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Evolution inclusions in non separable Banach spaces

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Abstract. We study a Cauchy problem for non-convex valued evolution inclusions in non separable Banach spaces under Filippov type assumptions. We establish existence and relaxation theorems.

Keywords: evolution inclusions, mild solutions, Lusin measurable multifunctions, Banach spaces, relaxation

Classification: 34A60, 34G20

1. Introduction

Let $\mathbb{E}$ be a real Banach space with norm $\|\cdot\|$, and let $\mathcal{C}(\mathbb{E})$ be the space of all closed bounded nonempty subsets of $\mathbb{E}$ endowed with the Pompeiu-Hausdorff distance h. Let $I = [0,1]$.

In this paper we consider the Cauchy problem for evolution inclusions of the form

$$(C_{a,F}) \qquad \begin{cases} x'(t) \in Ax(t) + F(t, x(t)) \\ x(0) = a. \end{cases}$$

Here A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \geq 0$, of bounded linear operators on $\mathbb{E}$, F is a multifunction from $I \times \mathbb{E}$ to $\mathcal{C}(\mathbb{E})$, and $a \in \mathbb{E}$.

When $\mathbb{E}$ is finite dimensional, Filippov [4] (see also Hermes [6]) proved that the Cauchy problem $(C_{a,F})$, with $A = 0$, has solutions provided that F is continuous in (t, x) and Lipschitzian in x, i.e.

$$h\big(F(t,x),\, F(t,y)\big) \leq k(t)\|x - y\| \qquad (t,x),\ (t,y) \in I \times \mathbb{E},$$

for some $k \in L^1(I)$. The more general case in which F is Carathéodory-Lipschitz, i.e. F is measurable in t and Lipschitzian in x, was studied by Himmelberg and Van Vleck [9]. It is worth while to observe that a crucial step in the proof of Filippov theorem is the construction, for a $\mathcal{C}(\mathbb{E})$ valued multifunction, of a measurable selector, which is usually obtained by virtue of a selection theorem of Kuratowski and Ryll-Nardzewski [12]. More recently Frankowska [5], Tolstonogov [16] and Papageorgiou [13] have shown that if $\mathbb{E}$ is infinite dimensional, Filippov's ideas can be suitably adapted in order to prove the existence of mild solutions to the

Cauchy problem $(C_{a,F})$, provided that $\mathbb{E}$ is separable. This restriction is actually unavoidable if one has to apply in an infinite dimensional setting either selection theorem, of Kuratowski and Ryll-Nardzewski [12] or of Bressan and Colombo [1].

In the present paper we will establish the existence of mild solutions for the Cauchy problem $(C_{a,F})$ in an arbitrary, not necessarily separable, Banach space $\mathbb{E}$, under assumptions on F of Filippov type. Our approach follows essentially the pattern introduced by Filippov [4] and developed by Frankowska [5], Tolstonogov [16], and Papageorgiou [13], however with the basic difference that measurable selectors of multifunctions, when needed, will be constructed without relying on either of the above mentioned selection theorems. Actually our existence result (see Theorem 3.1) covers also the case of F Carathéodory-Lipschitz, where measurability in t is understood in the sense of Lusin. Furthermore, for the Cauchy problem $(C_{a,F})$ we shall prove a corresponding relaxation result (see Theorem 4.1) without assuming the Banach space $\mathbb{E}$ to be separable. This is made possible by an argument which, unlike the ones of [5], [16], [13], again does not depend on the above mentioned selection theorems.

Our existence and relaxation results for the Cauchy problem $(C_{a,F})$ are only a partial generalization of analogous results proved by Frankowska [5], Tolstonogov [16] and Papageorgiou [13] under slightly different assumptions on A and F, in separable Banach spaces. So far it is not clear if an analogous existence and relaxation theory, in absence of separability assumptions, might hold also for more general classes of systems, of the type considered by Papageorgiou [14] and by Hu, Lakhsmikantham and Papageorgiou [10].

This paper consists of four sections. Notation and some properties of Lusin measurable multifunctions are contained in Section 2. The existence and relaxation theorems for the Cauchy problem $(C_{a,F})$ are discussed in Section 3 and Section 4, respectively.

2. Lusin measurable multifunctions

Throughout this paper $\mathbb{E}$ denotes an arbitrary real Banach space with norm $\| \;\|$, and $\mathcal{C}(\mathbb{E})$ the space of all closed bounded nonempty subsets of $\mathbb{E}$. For $x \in \mathbb{E}$ and $A \subset \mathbb{E}$, $A \neq \emptyset$, set $d(x, A) = \inf_{a\in A} \|x - a\|$. $\mathcal{C}(\mathbb{E})$ is endowed with the *Pompeiu-Hausdorff metric*

$$h(A,B) = \max\{e(A,B),\, e(B,A)\} \qquad A, B \in \mathcal{C}(\mathbb{E}).$$

Here $e(A,B)$ is the *metric excess of A over B* and $e(B,A)$ the *metric excess of B over A*, that is $e(A,B) = \sup_{a\in A} d(a,B)$ and $e(B,A) = \sup_{b\in B} d(b,A)$

If $A \subset \mathbb{E}$, $A \neq \emptyset$, and $r \geq 0$ we set $N(A,r) = \{x \in \mathbb{E} | d(x,A) \leq r\}$. Clearly $N(A,r)$ is closed in $\mathbb{E}$.

We recall below some properties of the metric excess functions, that we shall use later.

Let $A, B, C \in \mathcal{C}(\mathbb{E})$. We have: (a_1) $e(A,B) = 0$ if and only if $A \subset B$; (a_2) $e(A,B) \leq e(A,C) + e(C,B)$ (a_3) $e(A,C) \leq e(B,C)$ and $e(C,A) \geq e(C,B)$, if $A \subset B$; (a$_4$) $e(N(A,r),\ C) \le e(A,C) + r$ and $e(C,\ N(A,r)) \ge e(C,A) - r$; (a$_5$) $e(A,B) \le r$ if and only if $A \subset N(B,r)$, $r \ge 0$.

For $A \subset \mathbb{E}$, by $\operatorname{co} A$ and $\overline{\operatorname{co}}\, A$, we mean respectively the convex hull and the closed convex hull of A.

Let X be a metric space. An open (resp. closed) ball in X with center x and radius r is denoted by $U_X(x,r)$ (resp. $\tilde{U}_X(x,r)$). For any set $A \subset X$, $\operatorname{int} A$ and $\overline{A}$ stand, respectively, for the interior of A, and the closure of A in X. For convenience we set $U = U_{\mathbb{E}}(0,1)$ and $I = [0,1]$.

A multifunction $F : X \to \mathcal{C}(\mathbb{E})$ is said to be *h-upper semicontinuous* (resp. *h-lower semicontinuous*, *h-continuous*) at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in U_X(x_0,\delta)$ we have $e(F(x),\ F(x_0)) \le \varepsilon$ (resp. $e(F(x_0),\ F(x)) \le \varepsilon$, $h(F(x),\ F(x_0)) \le \varepsilon$). For brevity we write h-u.s.c. and h-l.s.c. to mean, respectively, h-upper semicontinuous and h-lower semicontinuous. F is called h-u.s.c. (resp. h-l.s.c., h-continuous) if it is so at each point $x_0 \in X$.

Let $\mathcal{L}$ be the σ-algebra of the (Lebesgue) measurable subsets of $\mathbb{R}$ and, for $A \in \mathcal{L}$, let $\mu(A)$ be the Lebesgue measure of A.

For any set $A \subset X$, we denote by χ_A the characteristic function of A.

Let $A \in \mathcal{L}$, with $\mu(A) < +\infty$. A multifunction $F : A \to \mathcal{C}(\mathbb{E})$ is said to be *Lusin measurable* if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset A$, with $\mu(A \setminus K_\varepsilon) < \varepsilon$, such that F restricted to K_ε is h-continuous.

It is clear that if $F, G : A \to \mathcal{C}(\mathbb{E})$ and $f : A \to \mathbb{E}$ are Lusin measurable, then so are F restricted to B ($B \subset A$ measurable), $F + G$, and $t \to d(f(t),\ F(t))$. Moreover, the uniform limit $F : A \to \mathcal{C}(\mathbb{E})$ of a sequence of Lusin measurable multifunctions $F_n : A \to \mathcal{C}(\mathbb{E})$ is also Lusin measurable.

Further details about other notions of measurability for multifunctions and their relations can be found in Castaing and Valadier [2], Himmelberg [8], Klein and Thompson [11], and in [3].

The above definitions of h-upper or h-lower semicontinuity, h-continuity, Lusin measurability are unchanged if the space $\mathcal{C}(\mathbb{E})$ is replaced by $\mathcal{P}(\mathbb{E})$, the space of all bounded nonempty subsets of $\mathbb{E}$ endowed with the Pompeiu-Hausdorff pseudometric h.

The following propositions show that h-u.s.c. and h-l.s.c. multifunctions are Lusin measurable.

Proposition 2.1. *If $F : I \to \mathcal{C}(\mathbb{E})$ is h-u.s.c., then F is Lusin measurable.*

PROOF: For $n \in \mathbb{N}$ set $I_i^n = \left[(i-1)/2^n,\ i/2^n\right[$, $i = 1, \dots, 2^n - 1$, $I_{2^n} = \left[(2^n - 1)/2^n, 1\right]$. The family $\{I_i^n\}_{i=1}^{2^n}$ is a partition of I. Now for $n \in \mathbb{N}$ define $G_n : I \to \mathcal{C}(\mathbb{E})$ by

$$G_n(t) = \sum_{i=1}^{2^n} \Big(\overline{\bigcup_{s \in I_i^n} F(s)}\Big) \chi_{I_i^n}(t).$$

It is clear that G_n is piecewise constant, and that $G_n(t) \in \mathcal{C}(\mathbb{E})$, for F is bounded on I. Moreover, we have:

(i) $G_1(t) \supset G_2(t) \supset \cdots \supset G_n(t) \supset \cdots \supset F(t)$ for every $t \in I$;

(ii) for each $n \in \mathbb{N}$, $t \to h\big(G_n(t),\ F(t)\big)$ is measurable;

(iii) $h\big(G_n(t),\ F(t)\big) \to 0$ as $n \to +\infty$, for every $t \in I$.

Property (i) follows immediately from the definition of G_n. To prove (ii), fix $n \in \mathbb{N}$ and let $t_0 \in I$, $t_0 \neq i/2^n$, $i = 0, 1, \ldots, 2^n$. Clearly $t_0 \in \operatorname{int} I_i^n$, for some $1 \leq i \leq 2^n$. Since F is h-u.s.c., given $\varepsilon > 0$ there is $\delta > 0$, with $U_I(t_0, \delta) \subset \operatorname{int} I_i^n$, such that $t \in U_I(t_0, \delta)$ implies $e\big(F(t),\ F(t_0)\big) \leq \varepsilon$. Hence for every $t \in U_I(t_0, \delta)$ we have

$$e\big(G_n(t_0),\ F(t_0)\big) \leq e\big(G_n(t_0),\ F(t)\big) + e\big(F(t),\ F(t_0)\big) \leq e\big(G_n(t),\ F(t)\big) + \varepsilon,$$

as G_n is constant on I_i^n. On the other hand, by (i), $e\big(F(t),\ G_n(t)\big) = 0$ for each $t \in I$. Consequently, $h\big(G_n(t),\ F(t)\big) \geq h\big(G_n(t_0),\ F(t_0)\big) - \varepsilon$, for every $t \in U_I(t_0, \delta)$, and (ii) follows, as a lower semicontinuous function is measurable.

It remains to prove (iii). Let $t_0 \in I$ and $\varepsilon > 0$ be arbitrary. Since F is h-u.s.c., there is a $\delta > 0$ such that $t \in U_I(t_0, \delta)$ implies $F(t) \subset N\big(F(t_0),\ \varepsilon\big)$. For every n large enough, say $n \geq n_0$, there is $1 \leq i_n \leq 2^n$ such that $t_0 \in I_{i_n}^n \subset U_I(t_0, \delta)$. Thus if $n \geq n_0$ we have

$$G_n(t) = \overline{\bigcup_{s \in I_{i_n}^n} F(s)} \subset N\big(F(t_0),\ \varepsilon\big) \qquad \text{for every} \ \ t \in I_{i_n}^n,$$

and hence, $e\big(G_n(t_0), F(t_0)\big) \leq \varepsilon$. On the other hand, from (i), $e\big(F(t_0), G_n(t_0)\big) = 0$ for every $n \in \mathbb{N}$, and so $h\big(G_n(t_0),\ F(t_0)\big) \leq \varepsilon$ for every $n \geq n_0$, and also (iii) is proved.

We are ready to show that F is Lusin measurable. Let $\sigma > 0$. Since each G_n is piecewise constant, there is a compact set $H_\sigma \subset I$, with $\mu(I \setminus H_\sigma) < \sigma/2$, such that each G_n restricted to H_σ is h-continuous. In view of (ii) and (iii), using Egoroff-Severini theorem, one can construct a compact set $K_\sigma \subset H_\sigma$, with $\mu(H_\sigma \setminus K_\sigma) < \sigma/2$, such that $h\big(G_n(t),\ F(t)\big) \to 0$ as $n \to +\infty$, uniformly on K_σ. Therefore F restricted to K_σ is h-continuous, as each G_n restricted to K_σ is so, and the convergence is uniform. Clearly $\mu(I \setminus K_\sigma) < \sigma$. Hence F is Lusin measurable, completing the proof. □

Proposition 2.2. *If $F : I \to \mathcal{C}(\mathbb{E})$ is h-l.s.c., then F is Lusin measurable.*

Proof: For $n \in \mathbb{N}$, let $\{I_i^n\}_{i=1}^{2^n}$ be as in the proof of Proposition 2.1.

We claim that for every $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that if $n \geq k$ we have

$$\bigcap_{t \in I_i^n} \big(F(t) + \varepsilon U\big) \neq \emptyset \qquad \text{for each} \ \ i = 1, \ldots, 2^n. \tag{2.1}$$

Indeed, in the contrary case, there is an $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ there exist $n_k \in \mathbb{N}$ and $1 \le i_{n_k} \le 2^{n_k}$ such that

$$\bigcap_{t \in I^{n_k}_{i_{n_k}}} \big(F(t) + \varepsilon U\big) = \emptyset. \tag{2.2}$$

Passing to a subsequence, without change of notation, we can suppose that $\{I^{n_k}_{i_{n_k}}\}$ converges to some point $\bar{t} \in I$. Since F is h-l.s.c., there is $\delta > 0$ such that $t \in U_I(\bar{t}, \delta)$ implies $F(\bar{t}) \subset F(t) + \varepsilon U$. But for k large enough, say $k \ge k_0$, $I^{n_k}_{i_{n_k}} \subset U_I(\bar{t}, \delta)$, and so $F(t) + \varepsilon U \supset F(\bar{t})$ for every $t \in I^{n_k}_{i_{n_k}}$. As this contradicts (2.2), the claim is proved.

Let $\varepsilon > 0$. Let k correspond to ε according to the claim, thus (2.1) holds with $n = k$. If $n > k$, each interval I^n_i, $1 \le i \le 2^n$, is contained exactly in one interval I^k_j, for some $1 \le j \le 2^k$, and hence

$$\bigcap_{t \in I^k_j} \big(F(t) + \varepsilon U\big) \subset \bigcap_{t \in I^n_i} \big(F(t) + \varepsilon U\big).$$

Now for each $n \ge k$ define $G^\varepsilon_n : I \to \mathcal{C}(\mathbb{E})$ by

$$G^\varepsilon_n(t) = \sum_{i=1}^{2^n} \left[\bigcap_{s \in I^n_i} \overline{\big(F(s) + \varepsilon U\big)} \right] \chi_{I^n_i}(t).$$

By definition each G^ε_n is piecewise constant. Moreover the sequence $\{G^\varepsilon_n\}_{n \ge k}$ has the following properties:

(i) $G^\varepsilon_k(t) \subset G^\varepsilon_{k+1}(t) \subset \ldots \subset G^\varepsilon_n(t) \subset \ldots \subset \overline{F(t) + \varepsilon U}$ for every $t \in I$;

(ii) for each $n \ge k$, $t \to h\big(G^\varepsilon_n(t), \overline{F(t) + \varepsilon U}\big)$ is measurable on I;

(iii) $h\big(G^\varepsilon_n(t), \overline{F(t) + \varepsilon U}\big) \to 0$ as $n \to +\infty$, for every $t \in I$.

Property (i) follows at once from the definition of G^ε_n. To prove (ii), fix $n \ge k$ and let $t_0 \in I$, $t_0 \ne i/2^n$, $i = 0, 1, \ldots, 2^n$. Clearly $t_0 \in \operatorname{int} I^n_i$, for some $1 \le i \le 2^n$. Since F is h-l.s.c., given $\sigma > 0$ there is $\delta > 0$, with $U_I(t_0, \delta) \subset \operatorname{int} I^n_i$, such that $t \in U_I(t_0, \delta)$ implies $e\big(F(t_0), F(t)\big) \le \sigma$. Hence for every $t \in U_I(t_0, \delta)$ we have:

$$\begin{aligned} e\big(\overline{F(t_0) + \varepsilon U},\ G^\varepsilon_n(t_0)\big) &\le e\big(\overline{F(t_0) + \varepsilon U},\ \overline{F(t) + \varepsilon U}\big) + e\big(\overline{F(t) + \varepsilon U},\ G^\varepsilon_n(t_0)\big) \\ &\le e\big(F(t_0),\ F(t)\big) + e\big(\overline{F(t) + \varepsilon U},\ G^\varepsilon_n(t_0)\big) \\ &\le \sigma + e\big(\overline{F(t) + \varepsilon U},\ G^\varepsilon_n(t)\big), \end{aligned}$$

for G^ε_n is constant on I^n_i. On the other hand, by (i), $e\big(G^\varepsilon_n(t), \overline{F(t) + \varepsilon U}\big) = 0$ for each $t \in I$. Consequently, $h\big(G^\varepsilon_n(t), \overline{F(t) + \varepsilon U}\big) \ge h\big(G^\varepsilon_n(t_0), \overline{F(t_0) + \varepsilon U}\big) - \sigma$ for every $t \in U_I(t_0, \delta)$, and hence (ii) follows, as a lower semicontinuous function is measurable.

It remains to prove (iii). Let $t_0 \in I$ and $0 < \sigma < \varepsilon$ be arbitrary. Since F is h-l.s.c., there is $\delta > 0$ such that $t \in U_I(t_0, \delta)$ implies $F(t_0) \subset F(t) + \sigma U$. For every n large enough, say $n \geq n_0 \geq k$, there is $1 \leq i_n \leq 2^n$ such that $t_0 \in I^n_{i_n} \subset U_I(t_0, \delta)$. Thus for every $n \geq n_0$ and $s \in I^n_{i_n}$ we have $F(t_0) + (\varepsilon - \sigma)U \subset F(s) + \sigma U + (\varepsilon - \sigma)U = F(s) + \varepsilon U$, which implies

$$F(t_0) + (\varepsilon - \sigma)U \subset \bigcap_{s \in I^n_{i_n}} (\overline{F(s) + \varepsilon U}) = G^\varepsilon_n(t_0).$$

Hence for every $n \geq n_0$, $F(t_0) + \varepsilon U \subset G^\varepsilon_n(t_0) + \sigma U$. This and (i) imply $h\big(G^\varepsilon_n(t_0), \overline{F(t_0) + \varepsilon U}\big) \leq \sigma$ for every $n \geq n_0$, and thus (iii) is proved.

We are ready to show that F is Lusin measurable. For each $j \in \mathbb{N}$ consider the sequence $\{G^{\varepsilon_j}_n\}_{n \geq k_j}$, where $\varepsilon_j = 1/j$ and k_j corresponds to ε_j. Each $G^{\varepsilon_j}_n$ is piecewise constant, thus there is a compact set $H_\sigma \subset I$ independent of j and n, with $\mu(I \smallsetminus H_\sigma) < \sigma/2$, such that every $G^{\varepsilon_j}_n$ restricted to H_σ is h-continuous. In view of (ii) and (iii), with $\varepsilon = \varepsilon_j$, using Egoroff-Severini theorem, one can find a compact set $K_\sigma \subset H_\sigma$ independent of j, with $\mu(H_\sigma \smallsetminus K_\sigma) < \sigma/2$, such that for each fixed $j \in \mathbb{N}$ we have

$$h\big(G^{\varepsilon_j}_n(t),\ \overline{F(t) + \varepsilon_j U}\big) \to 0 \qquad \text{as } \ n \to +\infty,$$

uniformly on K_σ. Since each $G^{\varepsilon_j}_n$ restricted to K_σ is h-continuous and the convergence is uniform, one has that the multifunction $t \to F(t) + \varepsilon_j U$ restricted to K_σ is h-continuous. But the sequence of these multifunctions, as $j \to +\infty$, converges to F uniformly on K_σ, hence also F restricted to K_σ is h-continuous. Clearly $\mu(I \smallsetminus K_\sigma) < \sigma$. Therefore F is Lusin measurable, completing the proof. $\square$

3. A Filippov type existence theorem

In this section we prove a theorem on the existence of mild solutions for the Cauchy problem $(C_{a,F})$ in an arbitrary (not necessarily separable) Banach space, under assumptions on F of Filippov type ([4]).

About the operator A and the multifunction $F : I \times \mathbb{E} \to \mathcal{C}(\mathbb{E})$, $I = [0, 1]$, we shall use the following assumptions.

(H_1) A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \geq 0$, of bounded linear operators from $\mathbb{E}$ into itself.

(H_2) For each $x \in \mathbb{E}$, $t \to F(t, x)$ is Lusin measurable on I.

(H_3) There exists a summable function $k : I \to [0, +\infty[$ such that

$$h\big(F(t, x),\ F(t, y)\big) \leq k(t)\|x - y\| \qquad \text{for every } (t, x),\ (t, y) \in I \times \mathbb{E}.$$

(H_4) There exists a summable function $q : I \to [0, +\infty[$ such that $F(t, 0) \subset \tilde{U}_{\mathbb{E}}\big(0, q(t)\big)$, for all $t \in I$.

As is well known (see Pazy [15, p. 4]), under the assumption (H_1) there is a constant $M \geq 1$ such that

$$\|S(t)\| \leq M \qquad \text{for every} \ \ t \in I.$$

Furthermore, if (H_3) is satisfied, we denote by $m : I \to [0, +\infty[$ the function given by

$$m(t) = \int_0^t k(s)\, ds.$$

Given a multifunction G defined on $I \times \mathbb{E}$ with nonempty values $G(t, x) \subset \mathbb{E}$, consider the Cauchy problem $(C_{a,G})$. By *mild solution* of the Cauchy problem $(C_{a,G})$ we mean a function $x : I \to \mathbb{E}$ satisfying the following conditions: (i) x is continuous on I with $x(0) = a$, (ii) there is a Lusin measurable function $v : I \to \mathbb{E}$ integrable in the sense of Bochner such that:

$$\begin{aligned} &v(t) \in G\big(t, x(t)\big) && \text{for each} \ \ t \in I \\ &x(t) = S(t)a + \int_0^t S(t-s)v(s)\, ds && \text{for each} \ \ t \in I. \end{aligned}$$

Remark 3.1. In the above definition the requirement that "$v : I \to \mathbb{E}$ is Lusin measurable" is equivalent to "$v : I \to \mathbb{E}$ is strongly measurable" (in the sense of Hille and Phillips [7, p. 72]). In fact if v is Lusin measurable then, by a standard iterative procedure one can easily construct a sequence of countably-valued functions converging to v a.e. in I, thus v is strongly measurable. Conversely, if v is strongly measurable then, by Hille and Phillips [7, Corollary 1, p. 73], v is the uniform limit a.e. of a sequence of countably valued functions, from which it follows that v is Lusin measurable.

Lemma 3.1. *Let $F_i : I \to \mathcal{P}(\mathbb{E})$, $i = 1, 2$, be two Lusin measurable multifunctions and let $\varepsilon_1, \varepsilon_2 > 0$ be such that*

$$G(t) = \big(F_1(t) + \varepsilon_1 U\big) \cap \big(F_2(t) + \varepsilon_2 U\big) \neq \emptyset \qquad \textit{for every} \ \ t \in I. \tag{3.1}$$

Then the multifunction $G : I \to \mathcal{P}(\mathbb{E})$ defined by (3.1) *has a Lusin measurable selector $v : I \to \mathbb{E}$*

Proof: Since F_1 and F_2 are Lusin measurable, one can construct a sequence $\{J_n\}$ of pairwise disjoint compact sets $J_n \subset I$ satisfying, for each $n \in \mathbb{N}$, the following properties:

(i) F_1 and F_2 restricted to J_n are h-continuous;

(ii) $J_{n+1} \subset I \setminus \bigcup_{i=1}^n J_i$;

(iii) $\mu\,(I \smallsetminus \bigcup_{i=1}^{n} J_i) < 1/2^n$.

Set $J_0 = I \smallsetminus \bigcup_n J_n$ and observe that, by (iii), $\mu(J_0) = 0$. It is evident that $\{J_n\}_{n\geq 0}$ is a partition of I.

We claim that for each $n = 0, 1, \ldots$ there is a Lusin measurable function $v_n : J_n \to \mathbb{E}$ which is a selector of the multifunction G restricted to J_n. To show this, fix an arbitrary $n \in \mathbb{N}$ (the case $n = 0$ is trivial). For each $t \in J_n$ pick out a point $u_t \in G(t)$. Since $G(t)$ is open and F_1 and F_2 restricted to J_n are h-continuous, there is a $\delta_t > 0$ such that

$$u_t \in \big(F_1(s) + \varepsilon_1 U\big) \cap \big(F_2(s) + \varepsilon_2 U\big) \qquad \text{for every } \; s \in U_{J_n}(t, \delta_t). \tag{3.2}$$

The family $\big\{U_{J_n}(t, \delta_t)\big\}_{t \in J_n}$ is an open covering of J_n. As J_n is compact, it admits a finite subcovering, say $\big\{U_{J_n}(t_k, \delta_{t_k})\big\}_{k=1}^{q}$. Now, consider the partition $\{I_k\}_{k=1}^{q}$ of J_n given by

$$I_1 = U_{J_n}(t_1, \delta_{t_1}) \qquad I_k = U_{J_n}(t_k, \delta_{t_k}) \smallsetminus \bigcup_{i=1}^{k-1} I_i \qquad\qquad 2 \leq k \leq q,$$

and define $v_n : J_n \to \mathbb{E}$ by

$$v_n(t) = \sum_{k=1}^{q} u_{t_k} \chi_{I_k}(t).$$

It is evident that v_n is Lusin measurable. Further, v_n is a selector of the multifunction G restricted to J_n. In fact let $s \in J_n$ be arbitrary, thus $s \in I_k$ for some $1 \leq k \leq q$. Since $s \in I_k \subset U_{J_n}(t_k, \delta_{t_k})$, in view of (3.2) (with $t = t_k$) we have

$$u_{t_k} \in \big(F_1(s) + \varepsilon_1 U\big) \cap \big(F_2(s) + \varepsilon_2 U\big),$$

thus $v_n(s) \in G(s)$, for $v_n(s) = u_{t_k}$. Hence v_n is a Lusin measurable selector of G restricted to J_n. Then the function $v : I \to \mathbb{E}$ given by

$$v(t) = \sum_{n \geq 0} v_n(t) \chi_{J_n}(t)$$

is a Lusin measurable selector of G, completing the proof. □

Lemma 3.2. *Let $F : I \times \mathbb{E} \to \mathcal{C}(\mathbb{E})$ satisfy the hypotheses (H_2) and (H_3). Then for arbitrary $x : I \to \mathbb{E}$ continuous, $u : I \to \mathbb{E}$ Lusin measurable, and $\varepsilon > 0$ we have:*

(a$_1$) *the multifunction $t \to F(t, x(t))$ is Lusin measurable on I;*

(a$_2$) *the multifunction $G : I \to \mathcal{P}(\mathbb{E})$ defined by*

$$G(t) = \big(F(t, x(t)) + \varepsilon U\big) \cap U_{\mathbb{E}}\big(u(t),\; d(u(t),\; F(t, x(t))) + \varepsilon\big)$$

has a Lusin measurable selector $v : I \to \mathbb{E}$.

Proof: (a$_1$) Let $\{x_n\}$ be a sequence of piecewise constant functions $x_n : I \to \mathbb{E}$ converging to x uniformly on I. Given $\varepsilon > 0$, let $K_\varepsilon \subset I$ be a compact set, with $\mu(I \smallsetminus K_\varepsilon) < \varepsilon$, such that k restricted to K_ε is continuous and, for each $n \in \mathbb{N}$, the multifunction $t \to F\big(t, x_n(t)\big)$ restricted to K_ε is h-continuous. Set $M_\varepsilon = \sup_{t\in K_\varepsilon} k(t)$.

Let t_0, $t \in K_\varepsilon$ be arbitrary. We have:

$$\begin{aligned} &h\big(F(t,x(t)),\ F(t_0,\ x(t_0))\big) \\ &\quad\le h\big(F(t,x(t)),\ F(t,x_n(t))\big) + h\big(F(t,x_n(t)),\ F(t_0,x_n(t_0))\big) \\ &\qquad\qquad\qquad + h\big(F(t_0,x_n(t_0)),\ F(t_0,x(t_0))\big) \\ &\quad\le M_\varepsilon\|x_n(t)-x(t)\| + h\big(F(t,x_n(t)),\ F(t_0,x_n(t_0))\big) + M_\varepsilon\|x_n(t_0)-x(t_0)\| \\ &\quad\le 2M_\varepsilon\sigma_n + h\big(F(t,x_n(t)),\ F(t_0,x_n(t_0))\big), \end{aligned}$$

where $\sigma_n = \sup_{t\in I}\|x_n(t)-x(t)\|$. Since $\sigma_n \to 0$ as $n \to +\infty$, and $t \to F\big(t,x_n(t)\big)$ restricted to K_ε is h-continuous, the multifunction $t \to F\big(t,x(t)\big)$ restricted to K_ε is h-continuous, and (a$_1$) is proved.

(a$_2$) For $t \in I$ set $G_1(t) = F\big(t,x(t)\big)$, $G_2(t) = \tilde{U}_{\mathbb{E}}\big(u(t),\ d(u(t),\ G_1(t))\big)$, and observe that G_1 and G_2 are Lusin measurable on I. Furthermore, for each $t \in I$ we have $G(t) = \big(G_1(t)+\varepsilon U\big)\cap\big(G_2(t)+\varepsilon U\big)$ and $G(t) \neq \emptyset$. Hence, by Lemma 3.1, G has a Lusin measurable selector $v : I \to \mathbb{E}$, thus also (a$_2$) holds, and the proof is complete. □

Theorem 3.1. *If (H_1)–(H_4) are satisfied, then for every $a \in \mathbb{E}$ the Cauchy problem $(C_{a,F})$ has a mild solution $x : I \to \mathbb{E}$.*

Proof: We will adapt a construction due to Filippov [4]. First we observe that if $z : I \to \mathbb{E}$ is continuous, then every Lusin measurable selector $u : I \to \mathbb{E}$ of the multifunction $t \to F\big(t,z(t)\big) + U$ is Bochner integrable on I. In fact, for each $t \in I$ we have

$$\|u(t)\| \le h\big(F(t,z(t))+U,\ 0\big) \le h\big(F(t,z(t)),\ F(t,0)\big) + h\big(F(t,0),\ 0\big) + 1$$

and hence, in view of (H_3) and (H_4),

$$\|u(t)\| \le k(t)\|z(t)\| + q(t) + 1, \qquad t \in I. \tag{3.3}$$

By Hille and Phillips [7, Theorem 3.7.4, p. 80], in view of Remark 3.1 and the above inequality (3.3), if follows that u is Bochner integrable on I.

Let $0 < \varepsilon < 1$ and, for $n \ge 0$, set $\varepsilon_n = \varepsilon/2^{n+2}$. Define $x_0 : I \to \mathbb{E}$ by

$$x_0(t) = S(t)a + \int_0^t S(t-s)v_0(s)\,ds, \tag{3.4}$$

where $v_0 : I \to \mathbb{E}$ is an arbitrary Lusin measurable function, integrable in the sense of Bochner. Since x_0 is continuous, by Lemma 3.2 there exists a Lusin measurable function, say $v_1 : I \to \mathbb{E}$, satisfying

$$v_1(t) \in \big(F(t, x_0(t)) + \varepsilon_1 U\big) \cap U_{\mathbb{E}}\big(v_0(t),\ d(v_0(t),\ F(t, x_0(t))) + \varepsilon_1\big) \qquad t \in I.$$

Clearly, by (3.3), v_1 is also Bochner integrable on I. Define $x_1 : I \to \mathbb{E}$ by

$$x_1(t) = S(t)a + \int_0^t S(t-s)v_1(s)\, ds.$$

Now by recurrence one can construct a sequence $\{x_n\}$ of continuous functions $x_n : I \to \mathbb{E}$, $n = 1, 2, \dots$, given by

$$x_n(t) = S(t)a + \int_0^t S(t-s)v_n(s)\, ds, \tag{3.5$_n$}$$

where $v_n : I \to \mathbb{E}$ is a Lusin measurable function satisfying

$(3.6)_n$

$$v_n(t) \in \big(F(t, x_{n-1}(t)) + \varepsilon_n U\big) \cap U_{\mathbb{E}}\big(v_{n-1}(t),\ d(v_{n-1}(t),\ F(t, x_{n-1}(t)))+ \varepsilon_n\big) \qquad t \in I.$$

Furthermore v_n is also Bochner integrable on I because, by $(3.6)_n$ and (3.3), we have

$$\|v_n(t)\| \le k(t)\|x_{n-1}(t)\| + q(t) + 1, \qquad t \in I. \tag{3.7}$$

Now from $(3.6)_n$, for $n = 2, 3, \dots$ and $t \in I$ we have

$$\begin{aligned}\|v_n(t) - v_{n-1}(t)\| &\le d\big(v_{n-1}(t),\ F(t, x_{n-1}(t))\big) + \varepsilon_n \\ &\le d\big(v_{n-1}(t),\ F(t, x_{n-2}(t))\big) \\ &\qquad + h(F(t, x_{n-2}(t)),\ F(t, x_{n-1}(t))\big) + \varepsilon_n \\ &\le \varepsilon_{n-1} + k(t)\|x_{n-1}(t) - x_{n-2}(t)\| + \varepsilon_n.\end{aligned}$$

Hence, for each $n = 2, 3, \dots$ and $t \in I$,

$$\|v_n(t) - v_{n-1}(t)\| \le \varepsilon_{n-2} + k(t)\|x_{n-1}(t) - x_{n-2}(t)\|, \tag{3.8$_n$}$$

as $\varepsilon_{n-1} + \varepsilon_n < \varepsilon_{n-2}$. Set $p_0(t) = d\big(v_0(t),\ F(t, x_0(t))\big)$, $t \in I$.

We claim that for each $n = 2, 3, \dots$ and $t \in I$ we have:

$$\begin{aligned}\|x_n(t) - x_{n-1}(t)\| \le \sum_{k=0}^{n-2} \int_0^t \varepsilon_{n-2-k} \frac{M^{k+1}\big(m(t) - m(u)\big)^k}{k!}\, du \\ + \varepsilon_0 \int_0^t \frac{M^n\big(m(t) - m(u)\big)^{n-1}}{(n-1)!}\, du + \int_0^t \frac{M^n\big(m(t) - m(u)\big)^{n-1}}{(n-1)!} p_0(u)\, du.\end{aligned} \tag{3.9$_n$}$$

First we verify the above inequality when $n = 2$. In view of $(3.5)_n$, $(3.8)_n$, (3.4) and $(3.6)_n$, for each $t \in I$ we have:

$$\begin{aligned}
\|x_2(t) - x_1(t)\| &\leq \int_0^t \|S(t-s)\|\|v_2(s) - v_1(s)\|\, ds \\
&\leq \int_0^t M\big[\varepsilon_0 + k(s)\|x_1(s) - x_0(s)\|\big]\, ds \\
&\leq \varepsilon_0 Mt + \int_0^t \left[Mk(s)\int_0^s \|S(s-u)\|\|v_1(u) - v_0(u)\|\, du\right] ds \\
&\leq \varepsilon_0 Mt + \int_0^t \left[M^2 k(s)\int_0^s \big(p_0(u) + \varepsilon_1\big)\, du\right] ds \\
&\leq \varepsilon_0 Mt + \int_0^t \left[M^2\big(p_0(u) + \varepsilon_0\big)\int_u^t k(s)\, ds\right] du \\
&= \varepsilon_0 Mt + \varepsilon_0 \int_0^t M^2\big(m(t) - m(u)\big)\, du \\
&\qquad + \int_0^t M^2\big(m(t) - m(u)\big)p_0(u)\, du,
\end{aligned}$$

and so $(3.9)_2$ is verified.

Now, assuming $(3.9)_n$, we shall show that $(3.9)_{n+1}$ holds. In view of $(3.8)_n$ and $(3.9)_n$, for each $t \in I$ we have:

$$\begin{aligned}
\|x_{n+1}(t) - x_n(t)\| &\leq \int_0^t \|S(t-s)\|\|v_{n+1}(s) - v_n(s)\|\, ds \\
&\leq \int_0^t M\Big[\varepsilon_{n-1} + k(s)\|x_n(s) - x_{n-1}(s)\|\Big] ds \\
&\leq \varepsilon_{n-1} Mt + \int_0^t k(s)\left[\sum_{k=0}^{n-2}\int_0^s \varepsilon_{n-2-k}\frac{M^{k+2}\big(m(s)-m(u)\big)^k}{k!}\, du\right. \\
&\qquad + \varepsilon_0 \int_0^s \frac{M^{n+1}\big(m(s)-m(u)\big)^{n-1}}{(n-1)!}\, du \\
&\qquad \left. + \int_0^s \frac{M^{n+1}\big(m(s)-m(u)\big)^{n-1}}{(n-1)!} p_0(u)\, du\right] ds \\
&= \varepsilon_{n-1} Mt + \sum_{k=0}^{n-2}\int_0^t \left[\int_0^s \varepsilon_{n-2-k}\frac{M^{k+2}\big(m(s)-m(u)\big)^k}{k!} k(s)\, du\right] ds \\
&\qquad + \varepsilon_0 \int_0^t \left[\int_0^s \frac{M^{n+1}\big(m(s)-m(u)\big)^{n-1}}{(n-1)!} k(s)\, du\right] ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\left[\int_0^s \frac{M^{n+1}\big(m(s)-m(u)\big)^{n-1}}{(n-1)!}k(s)p_0(u)\,du\right]ds\\
&=\varepsilon_{n-1}Mt+\sum_{k=0}^{n-2}\int_0^t\left[\int_u^t \varepsilon_{n-2-k}\frac{M^{k+2}\big(m(s)-m(u)\big)^{k}}{k!}k(s)\right]du\\
&\quad+\varepsilon_0\int_0^t\left[\int_u^t \frac{M^{n+1}\big(m(s)-m(u)\big)^{n-1}}{(n-1)!}k(s)\,ds\right]du\\
&\quad+\int_0^t\left[\int_u^t \frac{M^{n+1}\big(m(s)-m(u)\big)^{n-1}}{(n-1)!}k(s)\,ds\right]p_0(u)\,du\\
&=\varepsilon_{n-1}Mt+\sum_{k=0}^{n-2}\int_0^t \varepsilon_{n-2-k}\frac{M^{k+2}\big(m(t)-m(u)\big)^{k+1}}{(k+1)!}\,du\\
&\quad+\varepsilon_0\int_0^t \frac{M^{n+1}\big(m(t)-m(u)\big)^{n}}{n!}\,du\\
&\quad+\int_0^t \frac{M^{n+1}\big(m(t)-m(u)\big)^{n}}{n!}p_0(u)\,du\\
&=\sum_{k=0}^{n-1}\int_0^t \varepsilon_{n-1-k}\frac{M^{k+1}\big(m(t)-m(u)\big)^{k}}{k!}\,du\\
&\quad+\varepsilon_0\int_0^t \frac{M^{n+1}\big(m(t)-m(u)\big)^{n}}{n!}\,du\\
&\quad+\int_0^t \frac{M^{n+1}\big(m(t)-m(u)\big)^{n}}{n!}p_0(u)\,du.
\end{aligned}
$$

Thus $(3.9)_{n+1}$ holds true, and the claim is proved.

Now from $(3.9)_n$, for $n=2,3,\dots$ and every $t\in I$, we have

$$\|x_n(t)-x_{n-1}(t)\|\le a_n, \tag{3.10}$$

where

$$a_n=\sum_{k=0}^{n-2}\varepsilon_{n-2-k}\frac{M^{k+1}L^k}{k!}+\varepsilon_0\frac{M^nL^{n-1}}{(n-1)!}+\frac{M^nL^{n-1}}{(n-1)!}\int_0^1 p_0(u)\,du \tag{3.11}$$

and $L=m(1)$.

Clearly the series whose nth term is the first quantity on the right side of (3.11) is convergent, as Cauchy product of absolutely convergent series. Thus the series whose nth term is a_n converges as well. From this and (3.10) it follows that the sequence $\{x_n\}$ converges uniformly on I to a continuous function, say $x : I \to \mathbb{E}$. On the other hand, in view of $(3.8)_n$, for $n = 3, 4, \dots$ and every $t \in I$

$$\|v_n(t) - v_{n-1}(t)\| \le \varepsilon_{n-2} + k(t)a_{n-1},$$

which implies that $\{v_n\}$ converges on I to a Lusin measurable function, say $v : I \to \mathbb{E}$. As $\{x_n\}$ is bounded by a constant, say H, (3.7) yields $\|v_n(t)\| \le k(t)H + q(t) + 1$ for $n = 1, 2, \dots$ and each $t \in I$, and hence v is also Bochner integrable on I. Then from $(3.5)_n$, letting $n \to +\infty$ and using Lebesgue dominated convergence theorem, we obtain

$$x(t) = S(t)a + \int_0^t S(t-s)v(s)\,ds \qquad \text{for each } \ t \in I.$$

On the other hand, by $(3.6)_n$, $v_n(t) \in F\big(t, x_{n-1}(t)\big) + \varepsilon_n U$ for $n = 1, 2, \dots$ and $t \in I$, whence letting $n \to +\infty$ we have

$$v(t) \in F\big(t, x(t)\big) \qquad \text{for each } \ t \in I.$$

Therefore x is a mild solution of the Cauchy problem $(C_{a,F})$. This completes the proof. □

When $A = 0$ the Cauchy problem $(C_{a,F})$ takes the form

$$(D_{a,F}) \qquad \begin{cases} x'(t) \in F\big(t, x(t)\big) \\ x(0) = a. \end{cases}$$

By *solution* of the Cauchy problem $(D_{a,F})$ we mean a continuous function $x : I \to \mathbb{E}$ such that there exists a Lusin measurable function $v : I \to \mathbb{E}$, integrable in the sense of Bochner, satisfying:

$$v(t) \in F\big(t, x(t)\big) \qquad \text{for each } \ t \in I$$

$$x(t) = a + \int_0^t v(s)\,ds \qquad \text{for each } \ t \in I.$$

When $A = 0$, Theorem 3.1 yields the following:

Corollary 3.1. *If (H_2)–(H_4) are satisfied, then for every $a \in \mathbb{E}$ the Cauchy problem $(D_{a,F})$ has a solution $x : I \to \mathbb{E}$.*

4. A relaxation theorem

In this section we prove a relaxation theorem for the Cauchy problem $(C_{a,F})$. More precisely, we associate to $(C_{a,F})$ the convexified Cauchy problem

$$(C_{a,\overline{\text{co}}\,F}) \qquad \begin{cases} x'(t) \in Ax(t) + \overline{\text{co}}\,F\big(t, x(t)\big) \\ x(0) = a, \end{cases}$$

and we show that, if (H_1)–(H_4) are satisfied, then the set of the mild solutions of $(C_{a,F})$ is dense in the set of the mild solutions of $(C_{a,\overline{\text{co}}\,F})$.

Lemma 4.1. *Let $G : A \to \mathcal{C}(\mathbb{E})$ be a Lusin measurable multifunction defined on a measurable set $A \subset \mathbb{R}$, with $\mu(A) < +\infty$. Then G has a Lusin measurable selector $g : A \to \mathbb{E}$.*

Proof: By virtue of [3], Propositions 6 and 4, the statement holds true if A is compact. If A is measurable, it suffices to consider a countable partition $\{K_n\}_{n\geq 0}$ of A, where all K_n, $n \geq 1$, are compact and K_0 is of measure zero. □

The following lemma plays a crucial role in the proof of the relaxation theorem.

Lemma 4.2. *Let (H_1)–(H_4) be satisfied. Let $a \in \mathbb{E}$, and let $y : I \to \mathbb{E}$ be a mild solution of the convexified Cauchy problem $(C_{a,\overline{\mathrm{co}}\,F})$. Then given $0 < \varepsilon < 1$, there is a mild solution $x_0 : I \to \mathbb{E}$ of the Cauchy problem*

$$(C_{a,F+\varphi_\varepsilon U}) \qquad \begin{cases} x'(t) \in Ax(t) + F\big(t, x(t)\big) + \varphi_\varepsilon(t)U \\ x(0) = a, \end{cases}$$

where $\varphi_\varepsilon(t) = \varepsilon\,[k(t)/(L+1) + 1]$ and $L = \int_0^1 k(s)\,ds$, such that $\|x_0(t) - y(t)\| \leq \varepsilon/(L+1) \leq \varepsilon$ for every $t \in I$.

Proof: The proof, rather long, will be divided into four steps.

By hypothesis $y : I \to \mathbb{E}$ is a mild solution of $(C_{a,\overline{\mathrm{co}}\,F})$. Thus y is continuous, and there is a Lusin measurable function $u : I \to \mathbb{E}$, integrable in the sense of Bochner, satisfying

$$(4.1) \qquad u(t) \in \overline{\mathrm{co}}\,F\big(t, y(t)\big) \qquad t \in I$$

$$(4.2) \qquad y(t) = S(t)a + \int_0^t S(t-s)u(s)\,ds \qquad t \in I.$$

Let $\varepsilon > 0$. Our aim is to construct a Lusin measurable function $v_0 : I \to \mathbb{E}$ integrable in the sense of Bochner, and a continuous function $x_0 : I \to \mathbb{E}$ satisfying

$$v_0(t) \in F\big(t, x_0(t)\big) + \varphi_\varepsilon(t)U \qquad t \in I$$

$$x_0(t) = S(t)a + \int_0^t S(t-s)v_0(s)\,ds \qquad t \in I,$$

such that $\|x_0(t) - y(t)\| \leq \varepsilon/(L+1)$ for every $t \in I$.

Step 1. *Construction of v_0 and x_0.*

Let $0 < \varepsilon < 1$ be arbitrary. Fix δ such that

$$(4.3) \qquad 0 < \delta < \frac{\varepsilon}{4(M+1)^2(L+1)},$$

where $M \geq 1$ is a constant satisfying $\|S(t)\| \leq M$ for every $t \in I$. Clearly $\delta < \varepsilon < 1$. Likewise in the proof of Theorem 3.1, one can show that each Lusin measurable selector $w : I \to \mathbb{E}$ of the multifunction $t \to \overline{\mathrm{co}}\,F\big(t, y(t)\big) + \delta U$ satisfies

$$(4.4) \qquad \|w(t)\| \leq \psi(t) \qquad t \in I,$$

where $\psi(t) = k(t)\|y(t)\| + q(t) + 1$. As ψ is summable, w is Bochner integrable on I.

Take $\alpha > 0$ such that for each measurable set $A \subset I$,

$$\mu(A) < \alpha \quad \text{implies} \quad \int_A \psi(t)\,dt < \delta. \tag{4.5}$$

The mappings $t \to u(t)$ and $t \to F\big(t, y(t)\big)$ are Lusin measurable, the latter by Lemma 3.2, thus there is a compact set $K \subset I$, with

$$\mu(I \setminus K) < \alpha, \tag{4.6}$$

such that, when restricted to K, u is continuous and $t \to F\big(t, y(t)\big)$ is h-continuous.

For $N \in \mathbb{N}$, denote by $\{I_i\}_{i=1}^N$ the partition of I given by

$$I_i = [t_{i-1}, t_i[\qquad i = 1, \dots, N-1 \qquad I_N = [t_{N-1}, t_N] \qquad \text{where } t_i = \frac{i}{N}\,.$$

Now fix $N \in \mathbb{N}$ large enough so that for each $i = 1, \dots, N$ we have: $\mu(I_i) < \alpha$ and, furthermore,

$$\|u(t') - u(t'')\| < \delta \text{ and } h\big(F(t', y(t')),\ F(t'', y(t''))\big) < \delta, \qquad \text{for every } \ t', t'' \in I_i \cap K. \tag{4.7}$$

Set $\Im' = \{1 \le i \le N | I_i \cap K \ne \emptyset\}$ and $\Im'' = \{1 \le i \le N | I_i \cap K = \emptyset\}$. In each interval I_i, with $i \in \Im'$, choose a point $\tau_i \in I_i \cap K$. Since $u(\tau_i) \in \overline{\text{co}}\, F\big(\tau_i, y(\tau_i)\big)$, there exists a finite set $\{e_n^i\}_{n=1}^{p_i}$ of points

$$e_n^i \in F\big(\tau_i, y(\tau_i)\big) \qquad n = 1, \dots, p_i, \tag{4.8}$$

and there exist p_i numbers $\lambda_n^i \ge 0$, with $\lambda_1^i + \dots + \lambda_{p_i}^i = 1$, such that

$$\|u(\tau_i) - \sum_{n=1}^{p_i} \lambda_n^i e_n^i\| < \delta. \tag{4.9}$$

By Pazy [15, Corollary 2.3, p. 4], for each $i \in \Im'$ the functions $t \to S(t_i - t)u(\tau_i)$ and $t \to S(t_i - t)e_n^i$, $n = 1, \dots, p_i$, are continuous on the compact interval $\overline{I}_i$. Consequently, for each $i \in \Im'$ we can construct a partition $\{J_j^i\}_{j=1}^{r_i}$ of I_i, where

$$J_j^i = [s_{j-1}^i, s_j^i[\quad j = 1, \dots, r_i, \qquad \text{and} \ \ s_j^i = t_{i-1} + \frac{j}{r_i N}\,,$$

(if $i = N$, $J^N_{r_N}$ is closed) so that the following inequalities are satisfied:

$$\|S(t_i - t)u(\tau_i) - \sum_{j=1}^{r_i} S(t_i - s^i_j)u(\tau_i)\chi_{J^i_j}(t)\| \leq \delta \quad \text{for each} \quad t \in I_i,\ i \in \Im' \tag{4.10}$$

$$\|S(t_i - t)e^i_n - \sum_{j=1}^{r_i} S(t_i - s^i_j)e^i_n\chi_{J^i_j}(t)\| \leq \delta \quad \text{for each} \quad t \in I_i, i \in \Im' \quad n = 1, \dots, p_i. \tag{4.11}$$

Furthermore, for $i \in \Im'$ and $1 \leq j \leq r_i$ consider a partition $\{K^{ij}_n\}^{p_i}_{n=1}$ of $J^i_j \cap K$ by measurable sets K^{ij}_n such that

$$\mu(K^{ij}_n) = \lambda^i_n \mu(J^i_j \cap K) \qquad n = 1, \dots, p_i. \tag{4.12}$$

By Lemma 4.1, the multifunction $t \to F\big(t, y(t)\big)$ restricted to $I \setminus K$ admits a Lusin measurable selector, say $w_0 : I \setminus K \to \mathbb{E}$. Moreover, for each $i \in \Im'$, denote by $v_i : I_i \cap K \to \mathbb{E}$ the function given by

$$v_i(t) = \sum_{j=1}^{r_i} \sum_{n=1}^{p_i} e^i_n \chi_{K^{ij}_n}(t).$$

Now define $v_0 : I \to \mathbb{E}$ and $x_0 : I \to \mathbb{E}$ by

$$v_0(t) = \sum_{i \in \Im'} v_i(t)\chi_{I_i \cap K}(t) + w_0(t)\chi_{I \setminus K}(t) \qquad t \in I \tag{4.13}$$

$$x_0(t) = S(t)a + \int_0^t S(t-s)v_0(s)\,ds \qquad t \in I. \tag{4.14}$$

Clearly v_0 is Lusin measurable, and also Bochner integrable, because

$$v_0(t) \in F\big(t, y(t)\big) + \delta U \qquad t \in I. \tag{4.15}$$

To show (4.15) let $t \in I$ be arbitrary, thus $t \in I_i$, for some $1 \leq i \leq N$. If $t \in I \setminus K$, we have $v_0(t) = w_0(t) \in F\big(t, y(t)\big)$. If $t \in I_i \cap K$, then $t \in K^{ij}_n$ for some $1 \leq j \leq r_i$ and $1 \leq n \leq p_i$, hence $v_0(t) = e^i_n \chi_{K^{ij}_n}(t) = e^i_n \in F\big(\tau_i, y(\tau_i)\big)$, by (4.8). Since t, $\tau_i \in I_i \cap K$, (4.7) implies $F\big(\tau_i, y(\tau_i)\big) \subset F\big(t, y(t)\big) + \delta U$. Whence if $t \in I_i \cap K$, we have $v_0(t) \in F\big(t, y(t)\big) + \delta U$ and (4.15) is proved.

Step 2. *For each $i \in \Im'$ we have*:

$$\left\| \int_{I_i \cap K} S(t_i - s)u(s)\,ds - \int_{I_i \cap K} S(t_i - s)v_0(s)\,ds \right\| \leq 2(M+1)\delta\mu(I_i). \tag{4.16}$$

Denoting by Λ_i the quantity on the left side of (4.16), we have

$$\begin{aligned}\Lambda_i \le {}& \left\| \int_{I_i\cap K} S(t_i-s)u(s)\,ds - \int_{I_i\cap K} S(t_i-s)u(\tau_i)\,ds \right\| \\ & + \left\| \int_{I_i\cap K} S(t_i-s)u(\tau_i)\,ds - \sum_{j=1}^{r_i} \int_{J_j^i\cap K} S(t_i-s_j^i)u(\tau_i)\,ds \right\| \\ & + \left\| \sum_{j=1}^{r_i} \int_{J_j^i\cap K} S(t_i-s_j^i)u(\tau_i)\,ds - \sum_{j=1}^{r_i}\sum_{n=1}^{p_i} \int_{K_n^{ij}} S(t_i-s_j^i)v_i(s)\,ds \right\| \\ & + \left\| \sum_{j=1}^{r_i}\sum_{n=1}^{p_i} \int_{K_n^{ij}} S(t_i-s_j^i)v_i(s)\,ds - \int_{I_i\cap K} S(t_i-s)v_0(s)\,ds \right\|. \end{aligned} \tag{4.17}$$

Let $\Lambda_{i,\ldots}^I, \Lambda_i^{IV}$ be the first, ... , fourth term on the right side of (4.17). Clearly, by virtue of (4.7), we have

$$\Lambda_i^I \le \int_{I_i\cap K} \|S(t_i-s)\|\|u(s)-u(\tau_i)\|\,ds \le M\delta\mu(I_i). \tag{4.18}$$

Further,

$$\Lambda_i^{II} = \left\| \int_{I_i\cap K} S(t_i-s)u(\tau_i)\,ds - \int_{I_i\cap K}\left[\sum_{j=1}^{r_i} S(t_i-s_j^i)u(\tau_i)\chi_{J_j^i}(s)\right]ds \right\|$$

and so, by (4.10), we have

$$\Lambda_i^{II} \le \int_{I_i\cap K} \left\| S(t_i-s)u(\tau_i) - \sum_{j=1}^{r_i} S(t_i-s_j^i)u(\tau_i)\chi_{J_j^i}(s) \right\| ds \le \delta\mu(I_i). \tag{4.19}$$

As far as Λ_i^{III} is concerned we have:

$$\begin{aligned}\Lambda_i^{III} \le {}& \left\| \sum_{j=1}^{r_i} \int_{J_j^i\cap K} S(t_i-s_j^i)u(\tau_i)\,ds - \sum_{j=1}^{r_i} S(t_i-s_j^i)\sum_{n=1}^{p_i}\lambda_n^i e_n^i \mu(J_j^i\cap K) \right\| \\ & + \left\| \sum_{j=1}^{r_i} S(t_i-s_j^i)\sum_{n=1}^{p_i}\lambda_n^i e_n^i \mu(J_j^i\cap K) - \sum_{j=1}^{r_i}\sum_{n=1}^{p_i}\int_{K_n^{ij}} S(t_i-s_j^i)v_i(s)\,ds \right\| \\ \le {}& \left\| \sum_{j=1}^{r_i} S(t_i-s_j^i)\left(u(\tau_i) - \sum_{n=1}^{p_i}\lambda_n^i e_n^i\right)\mu(J_j^i\cap K) \right\| \\ & + \left\| \sum_{j=1}^{r_i}\sum_{n=1}^{p_i} S(t_i-s_j^i)\lambda_n^i e_n^i \mu(J_j^i\cap K) - \sum_{j=1}^{r_i}\sum_{n=1}^{p_i} S(t_i-s_j^i)e_n^i\mu(K_n^{ij}) \right\|. \end{aligned}$$

The last term on the right side of the above inequality is zero because, by (4.12), $\mu(K_n^{ij}) = \lambda_n^i \mu(J_j^i \cap K)$ for every $j = 1, \dots, r_i$ and $n = 1, \dots, p_i$. Thus, in view of (4.9), it follows:

$$\begin{aligned}\Lambda_i^{III} &\le \sum_{j=1}^{r_i} \|S(t_i - s_j^i)\| \|u(\tau_i) - \sum_{n=1}^{p_i} \lambda_n^i e_n^i\| \mu(J_j^i \cap K) \\ &\le M\delta \sum_{j=1}^{r_i} \mu(J_j^i \cap K) \le M\delta\mu(I_i).\end{aligned} \tag{4.20}$$

It remains to evaluate Λ_i^{IV}. Taking into account the definition of v_0, we have

$$\begin{aligned}&\int_{I_i \cap K} S(t_i - s) v_0(s)\, ds - \sum_{j=1}^{r_i} \sum_{n=1}^{p_i} \int_{K_n^{ij}} S(t_i - s_j^i) v_i(s)\, ds \\ &= \sum_{n=1}^{p_i} \sum_{j=1}^{r_i} \int_{K_n^{ij}} S(t_i - s) e_n^i\, ds - \sum_{n=1}^{p_i} \sum_{j=1}^{r_i} \int_{K_n^{ij}} S(t_i - s_j^i) e_n^i\, ds \\ &= \sum_{n=1}^{p_i} \int_{K_n^i} S(t_i - s) e_n^i\, ds - \sum_{n=1}^{p_i} \int_{K_n^i} \left[\sum_{j=1}^{r_i} S(t_i - s_j^i) e_n^i \chi_{K_n^{ij}}(s) \right] ds,\end{aligned}$$

where $K_n^i = \bigcup_{j=1}^{r_i} K_n^{ij}$. Thus

$$\Lambda_i^{IV} \le \sum_{n=1}^{p_i} \int_{K_n^i} \left\| S(t_i - s) e_n^i - \sum_{j=1}^{r_i} S(t_i - s_j^i) e_n^i \chi_{K_n^{ij}}(s) \right\| ds.$$

Now each $s \in K_n^i$ is in one set, say K_n^{ij}, for some $1 \le j \le r_i$, and thus $s \in J_j^i$. Hence by (4.11)

$$\Lambda_i^{IV} \le \delta \sum_{n=1}^{p_i} \mu(K_n^i) \le \delta\mu(I_i). \tag{4.21}$$

From (4.17), by virtue of (4.18)–(4.21), it follows

$$\Lambda_i \le M\delta\mu(I_i) + \delta\mu(I_i) + M\delta\mu(I_i) + \delta\mu(I_i) = 2(M+1)\delta\mu(I_i),$$

and Step 2 is proved.

Step 3. *We have* $\|x_0(t) - y(t)\| \le \varepsilon/(L+1)$ *for every* $t \in I$.

Let $t \in I$ be arbitrary, thus $t \in I_h$ for some $1 \le h \le N$. By virtue of (4.14) and (4.2) we have

$$\begin{aligned}
\|x_0(t) - y(t)\| &\le \left\| \int_0^{t_{h-1}} S(t-s)\big(v_0(s) - u(s)\big)\, ds \right\| \\
&\quad + \left\| \int_{t_{h-1}}^{t} S(t-s)\big(v_0(s) - u(s)\big)\, ds \right\| \\
&\le \left\| \sum_{i=1}^{h-1} S(t-t_i) \int_{t_{i-1}}^{t_i} S(t_i - s)\big(v_0(s) - u(s)\big)\, ds \right\| \\
&\quad + \int_{t_{h-1}}^{t_h} \|S(t-s)\| \big(\|v_0(s)\| + \|u(s)\|\big)\, ds,
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_0(t) - y(t)\| \le \sum_{i=1}^{h-1} \|S(t-t_i)\| \left\| \int_{t_{i-1}}^{t_i} S(t_i - s)\big(v_0(s) - u(s)\big)\, ds \right\| \\
+ M \int_{t_{h-1}}^{t_h} \big(\|v_0(s)\| + \|u(s)\|\big)\, ds.
\end{aligned} \tag{4.22}$$

The last term on the right side of (4.22) is not greater than $2M\delta$. In fact v_0 and u are selectors of the multifunction $t \to \overline{\text{co}}\, F\big(t, y(t)\big) + \delta U$, by (4.15) and (4.1), therefore satisfy (4.4) i.e. $\|v_0(s)\| \le \psi(s)$ and $\|u(s)\| \le \psi(s)$, $s \in I$. Further $\mu(I_h) < \alpha$, thus by virtue of (4.5) the statement holds true. From (4.22), in view of Step 2, we have

$$\begin{aligned}
\|x_0(t) - y(t)\| &\le M \sum_{i=1}^{h-1} \left\| \int_{I_i} S(t_i - s)\big(v_0(s) - u(s)\big)\, ds \right\| + 2M\delta \\
&\le M \sum_{\substack{i \in \Im' \\ i \le h-1}} \left\| \int_{I_i \cap K} S(t_i - s)\big(v_0(s) - u(s)\big)\, ds \right\| \\
&\quad + M \sum_{\substack{i \in \Im'' \\ i \le h-1}} \left\| \int_{I_i \setminus K} S(t_i - s)\big(v_0(s) - u(s)\big)\, ds \right\| + 2M\delta \\
&\le M \sum_{\substack{i \in \Im' \\ i \le h-1}} 2(M+1)\delta\mu(I_i)
\end{aligned}$$

$$+ M \int_{I \setminus K} \|S(t_i - s)\| \big(\|v_0(s)\| + \|u(s)\|\big)\, ds + 2M\delta$$

$$\leq 2M(M+1)\delta + 2M^2 \int_{I \setminus K} \psi(s)\, ds + 2M\delta.$$

By (4.6) $\mu(I \setminus K) < \alpha$, hence (4.5) implies that the latter integral is less than δ. Consequently,

$$\|x_0(t) - y(t)\| \leq 2M(M+1)\delta + 2M^2\delta + 2M\delta < 4(M+1)^2\delta < \frac{\varepsilon}{L+1}$$

for, by (4.3), $\delta < \varepsilon/[4(M+1)^2(L+1)]$. Since $t \in I$ is arbitrary, Step 3 is proved.

Step 4. *x_0 is a mild solution of the Cauchy problem* $(C_{a,F+\varphi_\varepsilon U})$.

In view of the definition of x_0 and v_0 (see (4.14) and (4.13)), x_0 is continuous on I, with $x(0) = a$, and v_0 is Lusin measurable and integrable in the sense of Bochner on I. To prove the statement we have only to show that

$$v_0(t) \in F\big(t, x_0(t)\big) + \varphi_\varepsilon(t)U \qquad t \in I. \tag{4.23}$$

Let $t \in I \setminus K$. From (4.13), $v_0(t) = w_0(t) \in F\big(t, y(t)\big)$, thus

$$d\big(v_0(t),\ F\big(t, x_0(t)\big)\big) \leq h\big(F(t, y(t)),\ F(t, x_0(t))\big) \leq k(t)\|y(t) - x_0(t)\|.$$

Since, by Step 3, $\|y(t) - x_0(t)\| \leq \varepsilon/(L+1)$, we have

$$d\big(v_0(t),\ F(t, x_0(t))\big) < \varepsilon[k(t)/(L+1) + 1] = \varphi_\varepsilon(t),$$

and hence (4.23) is satisfied, for each $t \in I \setminus K$.

Let $t \in K$. Then for some $i \in \Im'$, $1 \leq j \leq r_i$, and $1 \leq n \leq p_i$ we have $t \in K_n^{ij}$. By virtue of (4.13) and (4.8), $v_0(t) = e_n^i \in F\big(\tau_i, y(\tau_i)\big)$. On the other hand, by (4.7), $F\big(\tau_i, y(\tau_i)\big) \subset F\big(t, y(t)\big) + \delta U$ as τ_i, $t \in I_i \cap K$ and, consequently, $v_0(t) \in F\big(t, y(t)\big) + \delta U$. By virtue of Step 3 we have:

$$\begin{aligned} d\big(v_0(t),\ F\big(t, x_0(t)\big)\big) &\leq h\big(F(t, y(t)) + \delta U,\ F(t, x_0(t))\big) \\ &\leq h\big(F(t, y(t)),\ F(t, x_0(t))\big) + \delta \\ &\leq k(t)\|y(t) - x_0(t)\| + \delta \leq \varepsilon \frac{k(t)}{L+1} + \delta < \varphi_\varepsilon(t) \end{aligned}$$

as $\delta < \varepsilon$, by (4.3). It follows that (4.23) is satisfied also for $t \in K$, and Step 4 is proved. This completes the proof. □

Theorem 4.1. *Let (H_1)–(H_4) be satisfied. Let $a \in \mathbb{E}$, and let $y : I \to \mathbb{E}$ be an arbitrary mild solution of the convexified Cauchy problem $(C_{a,\overline{\mathrm{co}}\,F})$. Then, for every $\sigma > 0$, there exists a mild solution $x : I \to \mathbb{E}$ of the Cauchy problem $(C_{a,F})$ such that $\|x(t) - y(t)\| \le \sigma$ for every $t \in I$.*

Proof: Let $y : I \to \mathbb{E}$ be an arbitrary mild solution of the Cauchy problem $(C_{a,\overline{\mathrm{co}}\,F})$, and let $0 < \sigma < 1$. Fix ε so that

$$0 < \varepsilon < \frac{\sigma}{7Me^{LM}} ,$$

where $M \ge 1$ is a constant such that $M \ge \|S(t)\|$ for each $t \in I$, and $L = \int_0^1 k(t)\,dt$.

By Lemma 4.2, with the above choice of ε, there exists a mild solution $x_0 : I \to \mathbb{E}$ of the Cauchy problem $(C_{a,F+\varphi_\varepsilon U})$, where $\varphi_\varepsilon(t) = \varepsilon[k(t)/(L+1)+1]$, such that

$$\|x_0(t) - y(t)\| \le \frac{\varepsilon}{L+1} \le \varepsilon \qquad t \in I. \tag{4.24}$$

By definition of mild solution, x_0 is continuous, with $x_0(0) = a$, and there is a Lusin measurable function $v_0 : I \to \mathbb{E}$, integrable in the sense of Bochner, satisfying

$$\begin{aligned} &v_0(t) \in F\big(t, x_0(t)\big) + \varphi_\varepsilon(t)U && t \in I \\ &x_0(t) = S(t)a + \int_0^t S(t-s)v_0(s)\,ds && t \in I. \end{aligned} \tag{4.25}$$

With this choice of x_0 and v_0, following the argument and retaining the notation of Theorem 3.1, we can construct a sequence $\{x_n\}$ of continuous functions $x_n : I \to \mathbb{E}$, $n = 1, 2, \dots$ given by

$$x_n(t) = S(t)a + \int_0^t S(t-s)v_n(s)\,ds \qquad t \in I,$$

where $v_n : I \to \mathbb{E}$ is a Lusin measurable function, integrable in the sense of Bochner, such that

$(4.26)_n$

$$v_n(t) \in \big(F(t, x_{n-1}(t)) + \varepsilon_n U\big) \cap U_{\mathbb{E}}\big(v_{n-1}(t), d(v_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n\big) \qquad t \in I,$$

and $\varepsilon_n = \varepsilon/2^{n+2}$. Let $p_0 : I \to \mathbb{R}$ and $m : I \to \mathbb{R}$ be, respectively, given by

$$p_0(t) = d\big(v_0(t),\ F(t, x_0(t))\big) \qquad m(t) = \int_0^t k(s)\,ds.$$

Clearly, by (4.25), $p_0(t) \le \varphi_\varepsilon(t)$ for each $t \in I$, thus

$$\int_0^1 p_0(t)\,dt \le \int_0^1 \varphi_\varepsilon(t)\,dt = \varepsilon \int_0^1 \left(\frac{k(t)}{L+1} + 1\right) dt \le 2\varepsilon. \tag{4.27}$$

Now by virtue of (3.10), for every $N \ge 2$ and $t \in I$ we have

$$\begin{aligned} \|x_N(t) - x_0(t)\| &\le \sum_{n=2}^{N} \|x_n(t) - x_{n-1}(t)\| + \|x_1(t) - x_0(t)\| \\ &\le \sum_{n=2}^{N} a_n + \|x_1(t) - x_0(t)\|, \end{aligned} \tag{4.28}$$

where a_n is given by (3.11). Observe that, in view of (3.11) and (4.27),

$$\begin{aligned} \sum_{n=2}^{N} a_n = M \sum_{n=2}^{N} \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \frac{(LM)^k}{k!} \\ + \varepsilon_0 M \sum_{n=2}^{N} \frac{(LM)^{n-1}}{(n-1)!} + 2\varepsilon M \sum_{n=2}^{N} \frac{(LM)^{n-1}}{(n-1)!} \\ \le M \left(\sum_{k=0}^{+\infty} \varepsilon_k\right) \left(\sum_{k=0}^{+\infty} \frac{(LM)^k}{k!}\right) + \varepsilon_0 M e^{LM} + 2\varepsilon M e^{LM} \\ \le \frac{\varepsilon}{2} M e^{LM} + \frac{\varepsilon}{4} M e^{LM} + 2\varepsilon M e^{LM} < 3\varepsilon M e^{LM}. \end{aligned} \tag{4.29}$$

On the other hand, for every $t \in I$ we have:

$$\|x_1(t) - x_0(t)\| \le \int_0^t \|S(t-s)\| \|v_1(s) - v_0(s)\|\,ds \le M \int_0^1 \|v_1(s) - v_0(s)\|\,ds.$$

Since, by $(4.26)_1$, $\|v_1(s) - v_0(s)\| \le p_0(s) + \varepsilon_1$, in view of (4.27) for every $t \in I$ we have

$$\|x_1(t) - x_0(t)\| < 3\varepsilon M. \tag{4.30}$$

Then from (4.28), by virtue of (4.29) and (4.30), for every $N \ge 2$ and $t \in I$ it follows that

$$\|x_N(t) - x_0(t)\| < 6\varepsilon M e^{LM}.$$

Let $x : I \to \mathbb{E}$ be the uniform limit of $\{x_N\}$. As shown in Theorem 3.1, this limit exists and is a mild solution of the Cauchy problem $(C_{a,F})$. Clearly,

$$\|x(t) - x_0(t)\| \le 6\varepsilon M e^{LM} \qquad t \in I.$$

Combining the latter inequality and (4.24) gives, for each $t \in I$,

$$\|x(t) - y(t)\| \leq \|x(t) - x_0(t)\| + \|x_0(t) - y(t)\| \leq 6\varepsilon M e^{LM} + \varepsilon \leq 7\varepsilon M e^{LM} < \sigma,$$

for $\varepsilon < \sigma/(7Me^{LM})$. This completes the proof. □

Now consider the Cauchy problem $(D_{a,F})$, obtained by $(C_{a,F})$ by letting $A = 0$. We associate with $(D_{a,F})$ the convexified Cauchy problem

$$(D_{a,\overline{\mathrm{co}}\,F}) \qquad \begin{cases} x'(t) \in \overline{\mathrm{co}}\,F\big(t, x(t)\big) \\ x(0) = a. \end{cases}$$

By Theorem 4.1, with $A = 0$, we have the following:

Corollary 4.1. *Let (H_2)–(H_4) be satisfied. Let $a \in \mathbb{E}$, and let $y : I \to \mathbb{E}$ be an arbitrary solution of the convexified Cauchy problem $(D_{a,\overline{\mathrm{co}}\,F})$. Then, for every $\sigma > 0$, there exists a solution $x : I \to \mathbb{E}$ of the Cauchy problem $(D_{a,F})$ such that $\|x(t) - y(t)\| \leq \sigma$ for every $t \in I$.*

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