Decreasing (G) spaces

Ian Stares

Abstract. We consider the class of decreasing (G) spaces introduced by Collins and Roscoe and address the question as to whether it coincides with the class of decreasing (A) spaces. We provide a partial solution to this problem (the answer is yes for homogeneous spaces). We also express decreasing (G) as a monotone normality type condition and explore the preservation of decreasing (G) type properties under closed maps. The corresponding results for decreasing (A) spaces are unknown.

Keywords: decreasing (G), decreasing (A), homogeneous, monotone normality, closed map

Classification: 54E20, 54D70, 54C10

1. Introduction

In [3], Collins and Roscoe introduced their now well-known structuring mechanism. Assume that X is a topological space and for each $x \in X$ there is a family $\mathcal{W}(x) = \{W(n,x) : n \in \omega\}$ of subsets of X containing x. We say that X satisfies (G) if given an open set U containing a point x, there is an open set $V(x,U)$ containing x such that for each $y \in V(x,U)$, $x \in W(m,y) \subseteq U$ for some $m \in \omega$. If each $W(n,x)$ is open (a neighbourhood of x) then we say X satisfies open (neighbourhood) (G), and if $W(n+1,x) \subseteq W(n,x)$ for each n we say X satisfies decreasing (G). We may strengthen (G) by not allowing the natural number m to vary with y, that is, for each open set U which contains x there is an open set $V(x,U)$ and a natural number $m = m(x,U)$ such that $x \in W(m,y) \subseteq U$ for all $y \in V(x,U)$. In this case we say that X satisfies (A). Open, neighbourhood, and decreasing, (A) are all defined as before. The importance of these properties is illustrated by the following theorems. There is also the open question as to whether X satisfying open (G) is equivalent to X possessing a point-countable base.

Theorem 1.1 ([3], [4]). *The following are equivalent for a space X*

(a) *X is metrizable,*
(b) *X satisfies decreasing open (A),*
(c) *X satisfies decreasing open (G),*
(d) *X satisfies decreasing neighbourhood (A).*

Theorem 1.2 ([1], [4]). *The following are equivalent for a space X*

(a) *X is stratifiable,*

(b) *X satisfies decreasing (G) and X has countable pseudo-character,*

(c) *X satisfies decreasing (A) and X has countable pseudo-character.*

So decreasing open (G) coincides with decreasing open (A) but as noted in [3], McAuley's bow-tie space satisfies decreasing neighbourhood (G) but is not metrizable and therefore does not satisfy decreasing neighbourhood (A). In the light of these comments, it seems natural to ask whether all decreasing (G) spaces satisfy decreasing (A). In Section 2 we investigate the structure of decreasing (G) spaces and show that each decreasing (G) space can be broken up into two subspaces each of which is a decreasing (A) space in a strong way. From this we deduce a partial solution to the above question by showing that all homogeneous decreasing (G) spaces are decreasing (A).

In [11] the author showed that decreasing (A) spaces could be characterised as spaces satisfying a strong monotone normality type condition as follows.

Definition 1.3. A space X is said to be *Borges normal* if for each point x and open set U containing x there is an open set $H(x,U)$ and a natural number $n(x,U)$ such that if $H(x,U)\cap H(y,V)\neq\emptyset$ and $n(x,U)\leq n(y,V)$ then $y\in U$. We refer to H and n as BN operators.

Theorem 1.4. *A space X satisfies decreasing (A) if and only if X is Borges normal.*

In Section 3 we prove an analogue of this result for decreasing (G) spaces. Finally in Section 4 we sketch the proofs of some previously unpublished results due to the author and Philip Moody. These results first appeared in the author's D.Phil thesis. Some of them have since been proved by Gao [6]. They concern the preservation under various types of maps of conditions like (G). Since the corresponding results for (A) are not known this may be one way of distinguishing between decreasing (A) and decreasing (G).

Before we proceed we recall the definitions of monotone normality and K_1-spaces. We note that it is known that all monotonically normal spaces are K_1.

Definition 1.5 ([2]). A space X is said to be monotonically normal if for each point x and open set U containing x we can assign an open set $H(x,U)$ containing x such that, if $H(x,U)\cap H(y,V)\neq\emptyset$ then either $y\in U$ or $x\in V$.

Definition 1.6. A space X is a K_1-space if for each subspace F of X there is a function $k:\tau F\to\tau X$, known as a K_1-function, such that $k(U)\cap F=U$ for all U open in F and if $U\cap V=\emptyset$ then $k(U)\cap k(V)=\emptyset$. (Note for a space Y, τY denotes the topology on Y.)

2. Decreasing (G) = decreasing (A)?

For a space A, $A^{(n)}$ will denote the nth derived set of A (see [5, 1.3.C] for further details).

Lemma 2.1. *If X satisfies decreasing (G), then $X = S \cup \bigcup_{m\in\omega} L_m$ where S is a stratifiable subspace and each L_m is closed in X, $L_m^{(m)} = \emptyset$ for all m and $L_m \subseteq L_n$ whenever $m \leq n$.*

Proof: The proof uses Balogh's construction (see Lemma 2.1 [1]) to create the subspaces S and L_m ($m \in \omega$). Assume X satisfies decreasing (G). For every $m \in \omega$, let

$$M_m = \left\{\langle x, y\rangle \in X^2 : W(m,x) \cap W(m,y) = \emptyset\right\}.$$

Now, define $L_m = \{x \in X : \langle x, x\rangle \in \overline{M_m}\}$. Balogh has shown that, for each $m \in \omega$, $L_m^{(m)} = \emptyset$ and that $S = X \setminus (\bigcup_{m\in\omega} L_m)$ is a stratifiable subspace of X. It remains to show that each L_m is closed (the nesting is obvious).

If $x \notin L_m$, then $\langle x, x\rangle \notin \overline{M_m}$ so there exist open U and V in X such that $\langle x, x\rangle \in U \times V \subseteq X^2 \setminus \overline{M_m}$. It is obvious that $x \in U \cap V \subseteq X \setminus L_m$. □

Lemma 2.2. *If X is monotonically normal and $X^{(m)} = \emptyset$ for some $m \in \mathbb{N}$, then X is Borges normal and, without loss of generality, the BN operator n satisfies $n(x, U) \leq m$ for all points x and open sets U.*

Proof: Since $X^{(m)} = \emptyset$, we have that $X = Y_0 \cup Y_1 \cup \ldots \cup Y_{m-1}$, where $Y_i = X^{(i)} \setminus X^{(i+1)}$ for $i = 0, 1, 2, \ldots, m-1$, each Y_i is relatively discrete, and the union is disjoint. Note, for all $j \leq m-1$, $\bigcup_{i\leq j} Y_i$ is open in X. We shall define BN operators H and n on X. Let κ_j denote a K_1-function from τY_j to τX and let M be a monotone normality operator for X. For open U in X containing x, define,

$$\begin{aligned} H(x, U) &= M(x, U \cap (\cup_{i\leq j} Y_i)) \cap \kappa_j(\{x\}) &&\text{for } x \in Y_j \\ n(x, U) &= m - j &&\text{for } x \in Y_j. \end{aligned}$$

Now assume that $H(x, U) \cap H(y, V) \neq \emptyset$ and $n(x, U) \leq n(y, V)$. If $n(x, U) = n(y, V) = m - j$, then $\kappa_j(\{x\}) \cap \kappa_j(\{y\}) \neq \emptyset$ and, therefore $x = y$. If $n(x, U) < n(y, V)$, then $x \in Y_k$ and $y \in Y_j$ with $j < k$ and either, $x \in V \cap (\cup_{i\leq j} Y_i)$ or $y \in U \cap (\cup_{i\leq k} Y_i)$ (since M is a monotone normality operator). But, by assumption, $x \notin \cup_{i\leq j} Y_i$. Consequently, $y \in U$, as required. □

Theorem 2.3. *If X satisfies decreasing (G), then X is the disjoint union of S and L, where S is a stratifiable subspace and L is an F_σ subset of X satisfying decreasing (A), such that $L = \bigcup_{m\in\omega} L_m$ and each L_m is closed in X, $L_m^{(m)} = \emptyset$ for all m and $L_m \subseteq L_n$ whenever $m \leq n$.*

Proof: To prove this result we only need to prove that the subspace $L = \bigcup_{m\in\omega} L_m$ constructed in Lemma 2.1 satisfies decreasing (A). We shall show that L is Borges normal.

In what follows, Δ_n shall denote the nth triangle number ($\Delta_1 = 1$, $\Delta_2 = 3$, $\Delta_3 = 6$ etc.). We note that, in the proof of Lemma 2.2 it was not important which m natural numbers the BN operator for X ranged over, only that there were m

of them. Thus we can construct, for each $m \in \omega$, BN operators H_m and n_m for L_m such that, for all U open in L_m and $x \in U$, $\Delta_{m-1} < n_m(x,U) \le \Delta_m$. As a subspace of a decreasing (G) space L is at least monotonically normal. Assume M is a monotone normality operator on L and, for $m \in \omega$, let κ_m denote a K_1-function from τL_m to τL.

Since the L_m form an increasing chain, for each $x \in L$, let $i(x)$ denote the smallest natural number such that $x \in L_{i(x)}$. For x with $i(x) > 1$, choose an open set S_x in L such that $x \in S_x \subseteq L \setminus \bigcup_{m<i(x)} L_m$. We now define BN operators H and n for L as follows: if U is open in L and contains x then define,

$$H(x,U) = \kappa_{i(x)}\left(H_{i(x)}(x, U \cap L_{i(x)})\right) \cap M(x, U \cap S_x)$$
$$n(x,U) = n_{i(x)}(x, U \cap L_{i(x)}).$$

Suppose that $H(x,U) \cap H(y,V) \neq \emptyset$ and $n(x,U) \le n(y,V)$.

Case 1. $\Delta_{j-1} < n(x,U) \le n(y,V) \le \Delta_j$ for some j. By construction of the n_m, this implies $i(x) = i(y) = j$. We therefore have that $\kappa_j(H_j(x, U \cap L_j)) \cap \kappa_j(H_j(y, V \cap L_j)) \neq \emptyset$ and $n_j(x, U \cap L_j) \le n_j(y, V \cap L_j)$. Since κ_j is a K_1-function and H_j and n_j are BN operators, this implies that $y \in U$ as required.

Case 2. $\Delta_{j-1} < n(x,U) \le \Delta_j \le \Delta_{k-1} < n(y,V) \le \Delta_k$ for some j and k. Consequently, $i(x) = j$ and $i(y) = k$ with $j < k$. By assumption, $M(x, U \cap S_x) \cap M(y, V \cap S_y) \neq \emptyset$ and hence, $x \in V \cap S_y$ or $y \in U \cap S_x$. Since $S_y \subseteq L \setminus \bigcup_{m<i(y)} L_m$ and $i(x) < i(y)$, we have that $x \notin S_y$ and therefore, $y \in U$ as required. $\square$

Corollary 2.4. [1] *If X is an homogeneous space (in particular a topological group) and X satisfies decreasing (G), then either X is stratifiable or X satisfies decreasing A and $X = \bigcup_{n\in\omega} L_n$, where $L_n^{(n)} = \emptyset$, L_n is closed in X for each n and $L_m \subseteq L_n$ whenever $m \le n$.*

Proof: Applying the previous Theorem, if $S = \emptyset$ then we are done. Otherwise, choose $x \in S$. Since S is stratifiable, then $\{x\}$ is a G_δ-set in S which is a G_δ-set in X. Consequently, $\{x\}$ is a G_δ-set in X. Since X is homogeneous, this implies that all singletons in X are G_δ-sets in X and, therefore X has countable pseudo-character. By Theorem 1.2, X is stratifiable.

$\square$

Corollary 2.5. *Decreasing (G), homogeneous spaces satisfy decreasing (A).*

3. Decreasing (G) as a normality condition

If we compare Definition 1.3 with Definition 1.5, then we see that Borges normality is just a strengthening of monotone normality in which the natural numbers $n(x,U)$ are used to decide exactly which one of $x \in V$ or $y \in U$ holds when $H(x,U) \cap H(y,V) \neq \emptyset$. It is natural to ask whether a similar characterisation

[1]The author would like to thank Paul Gartside for suggesting this proof.

of decreasing (G) exists. We have the following theorem which tells us that we have a similar situation for decreasing (G) spaces, only now it also matters *which* points lie in the intersection $H(x,U) \cap H(y,V)$.

Theorem 3.1. *A space X satisfies decreasing (G) if and only if to each point $x \in X$ and open set U containing x we can assign an open set $H(x,U)$ containing x and for each point $a \in H(x,U)$ we can assign a natural number $n(a,x,U)$ such that if $a \in H(x,U) \cap H(y,V)$ and $n(a,x,U) \leq n(a,y,V)$ then $y \in U$.*

PROOF: If X satisfies decreasing (G), define $H(x,U) = V(x,U)$ and if $a \in H(x,U)$ then by (G) there is some $m \in \omega$ such that $x \in W(m,a) \subseteq U$. Let $n(a,x,U) = m$. It is straightforward to check that the monotone normality type condition is satisfied.

Conversely, without loss of generality $n(a,x,U) \geq 1$ for all a, x and U. For $a \neq x$ let $N(a,x) = \{n(a,x,U) : a \in H(x,U) \text{ for some open } U\}$. By assumption, this set of natural numbers is bounded above by $n(a,a,X \setminus \{x\})$. Let $n_a(x) = \max N(a,x)$ (or 0 if $N(a,x)$ is empty). Now define

$$W(n,a) = \{a\} \cup \{y : n_a(y) \geq n\}.$$

Clearly $a \in W(n,a)$ and $W(n+1,x) \subseteq W(n,x)$ for each $n \in \omega$. It remains to check the condition (G). To this end we first claim that if $a \in H(x,U) \setminus \{x\}$ then $x \in W(n_a(x),a) \subseteq U$.

It is obvious that $x \in W(n_a(x),a)$ by definition. So, assume that $y \in W(n_a(x),a)$. Then either, $y = a \in U$ or $y \neq a$ and $n_a(y) \geq n_a(x) \geq 1$. Choose open V such that $n_a(y) = n(a,y,V)$ and $a \in H(y,V)$. However we also have that $a \in H(x,U)$ and $n(a,y,V) \geq n_a(x) \geq n(a,x,U)$. Thus $y \in U$.

To complete our proof we need to check the case when $a = x$. We claim that $x \in W(n(x,x,U),x) \subseteq U$. So assume that $y \in W(n(x,x,U),x)$. Then either $y = x \in U$ or $n_x(y) \geq n(x,x,U)$. So for some open V, $x \in H(y,V)$ and $n_x(y) = n(x,y,V)$. But $x \in H(x,U)$ and consequently, $y \in U$. □

4. Preservation under mappings

In this section we investigate the stability of (G) type conditions under closed maps. As mentioned above, the work in this section is previously unpublished material which first appeared in the author's thesis (Oxford 1994). Later Gao [6] independently proved Theorem 4.1 (a)–(d). We first recall a generalisation of (G).

Assume that X is a topological space and for each $x \in X$ there is a family $\mathcal{W}(x)$ of subsets of X containing x. We say that X satisfies (F) if given an open set U containing a point x, there is an open set $V(x,U)$ containing x such that for each $y \in V(x,U)$, $x \in W \subseteq U$ for some $W \in \mathcal{W}(y)$. If each $\mathcal{W}(x)$ consists of open sets (neighbourhoods of x) then we say X satisfies open (neighbourhood) (F), and if each $\mathcal{W}(x)$ is well-ordered (a chain) under reverse inclusion then we say X satisfies well-ordered (chain) (F).

Some results are already known regarding closed images of spaces with these conditions. For example, it is known that the closed image of a space satisfying chain (F) satisfies chain (F) ([10]) and the closed image of a space satisfying well-ordered (F) satisfies well-ordered (F) ([7]). However, the proofs of these results follow by first reformulating the conditions in terms of monotone normality type conditions (acyclic monotone normality and acyclic Noetherian monotone normality respectively) and then appealing to standard proofs for normality type conditions to be preserved. Below, we show that a wide range of conditions are preserved by closed maps. We prove this directly and therefore provide alternative proofs for these results.

We shall not give definitions of some of the less common conditions mentioned below. We merely mention that *uniform* (G) is a weakening of decreasing (G) and Moody has shown ([9]) that a space X is metacompact and developable if and only if X satisfies uniform, open (G). Also *Noetherian of sub-infinite rank* (F) is a weakening of Noetherian bases (which were introduced by Lindgren and Nyikos [8]). In [7], Gartside and Moody showed that a more general theory holds for spaces satisfying Noetherian of sub-infinite rank (F). Since we show that Noetherian of sub-infinite rank (F) is preserved by closed mappings, this provides another reason why these spaces may be viewed as more satisfactory than those possessing Noetherian bases. Noetherian bases are not preserved by closed maps ([7, Example 13]).

If $f : X \to Y$ is a closed map and $A \subseteq X$, then $f^*(A)$ is defined by: $f^*(A) = \{y \in Y : f^{-1}(y) \subseteq A\}$. If f is closed, then $f^*(U)$ is open in Y for all U open in X.

Theorem 4.1 (Moody and Stares). *If X satisfies $\mathcal{P}$, where $\mathcal{P}$ is any of the following conditions, and $f : X \to Y$ is a continuous, onto, closed map then Y satisfies $\mathcal{P}$ also:* (a) *chain* (*F*)*;* (b) *well-ordered* (*F*)*;* (c) *decreasing* (*G*)*;* (d) *uniform* (*G*)*;* (e) *Noetherian of sub-infinite rank* (*F*).

Proof: Assume X satisfies $\mathcal{P}$ with corresponding families $\mathcal{W}_X(x)$ for each $x \in X$ and operator V_X. For each $y \in Y$ pick $x(y) \in f^{-1}(y)$. Let

$$\mathcal{W}_Y(y) = \{f(W) : W \in \mathcal{W}_X(x(y))\}$$

$$V_Y(y,U) = f^*\left(\bigcup\{V_X(x, f^{-1}(U)) : x \in f^{-1}(y)\}\right).$$

Clearly $V_Y(y,U)$ is an open set in Y containing y and if $z \in V_Y(y,U)$, then $x(z) \in V_X(x, f^{-1}(U))$ for some $x \in f^{-1}(y)$. Hence there exists $W \in \mathcal{W}_X(x(z))$ such that $x \in W \subseteq f^{-1}(U)$ and therefore $y = f(x) \in f(W) \subseteq U$ and so, since $f(W) \in \mathcal{W}_Y(z)$, Y satisfies (F). Parts (a) and (b) are now clear since the ordering on each $\mathcal{W}_X(x)$ goes over. Similarly for part (c) where $W_Y(n,y) = f(W_X(n, x(y)))$. The proof of (d) is straightforward given the definition of uniform (G). The proof of (e) follows by applying Lemma 9 in [7]. The details are omitted. $\square$

Corollary 4.2 (Moody and Stares)**.** *If X satisfies $\mathcal{P}$, where $\mathcal{P}$ is any of the following conditions, and $f : X \to Y$ is a continuous, onto, closed and open map then Y satisfies $\mathcal{P}$ also:* (a) *open* (*neighbourhood*) (*F*)*;* (b) *open* (*neighbourhood*) (*G*)*;* (c) *open* (*neighbourhood*) *chain* (*F*)*;* (d) *open* (*neighbourhood*) *well-ordered* (*F*)*;* (e) *open* (*neighbourhood*) *decreasing* (*G*)*;* (f) *uniform open* (*G*)*.*

Proof: Using the construction above, since f is an open map, the families $\mathcal{W}_Y(y)$ consist of open sets (or neighbourhoods of y). □

From the previous two results we immediately have new proofs of known results. The closed and open image of a metric space is metric (by Theorem 1.1 (c)). The closed image of a stratifiable space is stratifiable (by Theorem 1.2 and the easy result that countable pseudo-character is preserved by closed maps). The closed and open image of a metacompact, developable space is metacompact and developable (by Moody's result mentioned above).

We note that the proof of the above theorem does not work if we consider the condition, decreasing (A). We introduce the following condition which lies between decreasing (A) and decreasing (G).

Definition 4.3. If for each $x \in X$, $\mathcal{W}(x) = \{W(n,x) : n \in \omega\}$ is a collection of subsets of X containing x and $W(n+1,x) \subseteq W(n,x)$ for each n, then we say that X satisfies *weak decreasing* (*A*) if given $x \in U$ open in X, there is an open $V = V(x,U)$ containing x and a finite set of natural numbers $S(x,U)$ such that $y \in V$ implies $x \in W(s,y) \subseteq U$ for some $s \in S(x,U)$.

Proposition 4.4 (Moody and Stares)**.** *The perfect image of a weak decreasing* (*A*) *space satisfies weak decreasing* (*A*)*.*

Proof: As before construct the families $\mathcal{W}_Y(y)$ for each $y \in Y$. Let

$$V_Y(y,U) = f^*\left(\bigcup\{V_X(x_i, f^{-1}(U)) : i \le m\}\right)$$

where $x_i \in f^{-1}(y)$ are chosen such that $\{V_X(x_i, f^{-1}(U)) : i \le m\}$ is a finite cover of $f^{-1}(y)$. Let $S_Y(y,U) = \bigcup\{S_X(x_i, f^{-1}(U)) : i \le m\}$.

If $z \in V_Y(y,U)$, then $x(z) \in V_X(x_i, f^{-1}(U))$ for some $i \le m$. Thus $x_i \in W_X(s, x(z)) \subseteq f^{-1}(U)$ for some $s \in S_X(x_i, f^{-1}(U)) \subseteq S_Y(y,U)$. Consequently, $y = f(x_i) \in W_Y(s,z) \subseteq U$. □

The question still remains whether the closed image of a decreasing (A) space has decreasing (A). We only have the following result which shows that simple closed maps preserve decreasing (A).

Proposition 4.5. *If X is a decreasing* (*A*) *space and Y is obtained from X by identifying a closed subspace of X to a point, then Y satisfies decreasing* (*A*)*.*

Proof: Assume X is Borges normal, with operators H_X and n_X, and C is a closed subspace of X. (Without loss of generality $n_X(x,U) \ge 1$ for all x and U.) Let $Y = (X \setminus C) \cup \{p\}$ where $p \notin X$. Define $f : X \to Y$ by, $f(x) = x$ if $x \in X \setminus C$

and $f(x) = p$ if $x \in C$. Topologise Y by defining U to be open in Y if and only if $f^{-1}(U)$ is open in X. We shall show that Y is Borges normal. Define,

$$H(y,U) = \begin{cases} f(H_X(y, f^{-1}(U) \setminus C)) & \text{if } \ y \in X \setminus C \\ f\left(\bigcup_{a \in C} H_X(a, f^{-1}(U))\right) & \text{if } \ y = p \end{cases}$$

$$n(y,U) = \begin{cases} n_X(y, f^{-1}(U) \setminus C) & \text{if } \ y \in X \setminus C \\ 0 & \text{if } \ y = p. \end{cases}$$

Assume $H(y,U) \cap H(w,V) \neq \emptyset$ and $n(y,U) \leq n(w,V)$. We need to show that $w \in U$.

Case 1. $y \neq p$ and $w = p$. This is impossible as $1 \leq n(y,U) \leq n(w,V) = 0$.

Case 2. $y = p$ and $w \neq p$. If $H(y,U) \cap H(w,V) \neq \emptyset$ then for some $a \in C$, $f(H_X(a, f^{-1}(U))) \cap f(H_X(w, f^{-1}(V) \setminus C))$ for some $a \in C$. Thus $H_X(a, f^{-1}(U)) \cap H_X(w, f^{-1}(V) \setminus C) \neq \emptyset$ and hence $w \in f^{-1}(U)$. Consequently, $w \in U$ as required.

Case 3. $y \neq p$ and $w \neq p$. Thus $H_X(y, f^{-1}(U) \setminus C) \cap H_X(w, f^{-1}(V) \setminus C) \neq \emptyset$ and $n_X(y, f^{-1}(U) \setminus C) \leq n_X(w, f^{-1}(V) \setminus C)$. Hence, $w \in f^{-1}(U)$ and $w \in U$. □

The question as to whether decreasing (A) is preserved by closed maps remains open. Perhaps a negative answer may show us that decreasing (A) and decreasing (G) do not coincide. We end with a question.

Question 1. Are decreasing (A)/(G) preserved by domination or adjunction?

References

[1] Balogh Z., *Topological spaces with point-networks*, Proc. Amer. Math. Soc. **94** (1985), 497–501.

[2] Borges C.J.R., *A study of monotonically normal spaces*, Proc. Amer. Math. Soc. **38** (1973), 211–214.

[3] Collins P.J., Roscoe A.W., *Criteria for metrisability*, Proc. Amer. Math. Soc. **90** (1984), 631–640.

[4] Collins P.J., Reed G.M., Roscoe A.W., Rudin M.E., *A lattice of conditions on topological spaces*, Proc. Amer. Math. Soc. **94** (1985), 487–496.

[5] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[6] Gao Y.-Z., *A note concerning the Collins, Reed, Roscoe, Rudin metrization theorem*, Topology Appl. **74** (1996), 73–82.

[7] Gartside P.M., Moody P.J., *Well-ordered (F) spaces*, Topology Proc. **17** (1992), 111–130.

[8] Lindgren W.F., Nyikos P.J., *Spaces with bases satisfying certain order and intersection properties*, Pacific J. Math. **66** (1976), 455–476.

[9] Moody P.J., Reed G.M., Roscoe A.W., Collins P.J., *A lattice of conditions on topological spaces II*, Fund. Math. **138** (1991), 69–81.

[10] Moody P.J., Roscoe A.W., *Acyclic monotone normality*, Topology Appl. **47** (1992), 53–67.
[11] Stares I.S., *Concerning the Dugundji extension property*, Topology Appl. **63** (1995), 165–172.

Mathematical Institute, 24-29 St Giles', Oxford OX1 3LB, United Kingdom
E-mail: isstares@maths.ox.ac.uk

(*Received* November 20, 1997, *revised* March 9, 1998)