniform modules.*

In view of Proposition 2.10 above as a special case we obtain Theorem 1.9 of [3].

3.2 Corollary. *The weak Krull-Schmidt theorem holds for the class of uniserial modules.*

The following example showing that the class of uniserial modules is a proper subclass of the class of all cuniform module has been communicated to the author by G. Baccella.

3.3 Example. Let F be a field and consider the ring

$$R = \begin{pmatrix} F & F & F & F \\ 0 & F & 0 & F \\ 0 & 0 & F & F \\ 0 & 0 & 0 & F \end{pmatrix}$$

and the idempotent

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

The right R-module $M = eR$ can be obviously identified with $F^4 = (F, F, F, F)$ (as the right R-module). Now $K = (0, 0, 0, F)$ is the socle of M, $A = (0, 0, F, F)$ and $B = (0, F, 0, F)$ are incomparable submodules of M containing K and $L = A + B = (0, F, F, F)$ is the maximal submodule of M. Hence M is cuniform but not uniserial. $\square$

Recall that an abstract class $\mathfrak{M}$ is said to be *cohereditary* if it is closed under factor-modules. Example 3.3 shows that the class of all cuniform modules is neither hereditary, nor cohereditary. On the other hand, this class has the properties (CI) and (CS), which were very important in our proofs of the weak Krull-Schmidt theorem. We conclude this remark by showing that all hereditary and cohereditary classes of modules satisfying conditions (CI) and (CS) lay in the class of all uniserial modules.

3.4 Proposition. *The class $\mathfrak{M}$ of all uniserial modules is the largest hereditary and cohereditary class of modules satisfying conditions (CI) and (CS).*

Proof: Let $\mathfrak{M}$ be a hereditary and cohereditary class of modules containing the class of all uniserial modules. If $M \notin \mathfrak{M}$, then M contains two submodules A, B which are incomparable with respect to the inclusion. Then $\frac{A}{A\cap B}$ and $\frac{A+B}{A\cap B} = \frac{A}{A\cap B} \oplus \frac{B}{A\cap B}$ are in $\mathfrak{M}$, the composition $\frac{A}{A\cap B} \xrightarrow{\alpha} \frac{A}{A\cap B} \oplus \frac{B}{A\cap B} \xrightarrow{\beta} \frac{A}{A\cap B}$ of natural injection and projection is the identity map of $\frac{A}{A\cap B}$, but α is not surjective and β is not injective. $\square$

References

[1] Bican L., Kepka T., Němec P., *Rings, Modules and Preradicals*, Marcel Dekker, New York, Longman Scientific Publishing, London, 1982.

[2] Bican L., Torrecillas B., *QTAG torsionfree modules*, Comment. Math. Univ. Carolinae **33** (1994), 1–20.

[3] Facchini A., *Krull-Schmidt fails for serial modules*, Trans. Amer. Math. Soc. **348** (1996), 4561–4575.

[4] Golan J.S., *Torsion Theories*, Pitman Monographs and Surveys in Pure and Appl. Math., Longman Scientific Publishing, London, 1986.

[5] Herbera D., Shamsuddin A., *Modules with semi-local endomorphism rings*, Proc. Amer. Math. Soc. **123** (1995), 3593–3600.

[6] Stenström B., *Rings of Quotients*, Springer, Berlin, 1975.

[7] Varadarajan K., *Dual Goldie dimension*, Comm. Algebra **7** (1979), 565–610.

[8] Facchini A., *Module Theory. Endomorphism rings and direct decompositions in some classes of modules* (*Lecture Notes*), to appear.

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