

## An independency result in connectification theory

ALESSANDRO FEDELI, ATTILIO LE DONNE

*Abstract.* A space is called connectifiable if it can be densely embedded in a connected Hausdorff space.

Let  $\psi$  be the following statement: “a perfect  $T_3$ -space  $X$  with no more than  $2^c$  clopen subsets is connectifiable if and only if no proper nonempty clopen subset of  $X$  is feebly compact”.

In this note we show that neither  $\psi$  nor  $\neg\psi$  is provable in ZFC.

*Keywords:* connectifiable, perfect, feebly compact

*Classification:* 54D25, 54C25, 03E35

The problem of finding those spaces which can be densely embedded in a connected Hausdorff space has been extensively studied in the last years and many results have been obtained (see, e.g., [1], [2], [6], [10] and [13]).

Despite all the efforts, a characterization of connectifiable spaces is still unknown.

In this note we present a characterization of connectifiable perfect  $T_3$ -spaces with no more than  $2^c$  clopen subsets, which can be neither proved nor disproved in ZFC.

We recall that a space  $X$  is called:

- (i) perfect if every closed subset of  $X$  is a  $G_\delta$ -set;
- (ii) H-closed if every open cover of  $X$  has a finite subfamily whose union is dense, or equivalently,  $X$  is a closed subspace of every Hausdorff space in which it is contained;
- (iii) feebly compact if every countable open cover of  $X$  has a finite subfamily whose union is dense.

As usual,  $p$  will stand for the smallest cardinality of a maximal subfamily of  $[\omega]^\omega$  with the strong finite intersection property (see, e.g., [4] and [12]).

Regarding connectifiability observe that

- (1) A connectifiable space contains no proper nonempty open H-closed subset ([13]).
- (2) Let  $X$  be a Hausdorff space with no more than  $2^c$  clopen subsets. If every proper nonempty clopen subsets of  $X$  is not feebly compact, then  $X$  is connectifiable ([10]).

(3) There exists, in ZFC, a nonconnectifiable Hausdorff space of cardinality  $\mathfrak{c}$  with no proper nonempty H-closed subspace ([10]).

(4) It is consistent with ZFC that there is a nonconnectifiable normal Hausdorff space of cardinality  $\mathfrak{c}$  which has no proper nonempty H-closed subspace ([10]).

In our result we will make use of the following set-theoretic statements (which are consistent with ZFC):

(a)  $p > \omega_1$ ;

(b) (Jensen's Combinatorial Principle  $\diamond$ ) There are sets  $A_\alpha \subset \alpha$  for  $\alpha < \omega_1$  such that for every  $A \subset \omega_1$  the set  $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$  is stationary.

We refer the reader to [5] for topological terminology. For set-theoretic terminology see [8] and [4].

**Theorem.** *The following statement:*

*“a perfect  $T_3$ -space  $X$  with no more than  $2^{\mathfrak{c}}$  clopen subsets  $X$  is connectifiable if and only if no proper nonempty clopen subset of  $X$  is feebly compact”*

*is independent of ZFC.*

PROOF: First let us show that, under  $p > \omega_1$ , the above statement is true.

If  $X$  is not connectifiable then, by one of the above mentioned result, there is a proper nonempty feebly compact clopen subset  $A$  of  $X$ .

Now let us suppose that there is a proper nonempty feebly compact clopen subset  $A$  of  $X$ . Since  $X$  is  $T_3$  and perfect, it follows that  $A$  is a countably compact perfect  $T_3$ -space. So by a theorem of Weiss (here we use  $p > \omega_1$ )  $A$  is compact ([14], see also [12]).

Hence  $X$  is not connectifiable. Therefore the statement is consistent with ZFC.

Now let us prove the independency by showing that, under the Jensen's principle  $\diamond$ , there exists a connectifiable  $T_6$ -space with exactly two proper nonempty clopen subsets, each of which is feebly compact.

Let  $S$  be the Ostaszewskii's space, this space is, under  $\diamond$ , an example of a noncompact countably compact perfectly normal space ([9]).

Let  $Z$  be the cone over  $S$ , i.e., let  $Z$  be the quotient of  $S \times I$  obtained by identifying  $S \times \{1\}$  with a point.

Now  $Z$  is noncompact ( $S \times \{0\}$  is a noncompact space homeomorphic to a closed subspace of  $Z$ ), countably compact and perfectly normal ( $Z$  is the continuous closed image of the countably compact perfectly normal space  $S \times I$  under the natural mapping).

Now let  $X = Z \oplus Z$ ,  $X$  is a perfectly normal space. The only proper nonempty clopen subsets of  $X$  are the two copies of  $Z$ , which are countably compact (= feebly compact) but not compact.

Since a Hausdorff space with open components is connectifiable if and only if it has no proper nonempty open H-closed subspace ([7]), it follows that  $X$  is connectifiable.  $\square$

**Example.** It is worth noting that there is a ZFC example of a connectifiable perfect Hausdorff space, with no more than  $2^{\mathfrak{c}}$  clopen subsets, which has proper nonempty feebly compact clopen subsets.

In fact let  $\mathcal{F}$  be the set of all free ultrafilters on  $\omega$  and let  $Y$  be  $\omega \cup \mathcal{F}$  endowed with the topology generated by the points of  $\omega$  and all sets of the form  $G \cup \{p\}$  where  $G \in p \in \mathcal{F}$ . Now fix  $p \in \mathcal{F}$  and let  $X$  be the subspace  $Y \setminus \{p\}$  of  $Y$ .

$X$  is a Hausdorff space which is not H-closed (it is not closed in  $Y$ ).

Now let us show that  $X$  is feebly compact. By a result in [3] it is enough to show that every locally finite system of pairwise disjoint nonempty open subsets of  $X$  is finite.

Suppose that  $\mathcal{A} = \{A_n : n \in \omega\}$  is an infinite locally finite family of pairwise disjoint nonempty open subsets of  $X$ . Without loss of generality we may assume that, for every  $n$ ,  $A_n = \{\kappa_n\}$  for some  $\kappa_n \in \omega$ .

Let  $q$  be a free open ultrafilter on  $\omega$  such that  $q \neq p$  and  $\{\kappa_n : n \in \omega\} \in q$ . Since every neighbourhood of  $q$  in  $X$  meets infinitely many members of  $\mathcal{A}$ , we reach a contradiction. Therefore  $X$  is feebly compact.

Moreover  $X$  is perfect. In fact, every open subset  $A$  of  $X$ , is the union of the  $F_\sigma$ -set  $\omega \cap A$  and the closed set  $A \setminus \omega$  ( $A \setminus \omega$  is a subset of the closed discrete subspace  $X \setminus \omega$  of  $X$ ).

Now let  $C$  be the cone over  $X$  and set  $Z = C \oplus C$ .

$Z$  is a perfect Hausdorff space, and the only two proper nonempty clopen subsets of  $Z$  (namely the copies of  $X$ ) are feebly compact.

Nonetheless  $Z$  has open components and no proper nonempty H-closed subspaces, therefore  $Z$  is connectifiable.

**Remarks.** (i) If  $L$  is the long line, then  $X = L \oplus L$  is a ZFC example of a connectifiable hereditarily normal space of cardinality  $\mathfrak{c}$  which has proper nonempty feebly compact clopen subsets.

(ii) In [10] it is shown that, under  $MA + \neg CH$ , a disconnected perfectly normal space with no more than  $2^{\mathfrak{c}}$  clopen subsets is connectifiable if and only if no nonempty clopen subset is relatively pseudocompact.

#### REFERENCES

- [1] Alas O.T., Tkačenko M.G., Tkachuk V.V., Wilson R.G., *Connectifying some spaces*, Topology Appl. **71** (1996), 203–215.
- [2] Alas O.T., Tkačenko M.G., Tkachuk V.V., Wilson R.G., *Connectedness and local connectedness of topological groups and extensions*, Comment. Math. Univ. Carolinae, to appear.
- [3] Bagley R.W., Connell E.H., McKnight J.D., Jr., *On properties characterizing pseudocompact spaces*, Proc. A.M.S. **9** (1958), 500–506.
- [4] van Douwen E.K., *The integers and topology*, Handbook of Set-theoretic Topology, (Kunen K. and Vaughan J.E., eds.) Elsevier Science Publishers B.V., North Holland, 1984, pp. 111–167.
- [5] Engelking R., *General Topology*, Sigma series in Pure Mathematics 6, Heldermann Verlag, Berlin, 1989.
- [6] Fedeli A., Le Donne A., *On locally connected connectifications*, Topology Appl. (1998).
- [7] Fedeli A., Le Donne A., *Connectifications and open components*, submitted.

- [8] Kunen K., *Set Theory*, North Holland, 1980.
- [9] Ostaszewski A.J., *On countably compact, perfectly normal spaces*, J. London Math. Soc. (2) **14** (1976), 505–516.
- [10] Porter J.R., Woods R.G., *Subspaces of connected spaces*, Topology Appl. **68** (1996), 113–131.
- [11] Swardson M.A., *Nearly realcompact spaces and  $T_2$ -connectifiability*, Proceedings of the 8th Prague Topological Symposium, 1996, pp. 358–361.
- [12] Vaughan J.E., *Countably compact and sequentially compact spaces*, Handbook of Set-theoretic Topology, (Kunen K. and Vaughan J.E., eds.) Elsevier Science Publishers B.V., North Holland, 1984, pp. 569–602.
- [13] Watson S., Wilson R.G., *Embeddings in connected spaces*, Houston J. Math. **19** (1993), no. 3, 469–481.
- [14] Weiss W.A.R., *Countably compact spaces and Martin's axiom*, Canad. J. Math. **30** (1978), 243–249.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF L'AQUILA, 67100 L'AQUILA, ITALY

*E-mail*: fedeli@aquila.infn.it

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROME "LA SAPIENZA" 00100 ROME, ITALY

*E-mail*: ledonne@mat.uniroma1.it

(Received April 7, 1998)