

Equations with discontinuous nonlinear semimonotone operators

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Abstract. The aim of this paper is to present an existence theorem for the operator equation of Hammerstein type $x + KF(x) = 0$ with the discontinuous semimonotone operator F . Then the result is used to prove the existence of solution of the equations of Urysohn type. Some examples in the theory of nonlinear equations in $L_p(\Omega)$ are given for illustration.

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1. Introduction

Let X be a real Banach space and X^* be its dual which are uniformly convex. For the sake of simplicity, the norms of X and X^* will be denoted by one symbol $\|\cdot\|$. We write $\langle x^*, x \rangle$ instead of $x^*(x)$ for $x \in X^*$ and $x \in X$. Let $F : X \rightarrow X^*$ be a bounded, discontinuous and semimonotone operator and $K : X^* \rightarrow X$ a bounded (i.e. image of any bounded subset is bounded), linear and nonnegative operator.

Consider the nonlinear operator equation of Hammerstein type

$$(1.1) \quad x + KF(x) = 0.$$

Integral equations of Hammerstein type with a nonlinear smooth operator F are studied in [1]–[3], [6], [17]. When F is discontinuous, they are investigated in [5], [7], [15], [16] by introducing a new concept of solution. But, throughout this paper, the word ‘solution’ is meant in the classical sense. We shall prove an existence theorem for solution for discontinuous F . Using this result, we get a new result regarding the solvability of a class of nonlinear equations of Urysohn type

$$(1.2) \quad x + \sum_{j=1}^m K_j F_j(x) = 0,$$

where each K_j and F_j has the properties as K and F , respectively. Then, these theoretical results are applied to study the nonlinear integral equations in the spaces of type $L_p(\Omega)$. It should be mentioned that quasilinear elliptic equations

with nonlinear discontinuous part are usually used to describe the state of the systems with variable structure (see [10]). These equations are studied recently (see [12]–[14]) and can be transformed to equations of Hammerstein type (see [12]).

Below, the symbols \rightarrow and \rightharpoonup denote convergence in norm and weak convergence, respectively.

2. Main result

Definition 1 (see [13]). A point $x \in X$ is called a point of h-continuity of the operator $G : X \rightarrow X^*$ if

$$\forall l \in X \quad \lim_{t \rightarrow 0_+} \langle G(x + tl), l \rangle = \langle G(x), l \rangle.$$

A point $x \in X$ is called a point of discontinuity if x does not satisfy the condition in Definition 1.

Definition 2. A point of discontinuity x of G is called regular if

$$\exists l \in X : \lim_{t \rightarrow 0_+} \langle G(x + tl), l \rangle < 0.$$

Theorem 2.1. Assume that all the above conditions hold, all the points of discontinuity of F are regular and that there exists a positive constant r such that

$$\langle F(x), x \rangle > 0 \quad \text{if } \|x\| > r.$$

Then equation (1.1) has a solution x .

PROOF: As in [6], consider the regularized equation

$$(2.1) \quad x + B_n F(x) = 0, \quad B_n = B + \alpha_n V,$$

where V is the standard dual mapping of X^* , i.e. $V : X^* \rightarrow X$,

$$\langle V(x^*), x^* \rangle = \|V(x^*)\| \|x^*\| = \|x^*\|^2, \quad \forall x^* \in X^*,$$

and α_n is a sequence of positive real numbers such that $\alpha_n \rightarrow 0$ as $n \rightarrow +\infty$. Then $R(B_n) = X$, $B_n^{-1}(0) = 0$, B_n^{-1} is an one-to-one mapping and B_n^{-1} is continuous (see [4]). Therefore, all the points of discontinuity of F are points of discontinuity of $\tilde{B}_n + F$ and, conversely, all points of discontinuity of $\tilde{B}_n + F$ are points of discontinuity of F , where $\tilde{B}_n(x) = -B_n^{-1}(-x)$. Obviously, we can rewrite equation (2.1) in the form

$$(2.2) \quad \tilde{B}_n(x) + F(x) = 0.$$

By virtue of [17], equation (2.2) has a unique solution, henceforth denoted by x_n . Moreover, $\|x_n\| \leq r, \forall n$. As F is bounded, the sequence $\{F(x_n)\}$ is bounded, too. Without loss of generality, assume that

$$x_n \rightharpoonup x_0 \quad \text{and} \quad F(x_n) \rightharpoonup y_0^*.$$

From (2.1) it follows that

$$(2.3) \quad x_0 + By_0^* = 0.$$

Now, we have to prove that $y_0^* = F(x_0)$. Since F is semimonotone, we have $F = T + C$, with a monotone operator T and a compact operator C . Therefore,

$$\langle F(x) - C(x) - (F(x_n) - C(x_n)), x - x_n \rangle > 0, \quad \forall x \in X.$$

Hence,

$$\begin{aligned} \langle F(x) - C(x), x - x_n \rangle - \langle F(x_n) - C(x_n), x \rangle &\geq \langle F(x_n), BF(x_n) \rangle \\ &\quad - \langle C(x_n), x_n \rangle + \alpha_n \langle F(x_n), VF(x_n) \rangle. \end{aligned}$$

By passing $n \rightarrow +\infty$ in the last equality, because of

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle F(x_n), BF(x_n) \rangle &\geq \langle y_0^*, By_0^* \rangle, \\ \lim_{n \rightarrow +\infty} \alpha_n \langle F(x_n), VF(x_n) \rangle &= 0, \\ \lim_{n \rightarrow +\infty} \langle C(x_n), x_n \rangle &= \langle C(x_0), x_0 \rangle, \end{aligned}$$

and (2.3) we obtain

$$\langle F(x) - C(x), x - x_0 \rangle - \langle y_0^* - C(x_0), x \rangle \geq \langle y_0^*, By_0^* \rangle - \langle C(x_0), x_0 \rangle.$$

Thus,

$$(2.4) \quad \langle T(x) - (y_0^* - C(x_0)), x - x_0 \rangle \geq 0.$$

Replacing x by $x_0 + tl$ for any $l \in X$ and $t > 0$ in (2.4) we see that

$$\langle F(x_0 + tl) - (y_0^* + C(x_0)), l \rangle \geq 0, \quad \forall l \in X.$$

Hence, x_0 is a point of h-continuity of T . Consequently, from (2.4) and Minty's lemma (see [17]) $T(x_0) = y_0^* - C(x_0)$, i.e. $y_0^* = F(x_0)$. \square

Now, consider equation (2.1). Let the following conditions hold:

- $K_j : X^* \rightarrow X$ are linear and bounded operators satisfying the condition: $\sum_{j=1}^m \langle K_j x_j^*, x^* \rangle \geq 0, x^* = \sum_{i=1}^m x_i^*, x_i^* \in X^*$,
- $F_j : X \rightarrow X^*$ are bounded, discontinuous and semimonotone, and
- $\langle F_j(x), x \rangle \geq a_j \|x\|^2 - b_j \|x\| - c_j, a_j, b_j, c_j > 0$ (see [8]).

Operator equation (1.2) is investigated in [8]–[9], [11] with some smoothness property of F_j . Here, applying Theorem 2.1, we can prove the following result.

Theorem 2.2. *Under the above conditions on K_j and F_j , equation (1.2) has a solution in X .*

PROOF: Denote $Z = X \times \cdots \times X$ (m times). For $z = (x_1, \dots, x_m) \in Z$, let

$$\|z\| = \left(\sum_{j=1}^m \|x_j\|^2 \right)^{1/2}.$$

Then, Z is uniformly convex Banach space with respect to this norm with dual $Z^* = X^* \times \cdots \times X^*$. (x_1, \dots, x_m) means the column vector $(x_1, \dots, x_m)^T$. Let $K : Z^* \rightarrow Z$ and $F : Z \rightarrow Z^*$ be defined as follows

$$(2.5) \quad K = \begin{bmatrix} K_1 & K_2 & \cdots & K_m \\ K_1 & K_2 & \cdots & K_m \\ \vdots & \vdots & \ddots & \vdots \\ K_1 & K_2 & \cdots & K_m \end{bmatrix}, \quad F = \begin{bmatrix} F_1 & 0 & \cdots & 0 \\ 0 & F_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_m \end{bmatrix}.$$

Consider the Hammerstein equation

$$(2.6) \quad z + KF(z) = 0, \quad z \in Z$$

with K and F from (2.5). It is easy to see that K is a linear, bounded and nonnegative operator on Z^* and F is a semicontinuous operator on Z . Moreover,

$$\begin{aligned} \langle F(z), z \rangle &= \sum_{j=1}^m \langle F_j(x_j) \rangle \geq \sum_{j=1}^m (a_j \|x_j\|^2 - b_j \|x_j\| - c_j) \\ &\geq a \|z\|^2 - b \|z\| - c, \end{aligned}$$

where $a = \min a_j$, $b = \sqrt{m} \max b_j$ and $c = \max c_j$. Therefore, there exists a positive constant R such that $\langle F(z), z \rangle > 0$, if $\|z\| > R$. By virtue of Theorem 2.1, equation (2.6) has a solution $z_* = (x_{1*}, \dots, x_{m*})$. Consequently, equation (1.2) has a solution $x = x_{1*}$ ($= x_{2*} = \cdots = x_{m*}$). \square

3. Application

a. Consider the nonlinear integral equation of second kind

$$(3.1) \quad x(s) + \int_{\Omega} k(s, t) F(x(t)) dt = 0,$$

where the kernel function $k(s, t)$ is such that the operator K defined by

$$(Kx)(s) = \int_{\Omega} k(s, t) x(t) dt$$

is bounded, nonnegative and K acts from $L_q(\Omega)$ into $L_p(\Omega)$ with $\Omega \subset \mathbb{R}^n$ measurable and $p^{-1} + q^{-1} = 1$. The nonlinear function $f(t)$ satisfies the following conditions:

- (a) $f(t)t \geq a_0|t|^p + b_0|t|^\gamma + c_0$, $a_0 > 0$, $b_0 < 0$, $c_0 < 0$, $\gamma < p$ (see [14]),
- (b) $f(t)$ is nondecreasing, rightcontinuous and at any point of discontinuity t_0 $f(t_0 - 0) < 0$, $f(t_0) < 0$,
- (c) $|F(t)| \leq a_1 + b_1|t|^{p-1}$, $\forall t \in \mathbb{R}^1$, $a_1 + b_1 > 0$, $a_1 \geq 0$, $b_1 \geq 0$.

By virtue of (c) we can define the operator $F: X = L_p(\Omega) \rightarrow X^* = L_q(\Omega)$ as

$$F(x)(t) = F(x(t)), \quad \forall x(t) \in L_p(\Omega).$$

Then equation (3.1) can be rewritten in the form (1.1), where the defined operator F possesses all the properties from Section 1. Indeed, condition (a) guarantees the existence of r in Theorem 2.1, the monotone property and the regularity of all points of discontinuity of F follows from (b) (see [13]) and the remaining properties are verified on the base of (c). Therefore, equation (3.1) has a solution, and this solution is unique if one of the operators K , F is strictly monotone.

b. Consider the nonlinear integral equation

$$(3.3) \quad x(t) + \sum_{j=1}^m \int_{\Omega} k_j(t, s) f_j(x(s)) ds = 0.$$

If the operators K_j and F_j defined by

$$\begin{aligned} (K_j x)(t) &= \int_{\Omega} k_j(t, s) x(s) ds, \\ (F_j x)(t) &= f_j(x(t)), \end{aligned}$$

have the same properties as K and F in **a.**, where only instead of the nonnegativeness of K we assume that

$$\sum_{i=1}^m \int_{\Omega} x_i(t) \int_{\Omega} \sum_{j=1}^m k_j(t, s) x_j(s) ds dt \geq 0,$$

then (3.3) can be rewritten in the form (1.2). Therefore, equation (3.3) is solvable by Theorem 2.2.

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