A fixed point theorem for non-self multi-maps in metric spaces

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Abstract. A fixed point theorem is proved for non-self multi-valued mappings in a metrically convex complete metric space satisfying a slightly stronger contraction condition than in Rhoades [3] and under a weaker boundary condition than in Itoh [2] and Rhoades [3].

Keywords: metrically convex metric space, multi-valued non-self map, fixed point

Classification: 47H10, 54H25

Let (X, d) be a metric space. Then X is said to be metrically convex if for every pair $x, y \in X$, $x \neq y$, there is a point $z \in X$ such that $d(x, y) = d(x, z) + d(z, y)$. We need the following lemma in the sequel.

Lemma 1 ([1]). *Let K be a non-empty and closed subset of a metrically convex metric space X. Then for any $x \in K$ and $y \notin K$, there exists a point $z \in \partial K$ such that $d(x, y) = d(x, z) + d(z, y)$, where ∂K denotes the boundary of K.*

Let $CB(X)$ denote the family of all non-empty, closed and bounded subsets of X. Denote for $A, B \in CB(X)$

$$D(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$$

and

$$\delta(A, B) = \sup\{d(a, b) \mid a \in A, b \in B\}.$$

Note that $D(A, B) \leq H(A, B) \leq \delta(A, B)$, where $H(A, B)$ denotes the Hausdorff distance of A and B.

In [2] Itoh proved a fixed point theorem for the non-self maps $F : K \to CB(X)$ satisfying certain contraction condition in terms of Hausdorff metric H on $CB(X)$ under the boundary condition $F(\partial K) \subset K$. Recently Rhoades [3] generalized this result to a wider class of non-self multi-maps on K. In this paper we prove a fixed point theorem for non-self multi-maps on K satisfying a slightly stronger contraction condition than that in Rhoades [3] and under a weaker boundary condition.

Theorem 1. *Let (X,d) be a metrically convex complete metric space and K a non-empty closed subset of X. Let $F : K \to CB(X)$ be a multi-map satisfying*

$$\delta(Fx,Fy) \le \alpha \max\{d(x,y), D(x,Fx), D(y,Fy)\} + \beta[D(x,Fy)+D(y,Fx)] \tag{1}$$

for all $x,y \in K$, where $\alpha \ge 0$, $\beta \ge 0$ satisfy

$$2\alpha + 3\beta < 1. \tag{2}$$

Further, if $Fx \cap K \neq \emptyset$ for each $x \in \partial K$, then F has a unique fixed point $p \in K$ such that $Fp = \{p\}$ and F is continuous at p in the Hausdorff metric on X.

Proof: Let $x \in K$ be arbitrary and consider a sequence $\{x_n\}$ in K as follows: Let $x_0 = x$ and take a point $x_1 \in Fx_0 \cap K$ if $Fx_0 \cap K \neq \emptyset$. Otherwise choose a point $x_1 \in \partial K$ such that

$$d(x_0, x_1') = d(x_0, x_1) + d(x_1, x_1')$$

for some $x_1' \in Fx_0 \subset X \setminus K$.
Similarly pick $x_2 \in Fx_1 \cap K$ if $Fx_1 \cap K \neq \emptyset$, otherwise choose a point $x_2 \in \partial K$ such that

$$d(x_1, x_2) + d(x_2, x_2') = d(x_1, x_2')$$

for some $x_2' \in Fx_1 \subset X \setminus K$.
Continuing in this way we have

$$x_{n+1} \in Fx_n \cap K \quad \text{if} \quad Fx_n \cap K \neq \emptyset,$$

or $x_{n+1} \in \partial K$ satisfying

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1}') = d(x_n, x_{n+1}')$$

for some $x_{n+1}' \in Fx_n \subset X \setminus K$.
By the construction of $\{x_n\}$, we can write

$$\{x_n\} = P \cup Q \subset K,$$

where

$$P = \{x_n \in \{x_n\} : x_n \in Fx_{n-1}\}$$

and

$$Q = \{x_n \in \{x_n\} : x_n \in \partial K, x_n \notin Fx_{n-1}\}.$$

Then for any two consecutive terms x_n, x_{n+1} of the sequence $\{x_n\}$, we observe that there are only the following three possibilities:

(i) $x_n, x_{n+1} \in P$,
(ii) $x_n \in P$, $x_{n+1} \in Q$, and
(iii) $x_n \in Q$ and $x_{n+1} \in P$.

First we show that $\{x_n\}$ is a Cauchy sequence in K. Now for any $x_n, x_{n+1} \in \{x_n\}$, we have the following estimates:

Case I. Suppose that $x_n, x_{n+1} \in P$, then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \delta(Fx_{n-1}, Fx_n) \\ &\leq \alpha \max\{d(x_{n-1}, x_n), D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} \\ &\quad + \beta[D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})] \\ &\leq \alpha \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &\quad + \beta[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &= \alpha \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + \beta d(x_{n-1}, x_{n+1}) \\ &\leq \alpha \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &\quad + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &= \max\{(\alpha+\beta)d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}), \\ &\qquad (\alpha+\beta)d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n)\} \end{aligned}$$

and hence

$$d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n),$$

where $k = \max\{\frac{\alpha+\beta}{1-\beta}, \frac{\beta}{1-(\alpha+\beta)}\} < 1$, since $2\alpha + 3\beta < 1$.

Case II. Let $x_n \in P$ and $x_{n+1} \in Q$. Then

$$d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1})$$

for some $x'_{n+1} \in Fx_n$. Clearly,

$$\left\{ \begin{array}{l} d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) \\ d(x_n, x'_{n+1}) \leq \delta(Fx_{n-1}, Fx_n). \end{array} \right. \tag{3}$$

Now following arguments similar to those in Case I, we obtain

$$d(x_n, x'_{n+1}) \leq k d(x_{n-1}, x_n), \tag{4}$$

where again $k = \max\{\frac{\alpha+\beta}{1-\beta}, \frac{\beta}{1-(\alpha+\beta)}\} < 1$.

From (3) and (4) it follows that

$$d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n). \tag{5}$$

Case III. Suppose that $x_n \in Q$ and $x_{n+1} \in P$. Note that then $x_{n-1} \in P$ and there is a point $x'_n \in Fx_{n-1}$ such that

$$d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x_{n-1}, x'_n). \tag{6}$$

Now,

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq d(x_n, x'_n) + d(x'_n, x_{n+1})\\
&\leq d(x_n, x'_n) + \delta(Fx_{n-1}, Fx_n)\\
&\leq d(x_n, x'_n) + \alpha \max\{d(x_{n-1}, x_n), D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\}\\
&\quad + \beta[D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})]\\
&\leq d(x_n, x'_n) + \alpha \max\{d(x_{n-1}, x_n), d(x_{n-1}, x'_n), d(x_n, x_{n+1})\}\\
&\quad + \beta[d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)]\\
&\leq d(x_n, x'_n) + \alpha \max\{d(x_{n-1}, x'_n), d(x_n, x_{n+1})\}\\
&\quad + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x'_n)]\\
&= d(x_n, x'_n) + \alpha \max\{d(x_{n-1}, x'_n), d(x_n, x_{n+1})\}\\
&\quad + \beta[d(x_{n-1}, x'_n) + d(x_n, x_{n+1})]\\
&\leq d(x_{n-1}, x'_n) + \alpha \max\{d(x_{n-1}, x'_n), d(x_n, x_{n+1})\}\\
&\quad + \beta[d(x_{n-1}, x'_n) + d(x_n, x_{n+1})].
\end{aligned}$$

From (4) of Case II applied to $n-1$, we have $d(x_{n-1}, x'_n) \leq kd(x_{n-2}, x_{n-1})$ and hence

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq kd(x_{n-2}, x_{n-1}) + \max\{kd(x_{n-2}, x_{n-1}), d(x_n, x_{n+1})\}\\
&\quad + \beta[kd(x_{n-2}, x_{n+1}) + k(x_n, x_{n+1})]\\
&= \max\{(1+\alpha+\beta)kd(x_{n-2}, x_{n-1}) + \beta d(x_n, x_{n+1}),\\
&\qquad (1+\beta)kd(x_{n-2}, x_{n-1}) + (\alpha+\beta)d(x_n, x_{n+1})\}.
\end{aligned}$$

This implies

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \max\{(1+\alpha+\beta)k/(1-\beta), (1+\beta)k/[1-(\alpha+\beta)]\}d(x_{n-2}, x_{n-1})\\
&= qd(x_{n-2}, x_{n-1}),
\end{aligned}$$

where

$$\begin{aligned}
q &= \max\{(1+\alpha+\beta)k/(1-\beta), (1+\beta)k/[1-(\alpha+\beta)]\}\\
&= k\max\{(1+\alpha+\beta)/(1-\beta), (1+\beta)/[1-(\alpha+\beta)]\} = k(1+\beta)/[1-(\alpha+\beta)]\\
&= (1+\beta)/[1-(\alpha+\beta)]\max\{(\alpha+\beta)/(1-\beta), \beta/[1-(\alpha+\beta)]\}\\
&= \max\{(1+\beta)(\alpha+\beta)/[(1-\beta)(1-(\alpha+\beta))], \beta(1+\beta)/[1-(\alpha+\beta)]^2\}\\
&< 1.
\end{aligned}$$

To see this, the inequality (2) yields

$$\begin{aligned}
&\alpha+\beta < 1-2\beta-\alpha\\
&\Rightarrow \alpha+\beta+\alpha\beta+\beta^2 < 1-2\beta-\alpha+\alpha\beta+\beta^2\\
&\Rightarrow (\alpha+\beta+\alpha\beta+\beta^2)/(1-2\beta-\alpha+\alpha\beta+\beta^2) < 1\\
&\Rightarrow (1+\beta)(\alpha+\beta)/[(1-\beta)(1-\alpha-\beta)]\} < 1.
\end{aligned}$$

Similarly again from (2) we have

$$\begin{aligned}
&2\alpha + 3\beta < \alpha^2 + 2\alpha\beta + 1\\
&\Rightarrow \beta + \beta^2 < 1 - 2\alpha - 2\beta + \alpha^2 + 2\alpha\beta + \beta^2\\
&\Rightarrow \beta(1+\beta) < 1 - 2(\alpha+\beta) + (\alpha+\beta)^2\\
&\Rightarrow \beta(1+\beta) < [1-(\alpha+\beta)]^2\\
&\Rightarrow \beta(1+\beta)/[1-(\alpha+\beta)]^2 < 1.
\end{aligned}$$

Now for any $n \in N$, we have

$$d(x_{2n}, x_{2n+1}) \le q d(x_{2n-2}, x_{2n-1}) \le q^n d(x_0, x_1). \tag{7}$$

Since n is arbitrary, one has

$$d(x_n, x_{n+1}) \le q^n d(x_0, x_1). \tag{8}$$

Then from Cases I–III, it easily follows that $\{x_n\}$ is a Cauchy sequence in K. As K is closed it is complete and hence $\lim_n x_n = p$ exists. We show that p is a fixed point of F. Without loss of generality we may assume that $x_{n+1} \in Fx_n$ for some $n \in N$. Then

$$\begin{aligned}
D(p, Fp) &= \lim_n D(x_{n+1}, Fp)\\
&\le \lim_n \delta(Fx_n, Fp)\\
&\le \lim_n \max\{d(x_n, p), D(x_n, Fx_n), D(p, Fp)\}\\
&\quad + \beta \lim_n [D(x_n, Fp) + D(p, Fx_n)]\\
&= \alpha \lim_n \max\{d(x_n, p), d(x_n, x_{n+1}), D(p, Fp)\}\\
&\quad + \beta \lim_n [D(x_n, Fp) + d(p, x_{n+1})]\\
&= (\alpha + \beta) D(p, Fp)
\end{aligned}$$

which is possible only when $p \in Fp$.

Further, we have

$$\begin{aligned}
\delta(p, Fp) &\le \delta(Fp, Fp)\\
&\le \alpha \max\{d(p,p), D(p, Fp), D(p, Fp)\} + \beta[\delta(p, Fp) + D(p, Fp)]\\
&= \beta\delta(p, Fp)
\end{aligned}$$

and hence $Fp = \{p\}$.

To show the uniqueness of p, let q $(\neq p)$ be another fixed point of F. Then

$$\begin{aligned}
d(p, q) &\le \delta(Fp, Fq)\\
&\le \alpha \max\{d(p,q), D(p, Fp), D(q, Fq)\} + \beta[D(p, Fq) + D(q, Fp)]\\
&= (\alpha + 2\beta) d(p, q).
\end{aligned}$$

This is a contradiction since $\alpha + 2\beta < 1$ and hence $p = q$.

Finally we prove the continuity of F at p. Let $\{z_n\} \subset X$ by any sequence such that $z_n \to p$ as $n \to \infty$. Now

$$\begin{aligned}
\lim_n H(Fz_n, F) \leq\ & \lim_n \delta(Fz_n, Fp) \\
& \leq \alpha \lim_n \max\{d(z_n, p), D(z_n, Fz_n), D(p, Fp)\} \\
& \quad + \beta \lim_n [D(z_n, Fp) + D(p, Fz_n)] \\
& \leq \alpha \lim_n \max\{d(z_n, p), D(z_n, Fz_n)\} \\
& \quad + \beta \lim_n [d(z_n, p) + D(p, Fz_n)] \\
& = (\alpha + \beta) H(Fz_n, Fp)
\end{aligned}$$

where $\alpha + \beta < 1$. Therefore $\lim_n H(Fz_n, Fp) = 0$, showing that F is continuous at p. This completes the proof. □

The following fixed point theorem for non-self multi-maps on a complete convex metric space satisfying a slightly weaker contraction condition and under a stronger boundary condition than ours has been proved by Rhoades [3].

Theorem 2 ([3])**.** *Let (X, d) be a metrically convex metric space and K a non-empty closed subset of X.*

Let $F : K \to CB(X)$ satisfy

$$H(Fx, Fy) \leq \alpha d(x, y) + \beta \max\{D(x, Fx), D(y, Fy)\} + \gamma[D(x, Fy) + D(y, Fx)] \tag{9}$$

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ such that

$$\left(\frac{1 + \alpha + \gamma}{1 - \beta - \gamma}\right) \left(\frac{\alpha + \beta + \gamma}{1 - \gamma}\right) < 1. \tag{10}$$

Further if $Fx \subset K$ for each $x \in \partial K$, then there exists a $p \in K$ such that $p \in Fp$ and F is upper semi-continuous at p.

Proof: The existence of such a fixed point $p \in K$ follows from Theorem 1 of Rhoades [3]. We only show the upper semi-continuity of F at p.

Let $\{z_n\} \subset K$ be any sequence such that $z_n \to p$ as $n \to \infty$.

Let $\{y_n\}$ be a sequence in K such that $y_n \in Fx_n$ for each $n \in N$ and $y_n \to q$. To finish, we shall prove that $q \in Fp$. Now

$$\begin{aligned}
d(q, p) =\ & \lim_n d(y_n, p) \leq \lim_n H(Fz_n, Fp) \\
=\ & \lim_n d(z_n, p) + \beta \lim_n \max\{D(z_n, Fz_n), D(p, Fp)\} \\
& + \gamma \lim_n [D(z_n, Fp) + D(p, Fz_n)] \\
=\ & \beta \lim_n \max\{d(z_n, y_n), 0\} + \gamma \lim_n d(p, y_n) \\
=\ & \beta d(p, q) + \gamma d(p, q) = (\beta + \gamma) d(p, q)
\end{aligned}$$

which is possible only when $d(q,p) = 0$ as $\beta + \gamma < 1$. Hence $q \in Fp$ and the proof ot the theorem is complete. $\square$

Next we prove two fixed point theorems for multi-maps on a metric space satisfying a contractive condition more general than (1) and under certain compactness type conditions.

Theorem 3. *Let (X,d) be a complete metrically convex metric space and K a non-empty compact subset of X. Suppose that $F : K \to CB(X)$ is a continuous multi-map satisfying*

$$(11)\quad \delta(Fx,Fy) < \alpha \max\{d(x,y), D(x,Fx), D(y,Fy)\} + \beta[D(x,Fy) + D(y,Fx)]$$

for all $x,y \in K$, $x \notin Fx$, $y \notin Fy$, where $\alpha, \beta > 0$ satisfy $2\alpha + 3\beta \leq 1$. If $Fx \cap K \neq \emptyset$ for each $x \in \partial K$ then the multi-map F has a unique fixed point.

Proof: First we note that if the multi-map F has a fixed point then from condition (11) it follows that the fixed point is unique.

Since K is compact, both sides of the inequality (11) are bounded on K. Now there are two possibilities:

Case I. Suppose that the right hand side of (11) is zero for some $(x,y) \in K \times K$, then we have $x = y \in Fy$. Thus F has a fixed point and so it is unique.

Case II. Suppose that the right hand side of (11) is positive for all $x,y \in K$. Denote for brevity

$$M(x,y) = \alpha \max\{d(x,y), D(x,Fx), D(y,Fy)\} + \beta[D(x,Fy) + D(y,Fx)].$$

Now in the case when $2\alpha + 3\beta < 1$, the conclusion of Theorem 3 follows from Theorem 1. Therefore we treat only the case when $2\alpha + 3\beta = 1$.

Define a function $T : K^2 \to \mathbb{R}^+$ by

$$(12)\qquad T(x,y) = \frac{\delta(Fx,y)}{M(x,y)}\,.$$

Clearly the function T is well defined since $M(x,y) \neq 0$ for all $x,y \in K$.

Since F, D and δ are continuous, T is continuous and from the compactness of K it follows that there is a point $(u,v) \in K^2$ such that T attains its maximum at this point. Call the value c. From (11) we get $0 < c < 1$. By the definition of T, we obtain

$$\begin{aligned}\delta(Fx,Fy) &\leq cM(x,y)\\ &= \alpha' \max\{d(x,y), D(x,Fx), D(y,Fy)\} + \beta'[D(x,Fy) + D(y,Fx)]\end{aligned}$$

for all $x,y \in K$, where $2\alpha' + 3\beta' = c(2\alpha + 3\beta) < 1$. As K is compact, it is closed and so the desired conclusion follows by an application of Theorem 1. The proof is complete. $\square$

Theorem 4. *Let* (X, d) *be a complete metrically convex metric space and* K *a compact subset of* X*. Suppose that* $F : K \to CB(X)$ *is a continuous multi-map satisfying*

$$(13) \quad H(Fx, Fy) < \alpha d(x, y) + \beta \max\{D(x, Fx), D(y, Fy)\} + \gamma[D(x, Fy) + D(y, Fx)]$$

for all $x, y \in X$, $x \notin Fx$, $y \notin Fy$, *where* $\alpha, \beta, \gamma > 0$ *satisfy* $(\frac{1+\alpha+\gamma}{1-\beta-\gamma})\,(\frac{\alpha+\beta+\gamma}{1-\gamma}) \leq 1$. *If* $Fx \subset K$ *for each* $x \in \partial K$ *then the multi-map* F *has a fixed point.*

Proof: The proof is similar to Theorem 3 and now the desired conclusion follows by an application of Theorem 2. □

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References

[1] Blumenthal L.M., *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford, 1943.

[2] Itoh S., *Multi-valued generalized contraction and fixed point theorems*, Comment. Math. Univ. Carolinae **18** (1977), 247–248.

[3] Rhoades B.E., *A fixed point theorem for a multi-valued non-self mappings*, Comment. Math. Univ. Carolinae **37** (1996), 401–404.

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