Directional moduli of rotundity and smoothness

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Abstract. We study various notions of directional moduli of rotundity and when such moduli of rotundity of power type imply the underlying space is superreflexive. Duality with directional moduli of smoothness and some applications are also discussed.

Keywords: uniform rotundity, uniform smoothness, moduli of power type, superreflexive
Classification: 46B03, 46B20

1. Introduction

James' seminal work along with Enflo's renorming theorem culminated in the result that a Banach space is superreflexive if and only if it admits an equivalent uniformly rotund norm. Pisier later showed that every superreflexive Banach space can be renormed to have an equivalent uniformly rotund norm with a modulus of rotundity of power type. See [6, Chapter IV] for these and other related results. Recently Hájek [10] showed that a Banach space is an Asplund space provided it admits a weakly uniformly rotund norm whereas he showed there are nonreflexive dual spaces that admit dual weakly uniformly rotund norms. This motivated us to consider whether analogs of Pisier's theorem are valid in this or even more general settings. For instance, what can be said about a Banach space with an equivalent weakly uniformly rotund norm whose associated modulus of rotundity satisfies a power type condition?

Actually, the consideration of this type of question is not new. Indeed, on their way to showing some of the nice structural properties possessed by Banach spaces having certain smoothness properties and the Radon-Nikodym property, Deville *et al.* showed that a Banach space is superreflexive provided it admits a locally uniformly rotund norm with a pointwise modulus of rotundity of power type ([7, Proposition 2.1]). In the next section, we provide a systematic survey of what can be said about spaces with various types of rotund norms whose associated modulus of rotundity is of power type, in particular we give a directional improvement of the just cited result from [7] by combining their ideas with a dual version of an argument of Borwein and Noll [1, Proposition 2.2]. Some related results concerning directional moduli of smoothness of power type that were similarly developed by building on techniques from [1], [7] can be found in [11], [12].

Because of a recent work by Borwein *et al.* [2] which provides several applications in the field of optimisation of norms that are smooth with respect to subspaces of directions, the third section considers moduli of smoothness in a fixed set of directions. Many general notions of directional moduli of smoothness and their duality with directional moduli of rotundity were studied over 30 years ago by Cudia [4]. However, they were not studied in the context of power type, because of this and the applicability of smoothness in restricted directions illustrated in [2], we have included directional versions of some standard duality results in this context. We also provide a sufficient condition for producing an equivalent renorm that has a modulus of smoothness of power type with respect to a fixed set of directions, and then outline an application of such norms in fixed point theory.

2. Directional moduli of rotundity

For a Banach space X with dual X^*, let B_X denote the closed unit ball of X, and S_X the unit sphere. In the definitions that follow, D will always denote a bounded set. The *modulus of rotundity in the directions* $D \subset X^*$ is defined by

$$\delta(\varepsilon, D) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_X,\ |f(x-y)| \geq \varepsilon,\ f \in D \right\}.$$

Throughout, we use the convention that the infimum over the empty set is $+\infty$ in case there are no $x, y \in B_X$, $f \in D$ such that $|f(x-y)| \geq \varepsilon$. On occasion, we will write $\delta_{\|\cdot\|}(\varepsilon, D)$ or $\delta_X(\varepsilon, D)$, to emphasise the norm or space under consideration. If $D = B_{X^*}$, or $\sup\{|f(x)| : f \in D\} = \|x\|$ for all $x \in X$, we may suppress D in the modulus notation. Thus, X is *uniformly rotund* (UR) (*weakly uniformly rotund* (WUR)) if for each $\varepsilon > 0$, $\delta(\varepsilon) > 0$ ($\delta(\varepsilon, f) > 0$ for each $f \in S_{X^*}$).

Much of our focus will be on the following weaker notions. The *modulus of rotundity at* $x \in S_X$ *in the directions* $D \subset X^*$ is defined by

$$\delta(\varepsilon, x, D) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : y \in B_X,\ |f(x-y)| \geq \varepsilon, f \in D \right\}.$$

A Banach space X is *locally uniformly rotund* (LUR) if $\delta(\varepsilon, x) > 0$ for each $x \in S_X$ and $\varepsilon > 0$. The *modulus of extremality at* $x \in S_X$ *in the directions* $D \subset X^*$ is defined by

$$\delta_m(\varepsilon, x, D) := \inf \left\{ \left\| x - \frac{y+z}{2} \right\| : y, z \in B_X,\ |f(y-z)| \geq \varepsilon,\ f \in D \right\}.$$

A Banach space X is *mid-point locally uniformly rotund* (MLUR) (*weakly mid-point locally uniformly rotund* (WMLUR)) if for each $\varepsilon > 0$ and $x \in S_X$, $\delta(\varepsilon, x) > 0$ ($\delta(\varepsilon, x, f) > 0$ for each $f \in S_{X^*}$). Smith [15] has given a nice geometrical characterisation of WMLUR norms which has some interesting applications ([13, Theorem 3.3]).

A modulus of rotundity $\delta(\varepsilon, -)$ is said to be of *power type* if there are $K > 0$ and $p \geq 2$ such that $\delta(\varepsilon, -) \geq K\varepsilon^p$ for all $\varepsilon > 0$. Also, a subspace $Y \subset X^*$ is *norming* if there is a $\lambda > 0$ such that $\sup\{f(x) : f \in Y \cap B_{X^*}\} \geq \lambda \|x\|$ for all $x \in X$.

We begin by stating the main result of this section which provides a directional strengthening of [7, Proposition 2.1] by building on techniques from [1], [7].

Theorem 2.1. *Consider a Banach space X and S a residual subset of S_X and suppose that for each $x \in S$ there exists a closed norming subspace $Y_x \subset X^*$ where for each $f \in B_{Y_x}$ there exist $K_{x,f} > 0$ and $p_{x,f} > 0$ such that $\delta_m(\varepsilon, x, f) \geq K_{x,f}\varepsilon^{p_{x,f}}$ for all $\varepsilon > 0$. Then X can be equivalently renormed to be UR where there exist $K > 0$ and $p > 0$ such that $\delta(\varepsilon) \geq K\varepsilon^p$ for all $\varepsilon > 0$. Moreover, if $p_{x,f} \leq p_0$ for all $x \in S$ and $f \in B_{Y_x}$, then $p \leq p_0$.*

In particular, this theorem implies that a Banach space is superreflexive provided that it admits a WMLUR norm with a directional modulus of extremality of power type. The formulation in terms of norming subspaces enables it to apply to the analogous situations for w^*LUR and other types of norms which we have not discussed here. We now proceed with a few preliminary results needed in the proof of Theorem 2.1.

Lemma 2.2. *Consider a Banach space X and Y a closed norming subspace of X^*. Given $x \in S_X$ suppose for each $f \in B_Y$ there exist $K_f > 0$ and $p_f > 0$ such that $\delta_m(\varepsilon, x, f) \geq K_f\varepsilon^{p_f}$ for all $\varepsilon > 0$. Then there exist $K > 0$ and $p > 0$ such that $\delta_m(\varepsilon, x) \geq K\varepsilon^p$ for all $\varepsilon > 0$. Moreover, if $p_f \leq p_0$ for all $f \in B_Y$, then $p \leq p_0$.*

Proof: For each $n \in \mathbb{N}$, let $F_n = \{f \in B_Y : \delta_m(\varepsilon, x, f) \geq \varepsilon^n/n \text{ for } \varepsilon \geq 0\}$ (if $p_f \leq p_0$ for all $f \in B_Y$ use ε^{p_0}/n). Then $\cup_{n=1}^{\infty} F_n \supset \frac{1}{2}B_Y$ and so $\cup_{n=1}^{\infty} F_n$ is of the second category. To see that F_n is closed, suppose $f_k \in F_n$ and $f_k \to f$. For $\varepsilon > 0$ fixed, let $y, z \in B_X$ satisfy $|f(y-z)| \geq \varepsilon$. Then $|f_k(y-z)| \geq \varepsilon_k$ with $\varepsilon_k \to \varepsilon$, and so

$$\left\| x - \frac{y+z}{2} \right\| \geq \lim_{k\to\infty} \frac{1}{n}\varepsilon_k^n = \frac{1}{n}\varepsilon^n.$$

This implies that $f \in F_n$. Thus by Baire category theory, for some $p \in \mathbb{N}$, F_p contains $f_0 + rB_Y$ for some $r > 0$ and some $f_0 \in B_Y$. The definition of $\delta(\varepsilon, x, \cdot)$ implies $-f_0 + rB_Y \subset F_p$ as well. Because Y is norming, we fix $K > 0$ such that $\sup\{f(x) : f \in rB_Y\} \geq 2K\|x\|$. Now for $x \in X$, we choose $f \in rB_Y$ such that $f(x) \geq K\|x\|$. Thus $\max\{(\pm f_0 + f)(x)\} \geq K\|x\|$. Hence if $y, z \in B_X$ and $\|y-z\| \geq \varepsilon$ there is a $g \in F_p$ such that $g(y-z) \geq K\varepsilon$. Therefore, $\|x-(y+z)/2\| \geq \delta_m(K\varepsilon, x, g) \geq \frac{1}{p}(K\varepsilon)^p = \frac{K^p}{p}\varepsilon^p$. Consequently, $\delta_m(\varepsilon, x) \geq \frac{K^p}{p}\varepsilon^p$ for all $\varepsilon > 0$. □

Theorem 2.1 also relies on a useful renorming theorem of Day [5, Theorem 1]. We include a simple proof of a directionalised variant of this result for both completeness and the fact that we will need this directionalised version in the next section.

Lemma 2.3. *Let $S = \{x \in B_X : f(x) \geq 1 - 2\alpha\}$ be a slice of the unit ball B_X where $f \in S_{X^*}$ and $\alpha > 0$. Given a bounded $D \subset X^*$, define*

$$\delta_\varepsilon = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : |g(x-y)| \geq \varepsilon,\ g \in D,\ x, y \in S\right\}.$$

Then there is an equivalent norm ν on X such that $\delta_\nu(\varepsilon, D) \geq \delta_\varepsilon$. In the dual situation, if B_X or S is w^*-closed, then ν is a dual norm.

Proof: Let $x_0 \in$ interior S be such that $f(x_0) \geq 1-\alpha$. Now let $U = (x_0 - B_X) \cap (B_X - x_0)$. Then U is closed, convex, symmetric and has nonempty interior, and so U is the unit ball of an equivalent norm ν on X. We first claim that $U = (x_0 - S) \cap (S - x_0)$. Indeed, if $u \in U$, then $u \in x_0 - B_X$ and so $f(u) \geq -\alpha$. On the other hand, $u \in B_X - x_0$ and so we write $u = b - x_0$ for some $b \in B_X$. Thus $b = x_0 + u$ and so $f(b) \geq 1 - 2\alpha$ which means that $b \in S$. Hence $u \in S - x_0$, which implies that $-u \in x_0 - S$. Consequently, $U = (x_0 - S) \cap (S - x_0)$ as desired.

Now suppose that $x, y \in U$ and $|g(x-y)| \geq \varepsilon$ for some $g \in D$. We write $x = u - x_0 = x_0 - u_1$ and $y = v - x_0 = x_0 - v_1$ where $u, u_1, v, v_1 \in S$. Now $|g(u-v)| = |g(u_1 - v_1)| = |g(x-y)| \geq \varepsilon$, and so

$$\frac{u+v}{2} + \delta_\varepsilon B_X \subset B_X \text{ and } \frac{u_1+v_1}{2} + \delta_\varepsilon B_X \subset B_X.$$

Thus we obtain

$$\begin{aligned}\frac{x+y}{2} + \delta_\varepsilon B_X &= \frac{u+v}{2} + \delta_\varepsilon B_X - x_0 \subset B_X - x_0, \text{ and}\\ \frac{x+y}{2} + \delta_\varepsilon B_X &= x_0 - \left(\frac{u_1+v_1}{2} + \delta_\varepsilon B_X\right) \subset x_0 - B_X.\end{aligned}$$

This shows that $\frac{x+y}{2} + \delta_\varepsilon B_X \subset U$. Thus $\frac{x+y}{2} + \delta_\varepsilon U \subset U$, from which the theorem follows. The statement about dual norms is clear because U is w^*-closed when B_X or S is w^*-closed. □

The following simple fact will be useful.

Lemma 2.4. *Given $x_0 \in S_X$, a bounded $D \subset X^*$ and $0 < r < 1$, suppose that $\delta_m(\varepsilon, x, D) \geq K\varepsilon^p$ for all $x \in B_r(x_0) \cap S_X$ and $\varepsilon > 0$. If $y, z \in B_{\frac{r}{2}}(x_0) \cap B_X$ and $|g(y-z)| \geq \varepsilon$ for $g \in D$, then $1 - \left\|\frac{y+z}{2}\right\| \geq K\varepsilon^p$ for all $\varepsilon > 0$.*

Proof: Suppose $y, z \in B_{\frac{r}{2}}(x_0) \cap B_X$ and $g \in D$ satisfies $|g(y-z)| \geq \varepsilon$. Then $\frac{y+z}{2} \in B_{\frac{r}{2}}(x_0) \cap B_X$. Since $\left\|\frac{y+z}{2}\right\| \geq 1 - r/2$ we have $\frac{y+z}{\|y+z\|} \in B_r(x_0) \cap S_X$. Then

$$1 - \left\|\frac{y+z}{2}\right\| = \left\|\frac{y+z}{\|y+z\|} - \frac{y+z}{2}\right\| \geq K\varepsilon^p.$$

□

Proof of Theorem 2.1: By Lemma 2.2, there exist $K_x > 0$ and $p_x > 0$ so that $\delta_m(\varepsilon, x) \geq K_x\varepsilon^{p_x}$ for each $x \in S$ and all $\varepsilon > 0$. Now let $F_{n,p} = \{x \in S_X : \delta_m(\varepsilon, x) \geq \varepsilon^p/n \text{ for } \varepsilon > 0\}$ (use ε^{p_0}/n if $p_{x,f} \leq p_0$ for all $x \in S$ and $f \in B_{Y_x}$).

Then $\cup_{n,p=1}^{\infty} F_{n,p} \supset S$ and so $\cup_{n,p=1}^{\infty} F_{n,p}$ is of the second category. To see that $F_{n,p}$ is closed, suppose $x_k \in F_{n,p}$ and $x_k \to x$. If $\|y-z\| \geq \varepsilon$, then

$$\left\| x - \frac{y+z}{2} \right\| = \lim_{k\to\infty} \left\| x_k - \frac{y+z}{2} \right\| \geq \frac{1}{n}\varepsilon^p$$

and so $x \in F_{n,p}$. By Baire category theory, there are $x_0 \in S_X$, $0 < r < 1$, and $n, p \in \mathbb{N}$ such that $\delta_m(\varepsilon, x) \geq \frac{1}{n}\varepsilon^p$ for all $\varepsilon > 0$ and $x \in B_r(x_0) \cap S_X$. Then it follows from Lemma 2.4 that x_0 is a strongly exposed point of B_X, so there is a slice of B_X contained in $B_{r/2}(x_0)$. The conclusion of the theorem now follows from Lemma 2.4 and Day's renorming given in Lemma 2.3. □

From Theorem 2.1 it might seem reasonable that any space with any sort of uniformly rotund norm with modulus of power type should be superreflexive. However, we conclude this section by observing that standard renormings on separable Banach spaces show that Theorem 2.1 does not extend to cover the following well-known directional rotundity conditions. The *modulus of uniform rotundity in the direction* $z \in X \setminus \{0\}$ is defined by

$$\delta_d(\varepsilon, z) = \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_X, \quad x - y = \lambda z, \ \|x-y\| \geq \varepsilon \right\}.$$

A Banach space X is *uniformly rotund in every direction* (URED) if for each $z \in X \setminus \{0\}$, $\delta_d(\varepsilon, z) > 0$ for all $\varepsilon > 0$. Given a bounded $W \subset X \setminus \{0\}$, we define $\delta_d(\varepsilon, W) = \inf\{\delta_d(\varepsilon, w) : w \in W\}$. Following Smith [14], we will say that X is *uniformly rotund in weakly compact sets of directions* (URWC) (*uniformly rotund in weakly closed bounded sets of directions* (URWB)) if $\delta_d(\varepsilon, W) > 0$ for each nonempty weakly compact (weakly closed bounded) subset W of $X \setminus \{0\}$ and all $\varepsilon > 0$. We will need the following simple fact.

Lemma 2.5. *Suppose X is a Banach space and $\{f_i\}_{i=1}^{\infty} \subset B_{X^*}$. Then the equivalent norm $|||\cdot|||$ defined by $|||x|||^2 = \|x\|^2 + \sum_{i=1}^{\infty} \frac{f_i^2(x)}{2^i}$ has the property that for each $n \in \mathbb{N}$, $\delta(\varepsilon, f_n) \geq 2^{-n-3}\varepsilon^2$ for all $\varepsilon > 0$.*

PROOF: Given $n \in \mathbb{N}$, let $x, y \in X$ be such that $|||x||| \leq 1$ and $|||y||| \leq 1$ and $|f_n(x-y)| \geq \varepsilon$. Then

$$\begin{aligned} 4 - |||x+y|||^2 &\geq 2|||x|||^2 + 2|||y|||^2 - |||x+y|||^2 \\ &\geq 2^{-n}[2f_n^2(x) + 2f_n^2(y) - f_n^2(x+y)] \\ &= 2^{-n} f_n^2(x-y) \geq 2^{-n}\varepsilon^2. \end{aligned}$$

So $|||\frac{x+y}{2}|||^2 \leq 1 - 2^{-n-2}\varepsilon^2$ and thus $|||\frac{x+y}{2}||| \leq 1 - 2^{-n-3}\varepsilon^2$ as desired. □

If X is separable, then there is a countable norming subset of X^*, and so Lemma 2.5 shows that the norming subspaces in Lemma 2.2 and Theorem 2.1 cannot be replaced with arbitrary norming subsets. The next result sharpens [14, Corollary 2.4] and demonstrates that Theorem 2.1 does not extend to even URWB norms.

Theorem 2.6. (a) *If X is a Banach space such that X^* is weak* separable, then X can be equivalently renormed so that for each weakly compact set $W \subset X \setminus \{0\}$, there is a $K_W > 0$ such that $\delta_d(\varepsilon, W) \geq K_W \varepsilon^2$ for all $\varepsilon > 0$.*

(b) *Let X be a separable Banach space. Then X^* is separable if and only if X can be equivalently renormed such that for each weakly closed bounded $W \subset X \setminus \{0\}$, there is a $K_W > 0$ such that $\delta_d(\varepsilon, W) \geq K_W \varepsilon^2$ for all $\varepsilon > 0$.*

Proof: (a) Because X^* is weak* separable, we fix a countable subset $\{f_i\}_{i=1}^{\infty} \subset B_{X^*}$ such that $\sup_i f_i(x) > 0$ for each $x \in X \setminus \{0\}$. Let $|||\cdot|||$ be defined as in Lemma 2.5 using this set $\{f_i\}$. Given a weakly compact subset $W \subset X \setminus \{0\}$. For each $w \in W$, choose $f_{i_w} \in \{f_i\}$ such that $f_{i_w}(w) = \alpha_w > 0$. Now consider the weakly open sets $O_w = \{x : f_{i_w}(x) > \alpha_w/2\}$ that cover W. Let $O_{w_1}, \dots, O_{w_n}$ be a finite subcover, and $\alpha = \min\{\alpha_{w_k}/2 : 1 \leq k \leq n\}$. Now suppose $|||x||| \leq 1$, $|||y||| \leq 1$, $|||x-y||| \geq \varepsilon$ and $x - y = \lambda w$ for some $w \in W$. Then for some $1 \leq j \leq n$, we have $|f_{i_{w_j}}(x-y)| \geq |\lambda| \alpha_{w_j}/2 \geq \frac{\varepsilon\alpha}{|||w|||} \geq \varepsilon r \alpha$ where $r = \min\{|||w|||^{-1} : w \in W\}$. According to Lemma 2.5,

$$|||\frac{x+y}{2}||| \leq 1 - 2^{-i_{w_j}-3}(\varepsilon r \alpha)^2 \leq 1 - K_W \varepsilon^2$$

where $K_W = 2^{-N-3}\alpha^{-2} r^2$ and $N = \max\{i_{w_k} : 1 \leq k \leq n\}$. This $K_W > 0$ is independent of $w \in W$. Therefore $\delta_d(\varepsilon, W) \geq K_W \varepsilon^2$ for all $\varepsilon > 0$ as desired.

(b) Suppose X^* is separable. We fix a countable set $\{f_i\}_{i=1}^{\infty}$ norm dense in B_{X^*}, and define $|||\cdot|||$ as in Lemma 2.5. Fix a weakly closed and bounded set $W \subset X \setminus \{0\}$. Because $\{f_i\}_{i=1}^{\infty}$ is norm dense in B_{X^*}, there are $f_{k_1}, \dots f_{k_n}$ such that $W \subset \cup_{i=1}^{n} \{x : f_{k_i}(x) \geq \alpha_i\}$ where $\alpha_i > 0$ for each i. Proceeding as in (a), we find that $\delta_d(\varepsilon, W) \geq K_W \varepsilon^2$ for all $\varepsilon > 0$.

The converse follows from Smith's result [14, Theorem 2.2 (ii)] that X is WUR if and only if X is URWB, and Hájek's recent result [10, Theorem 1] that X^* is separable provided X is separable and has an equivalent WUR norm. $\square$

The following diagram indicates the relative strengths of the major rotundity conditions we have discussed.

$$\begin{tikzcd}
& \text{LUR} \arrow[r] \arrow[dr] & \text{MLUR} \arrow[r] & \text{WMLUR} \arrow[dr] & \\
\text{UR} \arrow[ur] \arrow[r] \arrow[dr] & \text{WUR} \arrow[r] \arrow[d, leftrightarrow] & \text{WLUR} \arrow[ur] & & \text{Rotundity} \\
& \text{URWB} \arrow[r] \arrow[ur] & \text{URWC} \arrow[r] & \text{URED} \arrow[ur] &
\end{tikzcd}$$

In particular, we have seen that a power type condition on the natural directional modulus associated with URWB norms does not imply superreflexivity, while a power type condition on the natural directional modulus for the equivalent WUR property or even the drastically weaker WMLUR property yields superreflexivity. However, in general, we do not know when the existence of a (dual) URED, URWC or URWB norm on a space implies that there is an equivalent such (dual) norm with an associated modulus $\delta_d(\varepsilon, -)$ of power type.

3. Directional moduli of smoothness

For a Banach space X and a bounded $D \subset X$, the *modulus of smoothness with respect to D* is given by

$$\rho(t, D) = \sup\left\{\frac{\|x+th\| + \|x-th\|}{2} - 1 : \|x\| = 1,\ h \in D\right\} \quad \text{for} \quad t > 0.$$

The *modulus of smoothness at $x \in S_X$ in the directions D* is defined by

$$\rho(t, x, D) = \sup\left\{\frac{\|x+th\| + \|x-th\|}{2} - 1 : h \in D\right\} \quad \text{for} \quad t > 0.$$

Occasionally we will use $\rho_{\|\cdot\|}(t, D)$ or $\rho_X(t, D)$ to emphasise the particular norm or space under consideration. We will say that $\rho(t, -)$ is of *power type* if there are $K > 0$ and $1 < p \leq 2$ such that $\rho(t, -) \leq Kt^p$ for all $t \geq 0$. If $D = B_X$, we may omit it in the notation and write $\rho(t)$ and $\rho(t, x)$.

The norm on X is *uniformly Gateaux* (*uniformly weak Hadamard, uniformly Fréchet*) *differentiable* if and only if $\lim_{t\to 0^+} \rho(t, D)/t = 0$ for each singleton D (each weakly compact set D, $D = B_X$) in X. The norm on X is *Lipschitz smooth* (*Fréchet differentiable, Gateaux differentiable*) at x if and only if there is a $C > 0$ such that $\rho(t, x) \leq Ct^2$ for all $t > 0$ ($\lim_{t\to 0^+} \rho(t, x)/t = 0$, $\lim_{t\to 0^+} \rho(t, x, y)/t = 0$ for each $y \in S_X$). See [4], [6], [8], [9] for more information on these and related smoothness notions.

Because exposedness is a dual notion to smoothness, given $x \in S_X$ with support functional $f_x \in S_{X^*}$ where $f_x(x) = 1$ and a bounded $D \subset X^*$, we define the *modulus of exposedness of x by f_x with respect to D* by

$$\Delta_{f_x}(\varepsilon, x, D) := \inf\left\{f_x(x) - \frac{f_x(y+z)}{2} : y, z \in B_X,\ |f(y-z)| \geq \varepsilon,\ f \in D\right\}.$$

The definitions involved imply that $\Delta_{f_x}(\varepsilon, x, D) \leq \delta_m(\varepsilon, x, D)$, and so Theorem 2.1 applies to moduli of exposedness of power type. It can also be verified that f_x *strongly exposes* $x \in B_X$ if and only if $\Delta_{f_x}(\varepsilon, x) > 0$ for each $\varepsilon > 0$, whereas f_x *Lipschitz exposes* $x \in B_X$ in the sense of Fabian [8, p. 115] if and only if there is a $K > 0$ such that $\Delta_{f_x}(\varepsilon, x) \geq K\varepsilon^2$ for all $\varepsilon > 0$.

The following proposition presents straightforward variants of Lindenstrauss' duality formula (see [6, Proposition IV.1.7]).

Proposition 3.1. *Let X be a Banach space with dual X^*. Then:*

(a) $\rho_{X^*}(t,D) = \sup\left\{\frac{t\varepsilon}{2} - \delta_X(\varepsilon,D) : \varepsilon \geq 0\right\}$ *for each bounded $D \subset X^*$;*
(b) $\rho_X(t,D) = \sup\left\{\frac{t\varepsilon}{2} - \delta_{X^*}(\varepsilon,D) : \varepsilon \geq 0\right\}$ *for each bounded $D \subset X$;*
(c) *given $x \in S_X$ and $f_x \in S_{X^*}$ where $f_x(x) = 1$*

$$\rho(t,x,D) = \sup\left\{\frac{t\varepsilon}{2} - \Delta_x(\varepsilon, f_x, D) : \varepsilon \geq 0\right\} \quad \text{for each bounded } D \subset X.$$

Proof: All the arguments are almost identical to the proof of [6, Proposition IV.1.7], so we prove only (c) for the sake of completeness. We first show that $\rho(t,x,D)$ is at least as large as the stated supremum. For this, let $y \in D$ and $f, g \in S_{X^*}$ with $|(f-g)(y)| \geq \varepsilon$, where we may assume that $(f-g)(y) \geq \varepsilon$. (If no such f, g, y exist, then $\Delta_x(\varepsilon, f_x, D) = \infty$, and so $\rho(t,x,D) \geq t\varepsilon/2 - \Delta_x(\varepsilon, f_x, D)$ which is what we want.) Then,

$$\begin{aligned} 2\rho(t,x,D) &\geq \|x+ty\| + \|x-ty\| - 2 \\ &\geq f(x+ty) + g(x+ty) - 2 \\ &= (f+g)(x) + t(f-g)(y) - 2 \\ &\geq (f+g)(x) + t\varepsilon - 2. \end{aligned}$$

It follows that

$$1 - \frac{\hat{x}(f+g)}{2} \geq \frac{t\varepsilon}{2} - \rho(t,x,D).$$

Consequently $\rho(t,x,D) \geq t\varepsilon/2 + \Delta_x(\varepsilon, f_x, D)$.

For the reverse inequality, let $y \in D$, $\eta > 0$ and choose $f, g \in S_{X^*}$ such that

$$f(x+ty) \geq \|x+ty\| - \eta \quad \text{and} \quad g(x-ty) \geq \|x-ty\| - \eta.$$

Now let $\varepsilon_0 = |(f-g)(y)|$. From this, we have

$$\|x+ty\| + \|x-ty\| \leq (f+g)(x) + t(f-g)(y) + 2\eta.$$

Now $(f+g)(x) \leq 2 - 2\Delta_x(\varepsilon_0, f_x, D)$ and so

$$\frac{\|x+ty\| + \|x-ty\| - 2}{2} \leq \frac{t\varepsilon_0}{2} - \Delta_x(\varepsilon_0, f_x, D) + 2\eta.$$

Because $\eta > 0$ was arbitrary, the proof is complete. □

Corollary 3.2. *Suppose X is a Banach space, D is a bounded subset of X and $x \in S_X$.*

(a) *For $1 < p \leq 2$ and $1/p + 1/q = 1$, there exists $C > 0$ such that $\rho(t,D) \leq Ct^p$ ($\rho(t,x,D) \leq Ct^p$) for all $t > 0$ if and only if there exists $K > 0$ such that $\delta_{X^*}(\varepsilon,D) \geq K\varepsilon^q$ ($\Delta_x(\varepsilon, f_x, D) \geq K\varepsilon^q$ for any $f_x \in S_{X^*}$ where $f_x(x) = 1$) for all $\varepsilon > 0$.*
(b) *$\lim_{t\to 0^+} \rho(t,x,D)/t = 0$ ($\lim_{t\to 0^+} \rho(t,D)/t = 0$) if and only if for all $\varepsilon > 0$, $\Delta_x(\varepsilon, f_x, D) > 0$ ($\delta_{X^*}(\varepsilon,D) > 0$).*

Proof: For (a), follow the details of [6, Proposition IV.1.12] using Proposition 3.1, and for (b), follow the details of [6, Proposition IV.1.11 (i)] using Proposition 3.1.

$\square$

In particular, Corollary 3.2 (a) with $p = q = 2$ recaptures the duality between pointwise Lipschitz smoothness and Lipschitz exposed points for the case of norms as provided in [8, Proposition 2.2]. The next result shows that [7, Corollary 3.8] actually is true for fixed arbitrary set of directions.

Theorem 3.3. *Let X be a Banach space such that the set S of support functionals of B_X is second category in S_{X^*}, and let $D \subset X$ be bounded. If for each $x \in S_X$ there exist $C_x > 0$ and $1 < p_x \leq 2$ such $\rho(t, x, D) \leq C_x t^{p_x}$ for all $t > 0$, then there exist $C > 0$ and $p > 0$ such that $\rho(t, D) \leq Ct^p$ for all $t > 0$. Moreover, if $p_x \geq p_0$ for all x, then $p \geq p_0$.*

The proof of this result relies on the directional version of [7, Proposition 2.1] which in turn uses the directional version of Day's renorming in Lemma 2.3.

Lemma 3.4. *Consider a Banach space X, S a residual subset of S_X and D a bounded subset of X^*. If for each $x \in S$, there exist $K_x > 0$ and $p_x > 0$ such that $\delta_m(\varepsilon, x, D) \geq K_x \varepsilon^{p_x}$ for all $\varepsilon > 0$, then X can be equivalently renormed so that for some $K > 0$ and $p > 0$, $\delta(\varepsilon, D) \geq K\varepsilon^p$ for all $\varepsilon > 0$. Moreover, if $p_x \leq p_0$ for all $x \in S$, then $p \leq p_0$.*

Proof: As in the proof of Theorem 2.1 we see that there exist $x_0 \in S_X$, $0 < r < 1$, $p > 0$ and $0 < C \leq 1$ such that $\delta_m(\varepsilon, x, D) \geq C\varepsilon^p$ for all $\varepsilon > 0$ and $x \in B_r(x_0) \cap S_X$ (where $p = p_0$ in the "moreover" situation). According to Lemma 2.4, if $y, z \in B_{\frac{r}{2}}(x_0) \cap B_X$ and $|g(y - z)| \geq \varepsilon$ then

$$1 - \left\| \frac{y + z}{2} \right\| \geq C\varepsilon^p. \tag{1}$$

If some point in the interior of $B_{\frac{r}{2}}(x_0) \cap S_X$ were strongly exposed, then Lemma 2.3 would complete the proof. To overcome this, define the equivalent norm $|||\cdot|||$ on X by $|||x|||^2 := \|x\|^2 + \mathrm{dist}^2(x, l(x_0))$ where $l(x_0)$ is the line through 0 and x_0. Then $|||\cdot|||$ is locally uniformly rotund at points on the line $l(x_0)$ and consequently x_0 is a strongly exposed point of $B_{|||\cdot|||}$. Moreover, $|||\cdot|||$ is a dual norm provided $\|\cdot\|$ is. It remains to verify that $|||\cdot|||$ satisfies (1), or equivalent. For this, we may assume that

$$r \leq 1/2M \quad \text{where} \quad M := \max\{\|f\| : f \in D\}. \tag{2}$$

For $K := \min\{1/(128M^2), C/2^{p+5}\}$, we will show that

$$1 - |||\frac{x + y}{2}||| \geq K\varepsilon^p \tag{3}$$

whenever $x, y \in \frac{r}{2}B_{|||\cdot|||}(x_0) \cap B_{|||\cdot|||}$ and $|g(x-y)| \geq \varepsilon$ for some $g \in D$. Without loss of generality, we assume $\|x\| \geq \|y\|$ and we let $t = \|y\|/\|x\|$; notice that $\|x\| \geq 1 - r/2 > 1/2$ and $1/2 \leq t \leq 1$. If (3) does not hold, then for some $g \in D$ with $|g(x-y)| = \varepsilon$, we have $|||x+y||| > 2 - 2K\varepsilon^p \geq 0$ by (2). Therefore

$$2|||x|||^2 + 2|||y|||^2 - |||x+y|||^2 < 8K\varepsilon^p \text{ which implies that}$$

$$2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 < 8K\varepsilon^p \text{ dividing by } \|x\|^2 \text{ implies that} \tag{4}$$

$$2\|\bar{x}\|^2 + 2\|t\bar{y}\|^2 - \|\bar{x}+t\bar{y}\|^2 < 32K\varepsilon^p \text{ since } \|x\| > 1/2$$

where $\bar{x} = x/\|x\|$ and $\bar{y} = y/\|y\|$. Now (4) implies that $(\|\bar{x}\| - \|t\bar{y}\|)^2 \leq 32K\varepsilon^p$ while (2) forces $\varepsilon \leq 1$. Consequently, $\|t\bar{y}\| \geq 1 - \sqrt{32K}\varepsilon$ and so

$$1 - t \leq \sqrt{32K}\varepsilon \leq \frac{\varepsilon}{2M} \text{ because } K \leq 1/(128M^2). \tag{5}$$

Now $g(\bar{x} - t\bar{y}) = \frac{1}{\|x\|}g(x-y) \geq \varepsilon$. Combining this with (5) and the fact that $\|g\| \leq M$, we conclude that $g(\bar{x} - \bar{y}) \geq \varepsilon/2$. Thus (1) implies that $\|\bar{x} + \bar{y}\| \leq 2[1 - C(\varepsilon/2)^p]$. Using this, and recalling that $1/2 \leq t \leq 1$, we estimate

$$\begin{aligned} \|\bar{x} + t\bar{y}\| &\leq \|\bar{x} - t\bar{x}\| + \|t\bar{x} + t\bar{y}\| \leq 1 - t + t\|\bar{x} + \bar{y}\| \\ &\leq (1-t) + 2t[1 - C(\varepsilon/2)^p] \leq 1 + t - C\varepsilon^p/2^p. \end{aligned} \tag{6}$$

Squaring (6), we obtain

$$\begin{aligned} \|\bar{x} + t\bar{y}\|^2 &\leq 2 + 2t^2 - C\varepsilon^p/2^{p-1} + C^2\varepsilon^{2p}/2^{2p} \\ &\leq 2 + 2t^2 - C\varepsilon^p/2^p \qquad \text{because } C \leq 1,\ \varepsilon \leq 1 \\ &\leq 2\|\bar{x}\|^2 + 2\|t\bar{y}\|^2 - 32K\varepsilon^p \qquad \text{because } K \leq C/2^{p+5} \end{aligned}$$

which contradicts (4). Thus (3) is valid and so Lemma 2.3 provides the desired equivalent norm. □

Proof of Theorem 3.3: If $f \in S$ supports $x \in S_X$, then $\Delta_x(\varepsilon, f, D)$ is of power type by Corollary 3.2 (a), and consequently $\delta_m(\varepsilon, f, D)$ is of power type. Now invoking Lemma 3.4 we obtain a dual norm ν^* on X^* such that $\delta_{\nu^*}(\varepsilon, D)$ is of power type. Corollary 3.2 (a) ensures that for the predual norm ν on X, $\rho_\nu(t, D)$ is of power type as desired.

Moreover, if $p_x \geq p_0$, then $\Delta_x(\varepsilon, f, D)$ is of power type q_0 where $1/q_0 + 1/p_0 = 1$. Then $\delta_{X^*}(\varepsilon, D)$ is of power type q_0 for the dual norm given in Lemma 3.4. By Corollary 3.2 (a), $\rho_X(t, D)$ is of power type p_0. □

With regard to the conditions in Theorem 3.3, notice that any Banach space with the RNP has the property that the support functionals of B_X form a set of the second category in S_{X^*}; see [3, Corollary 3.5.7].

The concept of *normal structure* plays a crucial role in fixed points of nonexpansive mappings on weakly compact convex sets; see [6, pp. 67, 68] and [16] for more information on this subject. We now show that norms as given in Theorem 3.3, or more generally norms that are uniformly smooth in fixed sets of directions have implications for the normal structure of certain subsets of the space.

Proposition 3.5. *Let X be a Banach space and $D \subset X$ be a closed balanced convex set such that for each $\alpha > 0$, $\limsup_{t\to 0^+} \rho(t, D_\alpha)/t < 1/2$ where $D_\alpha = \alpha D \cap B_X$. Then every weakly compact convex subset of D has normal structure.*

Proof: We appropriately modify Turett's argument from [17]. Let W be a weakly compact convex subset of D, and suppose that W does not have normal structure. By the Brodskii-Milman criterion (see [16, p. 203]), there is a sequence $\{x_n\} \subset W$ such that $\operatorname{dist}(x_n, \operatorname{conv}\{x_1, x_2, \dots x_{n-1}\}) \to r := \operatorname{diam}(\{x_n\}) > 0$. By passing to a subsequence if necessary, we may assume that $x_n \to x$ weakly for some x. Because D is closed, convex and balanced, $y_n = (x_n - x)/r \in (2/r)D$. Now observe that $y_n \to 0$ weakly, and has the property that $d_n := \operatorname{dist}(y_n, \operatorname{conv}\{y_1, \dots, y_{n-1}\}) \to \operatorname{diam}(\{y_n\}) = 1$. By the separation property, we choose $f_n \in S_{X^*}$ such that $f_n(y_n) > (\sup f_n(C_n)) + d_n - 1/n$, where $C_n := \operatorname{conv}\{y_1, \dots, y_{n-1}\}$. Since $y_n \to 0$ weakly we know that $\operatorname{dist}(0, C_n) \to 0$ and so $\limsup_{n\to\infty}(\sup f_n(C_n)) \geq 0$ which in turn implies that $f_n(y_n) \to 1$. This with the fact that $y_n \to 0$ weakly implies that

$$\lim_{n\to\infty} f_n(y_n - y_j) = 1 \quad \text{and} \quad \lim_{n\to\infty} -f_j(y_n - y_j) = f_j(y_j) \quad \text{for each } j.$$

Thus, given $k \geq 1$ we choose m_k and n_k such that $f_{n_k}(y_{n_k}) > 1 - 1/k$ and

$$(7) \qquad (a)\ f_{n_k}(y_{n_k} - y_{m_k}) > 1 - \frac{1}{k} \quad \text{and} \quad (b)\ - f_{m_k}(y_{n_k} - y_{m_k}) > 1 - \frac{1}{k}\,.$$

Now let $f_k := f_{n_k}$ and $g_k := -f_{m_k}$. Then by (7), $(f_k + g_k)(y_{n_k} - y_{m_k}) > 2 - 2/k$; this implies that $\|f_k + g_k\| \to 2$ because $\operatorname{diam}(\{y_n\}) = 1$. On the other hand (7b) implies that $g_k(y_{n_k}) < 1/k$ and thus $(f_k - g_k)(y_{n_k}) > 1 - 2/k$. Consequently, $\sup(f_k - g_k)(D_{2/r}) > 1 - 2/k$. This with $\|f_k + g_k\| \to 2$ shows that $\delta_{X^*}(\varepsilon, D_{2/r}) = 0$ for $\varepsilon < 1$. By Proposition 3.1 (b), $\rho(t, D) \geq t\varepsilon/2$ for $t > 0$ and all $\varepsilon < 1$ contradicting $\limsup_{t\to 0^+} \rho(t, D)/t < 1/2$. ☐

In particular, the following result is valid.

Corollary 3.6. (a) *If X is a Banach space such that $\limsup_{t\to 0^+} \rho(t, W)/t < 1/2$ for each weakly compact set W in B_X* (*in particular if the norm on X is uniformly weak Hadamard differentiable*), *then each weakly compact convex subset of X has normal structure.*

(b) *Suppose that D is a closed balanced convex subset of X such that $\lim_{t\to 0^+} \rho(t, D)/t = 0$. Then for each $r > 0$, every weakly compact convex set contained in a translate of rD has normal structure.*

Proof: (a) This follows directly from Proposition 3.5 because the closed balanced hull of a weakly compact set intersected with B_X is weakly compact.

(b) It is straightforward to check that for $D_{r\alpha} = r\alpha D \cap B_X$, the hypothesis on $\rho(t, D)$ implies that $\lim_{t\to 0^+} \rho(t, D_{r\alpha})/t = 0$ for each $\alpha > 0$. According to Proposition 3.5, every weakly compact convex subset of rD has normal structure. Hence the same is true of translates of rD. ☐

As for the applicability of Corollary 3.6 (a), let us recall [9, Theorem 1.4] shows that every L_1 space over a σ-finite measure has an equivalent uniformly weak Hadamard differentiable norm. However, we did not investigate the degree to which the results in this note can be extended to hold in more general contexts, such as for convex functions or bump functions (see [7], [11]). We close with some questions pertaining to directional smoothness.

Remarks and questions 3.7. (a) It is not clear to us whether the argument of Turett [17] can be used to provide generalisations of Ballion's theorem beyond that given in Corollary 3.6 (a). Thus it would be interesting to know whether a Banach space with an equivalent uniformly Gateaux differentiable norm satisfies the property that every nonexpansive mapping on a weakly compact convex set has a fixed point (cf. [6, p. 88]).

(b) It is not difficult to generalise Zajíček's [18, Proposition 7] for sets of directions $D \subset B_X$, and also one can obtain the following generalisation of (i) implies (ii) of [18, Theorem 3]. *Suppose* $\limsup_{t\to 0^+} \rho(t,D)/t = 0$ *for* $D \subset B_X$, *and suppose* $F \subset X$ *is closed and nonempty. Let* $f(x) = \operatorname{dist}(x,F)$. *Then for* $x \notin F$, *the right-hand derivative of* f *at* x $D'_+ f(x)(h)$ *exists for* $h \in D$ *and the difference quotient is uniform for* $h \in D$. However, it is not clear to us whether the converse holds outside of the uniformly Gateaux differentiable case where D is a singleton as was shown in [18, Theorem 3]. For example, is the norm on X uniformly Fréchet differentiable if $d(\cdot,F)$ has a "Fréchet" right-hand derivative for each closed nonempty subset F and each $x \notin F$?

(c) It is well known that if $\limsup_{t\to 0^+} \rho(t,B_X)/t < 1$ then X is superreflexive and has an equivalent uniformly smooth norm (see e.g. [17]). However, in the analogous situation when $\limsup_{t\to 0^+} \rho(t,W)/t < 1$ for each weakly compact $W \subset B_X$, we do not know what can be said about the existence of an equivalent uniformly weak Hadamard differentiable norm on X.

(d) The following question from [2] is of interest for many applications in optimisation. Suppose Y is a subspace of X and Y admits an equivalent Fréchet differentiable norm. Is there an equivalent norm on X such that $\lim_{t\to 0^+} \rho(t,B_Y)/t = 0$? Notice that the answer is positive if Y^* admits an equivalent dual LUR norm ([2]).

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(*Received* August 28, 1997)