On the extensibility of closed filters in $\mathbf{T}_1$ spaces and the existence of well orderable filter bases

Kyriakos Keremedis, Eleftherios Tachtsis

Abstract. We show that the statement CCFC = "*the character of a maximal free filter F of closed sets in a T_1 space (X,T) is not countable*" is equivalent to the *Countable Multiple Choice Axiom* CMC and, the axiom of choice AC is equivalent to the statement CFE_0 = "*closed filters in a T_0 space (X,T) extend to maximal closed filters*". We also show that AC is equivalent to each of the assertions:
"*every closed filter $\mathcal{F}$ in a T_1 space (X,T) extends to a maximal closed filter with a well orderable filter base*",
"*for every set $A \neq \emptyset$, every filter $\mathcal{F} \subseteq \mathcal{P}(A)$ extends to an ultrafilter with a well orderable filter base*" and
"*every open filter $\mathcal{F}$ in a T_1 space (X,T) extends to a maximal open filter with a well orderable filter base*".

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1. Definitions

Let (X,T) be a topological space and $\mathcal{E} \subseteq \mathcal{P}(X)\backslash\{\emptyset\}$. A non empty collection $\mathcal{F} \subseteq \mathcal{E}$ is an *$\mathcal{E}$-filter* iff

(i) if F_1, $F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$;

(ii) if $F \in \mathcal{F}$, $F \subset F'$ and $F' \in \mathcal{E}$, then $F' \in \mathcal{F}$.

In particular if $\mathcal{E}$ is the collection of all non empty closed sets then we say that $\mathcal{F}$ is a *closed filter*. Likewise, if $\mathcal{E}$ is the collection of all non empty open sets then we say that $\mathcal{F}$ is an *open filter*. If $\mathcal{E} = \mathcal{P}(X)\backslash\{\emptyset\}$ then an $\mathcal{E}$-filter is called simply a *filter* on X i.e. a filter on X is an open (closed) filter of the space X carrying the discrete topology. An $\mathcal{E}$-filter $\mathcal{F}$ is *free* iff $\bigcap \mathcal{F} = \emptyset$.

A non empty collection $\mathcal{B} \subseteq \mathcal{P}(X)\backslash\{\emptyset\}$ is a *filter base* for some filter $\mathcal{F}$ iff for every $B_1, B_2 \in \mathcal{B}$ there exists $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$.

The *character* of a filter (open, closed, etc.,) $\mathcal{F}$ is the minimum cardinality (if it exists) of a filter base $\mathcal{B}$ for $\mathcal{F}$.

A family $\mathcal{U}$ of subsets of X is *locally finite* if every point of X has a neighborhood meeting a finite number of elements of $\mathcal{U}$.

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The *Axiom of choice* AC is the statement:

> *For every family $\mathcal{A} = \{A_i : i \in k\}$ of disjoint non empty sets there exists a set C which consists of one and only one element from each element of $\mathcal{A}$.*

The *Axiom of multiple choice* MC is the statement:

> *For every family $\mathcal{A} = \{A_i : i \in k\}$ of disjoint non empty sets there exists a set $\mathcal{F} = \{F_i : i \in k\}$ of finite non empty sets such that $F_i \subseteq A_i$ for all $i \in k$.*

CAC and CMC stand for AC and MC respectively restricted to countable sets. The *Boolean Prime Ideal Theorem* BPI is the proposition:

> *Every Boolean algebra has a prime ideal.*

For notation and terminology used but not defined here the reader is referred to any standard text of General Topology such as [10].

2. Introduction and some preliminary results

If one goes carefully through the proof that AC implies Tychonoff's compactness theorem, see [10], he will realize that, in order to get that proof go through, he only needs two facts:

1. *in Tychonoff products of compact T_1 spaces every closed filter extends to a maximal closed filter, and*
2. *in Tychonoff products of compact T_1 spaces the projections are closed maps.*

Motivated by (1) and (2), it is plausible to define, as the authors in [6] did, the axiom of *closed filter extensibility* CFE by requiring:

- *closed filters in a T_1 space (X, T) extend to maximal closed filters.*

In [6] it has been proved that

Proposition 1. (i) *AC iff CFE + CAC.*
(ii) *CFE implies but it is not equivalent to BPI.*

There remains the question:

(A) *Does CFE imply AC?*

Since the statement CFE_0 given by

> *closed filters in a T_0 space (X, T) extend to maximal closed filters*

clearly implies CFE, one may expect that CFE_0 implies AC. Indeed, we have:

Theorem 2. *CFE_0 is equivalent to AC.*

PROOF: An easy application of Zorn's Lemma, shows that AC implies CFE_0. To see the converse let $\mathcal{A} = \{A_i : i \in k\}$ be a disjoint family of infinite sets. Topologize $X_i = A_i \cup \{\infty_i\}$ by declaring basic neighborhoods of points $x \in X_i$ to be all cofinite supersets of $\{x, \infty_i\}$. Let X be the Tychonoff product of the X_i's. CFE_0 implies that X has a maximal closed filter $\mathcal{G}$ and one more application of CFE_0 to X with the discrete topology extends $\mathcal{G}$ to an ultrafilter $\mathcal{F}$. We show first that $\bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$ (and consequently $\bigcap \mathcal{G} \neq \emptyset$). To this end it suffices to find, in view of the proof of the Tychonoff's compactness theorem given in [10], $c \in X$, with $c(i) \in B_i = \bigcap\{\overline{\pi_i(F)} : F \in \mathcal{F}\}$ for all $i \in k$. As X_i is compact and $\{\pi_i(F) : F \in \mathcal{F}\}$ is a filter of X_i, it follows that $B_i \neq \emptyset$. Without loss of generality we may assume that $B_i \neq X_i$ for all $i \in k$. If $B_i = X_i$ choose $c_i = \infty_i$ ($\overline{\{\infty_i\}} = X_i$). Put $B = \{B_i : i \in k\}$. Since CFE_0 clearly implies the axiom of choice restricted to families of finite sets, see [5], it follows that B has a choice function c as required.

$\mathcal{G}$ being a closed ultrafilter implies that c cannot be the element $c(i) = \infty_i$ for all $i \in k$ for the only closed set including c is X itself. In order to complete the proof of the theorem it suffices to show:

Claim. $c_i \neq \infty_i$ for all $i \in k$.

PROOF OF THE CLAIM: Indeed, if $c_i = \infty_i$ for some $i \in k$ then letting

$$Y = \prod_{i \in k} Y_i, Y_i = \begin{cases} \{c_i\} & \text{if } \ c_i \neq \infty_i \\ X_i & \text{if } \ c_i = \infty_i \end{cases}$$

carry the subspace topology, we see that $Y \in \mathcal{G}$ ($\mathcal{G}$ is a closed ultrafilter and Y is a closed subset of X meeting each member of $\mathcal{G}$). Now, the only closed subset of Y including c is Y itself. Thus $Y \subseteq \bigcap \mathcal{G}$ and we have reached a contradiction. (If $c_{i^*} = \infty_{i^*}$ then we pick a finite non empty subset K of A_{i^*} and let

$$H = \prod_{i \in k} H_i, H_i = \begin{cases} \{c_i\} & \text{if } \ c_i \neq \infty_i \\ X_i & \text{if } \ c_i = \infty_i \ \text{ and } \ i \neq i^* \\ K & \text{if } \ i = i^* \end{cases}.$$

It follows that $H \notin \mathcal{G}$ is a closed set included in every member of $\mathcal{G}$, contradicting the fact that $\mathcal{G}$ is a maximal closed filter.) Thus, c is a choice function finishing the proof of the claim and of the theorem. □

If X is a countable set, say $X = \omega$, and $\mathcal{F}$ is a maximal free filter on ω then one can verify, in the Zermelo-Fraenkel set theory without the axiom of choice, ZF-AC for abbreviation, that $\mathcal{F}$ cannot have a countable filter base. More generally, if X is any infinite set and $\mathcal{F}$ is a maximal free filter on X then one can easily verify in ZF-AC that $\mathcal{F}$ cannot have a well ordered nested filter base $\mathcal{B} = \{B_i : i \in \aleph\}$. (If

$C_i = B_i \backslash B_{i+1}$, $i \in \aleph$ and $S \subseteq \aleph$ a set of size $\aleph$ whose complement has also size $\aleph$ then $G = \bigcup\{C_i : i \in S\}$ meets, but does not include, all members of $\mathcal{B}$. As $\mathcal{B}$ is a filter base and the filter $\mathcal{F}$ is maximal, it follows that $G \in \mathcal{F}$. Hence, G includes a member of $\mathcal{B}$ and this is a contradiction). So, there remains the question what happens if X carries some topology other than the discrete one and the members of the filter are maximal with respect to some property, say closedness. That is:

(B) *Can maximal closed filters have a countable filter base*?

The research in this paper is motivated by questions (A) and (B). We show in Sections 3 and 4 that the answer to (B) depends on AC and the answer to (A) is related to (B).

Let CCFC, CCFC2 and COFC stand for the statements

- *no maximal closed free filter in a* T_1 *space has countable character*,
- *no maximal closed free filter in a* T_2 *space has countable character*,
- *no maximal open free filter in a* T_2 *space has countable character*

respectively.

Proposition 3. *In CCFC the* T_1 *separation axiom cannot be replaced by the* T_0 *axiom.*

Proof: $\mathcal{T} = \{\emptyset, \omega, [0, n) : n \in \omega\}$ is a T_0 topology on ω and $\mathcal{F} = \{[n, \infty) : n \in \omega\}$ is a countable free closed ultrafilter of $(\omega, \mathcal{T})$. □

Proposition 4. *In COFC the requirement that the space* X *be* T_2 *cannot be dropped out.*

Proof: Let $A = \{A_i : i \in \omega\}$ be a disjoint family of non empty sets. Topologize $X = \bigcup A$ by declaring basic neighborhoods of points $x \in X$ to be all sets of the form $V_{xn} = \{x\} \cup (\bigcup\{A_i : i \geq n\})$. It can be readily verified that X is a first countable T_1 space having a countable filter base $B = \{\bigcup\{A_i : i \geq n\} : n \in \omega\}$ for the open ultrafilter $\mathcal{F}$ of all non empty open sets independently of AC. □

Remark 1. *If each member of* A *is finite then* X *is a compact space. It follows that if no infinite subset of* A *has a choice function then* X *is not separable. Furthermore, as each sequence in* X *has a finite range, we see that even though* X *is first countable, the closure operator cannot be described sequentially.* ($W \subseteq X$ *is closed iff whenever* $(w_n)_{n\in\omega} \subseteq W$ *converges to* w *then* $w \in W$).

Proposition 5. *COFC can be proved in ZF-AC.*

Proof: Let (X, T) be a T_2 space and $\mathcal{F}$ a free maximal filter of open sets having a countable filter base $B = \{B_n : n \in \omega\}$. Without loss of generality we may assume that B is strictly descending. Put $b = \bigcap \bar{B}$, $\bar{B} = \{\bar{B}_n : n \in \omega\}$. Then any open set O meeting b is in $\mathcal{F}$. Hence, b can have at most one point. (If $x, y \in b$ then since X is a T_2 space it follows that there exist open disjoint sets O_x and O_y including x and y. Then both O_x, O_y are in $\mathcal{F}$ and this is a contradiction.) For every $n \in \omega$, put $O_n = B_n \backslash \bar{B}_{n+1}$.

Claim. There exists $m \in \omega$ such that $\forall n \geq m$, $O_n = \emptyset$.

Proof of the claim: If not then there exists an infinite subset $S \subseteq \omega$ such that $O_s \neq \emptyset$ for all $s \in S$. Without loss of generality we may assume that $S = \omega$. Put $O = \bigcup\{O_{2n} : n \in \omega\}$. Clearly O is an open set meeting but not including each B_n. Hence $O \in \mathcal{F}$ and consequently $B_{n^*} \subseteq O$ for some $n^* \in \omega$. This contradiction establishes the claim.

By the claim we conclude that $\bar{B}_m = \bar{B}_n$ for all $n \geq m$. Thus, $\bar{B}_m \subseteq b$ and we have arrived at a contradiction. □

Remark 2. *Since COFC is provable in ZF-AC it follows that CCFC2 is of interest only in case where some member of the filter is a* closed nowhere dense *set.*

In what follows we shall make use of the following results.

Proposition 6 ([8]). *CMC iff PCMC (= for every countable family B of disjoint infinite sets, some infinite subfamily $\bar{B}$ of B has a multiple choice).*

Proof: Mimic the proof of Lemma 1 (iv) given in [4]. □

Lemma 7 (Levy's Lemma [9]). *MC iff every set can be written as a well ordered union of disjoint finite sets.*

Below we give a list of the statements (all provable in ZF^0+ AC) that will be studied in the paper.

(1) Every closed filter $\mathcal{F}$ in a T_1 space (X, T) has a well orderable filter base.
(2) Every open filter $\mathcal{F}$ in a T_1 space (X, T) has a well orderable filter base.
(3) Every open filter $\mathcal{F}$ in a dense-in-itself T_1 space (X, T) has a well orderable filter base.
(4) Every open ultrafilter $\mathcal{F}$ in a T_1 space (X, T) has a well orderable filter base.
(5) For every set $A \neq \emptyset$, every filter $\mathcal{F} \subseteq \mathcal{P}(A)$ has a well orderable filter base.
(6) Every closed filter $\mathcal{F}$ in a dense-in-itself T_1 space (X, T) has a well orderable filter base.
(7) Every closed filter $\mathcal{F}$ in a dense-in-itself T_2 space (X, T) has a well orderable filter base.
(8) If (X, T) is a T_2 topological space and $\mathcal{B}$ is a lattice of closed sets then every maximal $\mathcal{B}$-filter $\mathcal{F}$ has a well orderable filter base.
(9) Every closed filter $\mathcal{F}$ in a T_1 space (X, T) extends to a maximal closed filter with a well orderable filter base.
(10) For every set $A \neq \emptyset$, every filter $\mathcal{F} \subseteq \mathcal{P}(A)$ extends to an ultrafilter with a well orderable filter base.
(11) Every open filter $\mathcal{F}$ in a T_1 space (X, T) extends to a maximal open filter with a well orderable filter base.
(12) Every closed filter $\mathcal{F}$ in a T_1 space (X, T) has a well orderable base and is included in a maximal closed filter $\mathcal{G}$.
(13) Every closed ultrafilter $\mathcal{F}$ in a T_1 space (X, T) has a well orderable filter base.

(14) For every set $A \neq \emptyset$, every ultrafilter $\mathcal{F} \subseteq \mathcal{P}(A)$ has a well orderable filter base.

3. Existence of well ordered filter bases

We begin this section with a list of equivalent forms of MC.

Theorem 8. *The following are equivalent: MC.*
(1) *Every closed filter $\mathcal{F}$ in a T_1 space (X,T) has a well orderable filter base.*
(2) *Every open filter $\mathcal{F}$ in a T_1 space (X,T) has a well orderable filter base.*
(3) *Every open filter $\mathcal{F}$ in a dense-in-itself T_1 space (X,T) has a well orderable filter base.*
(4) *Every open ultrafilter $\mathcal{F}$ in a T_1 space (X,T) has a well orderable filter base.*
(5) *For every set $A \neq \emptyset$, every filter $\mathcal{F} \subseteq \mathcal{P}(A)$ has a well orderable filter base.*
(6) *Every closed filter $\mathcal{F}$ in a dense-in-itself T_1 space (X,T) has a well orderable filter base.*
(7) *Every closed filter $\mathcal{F}$ in a dense-in-itself T_2 space (X,T) has a well orderable filter base.*
(8) *If (X,T) is a T_2 topological space and $\mathcal{B}$ is a lattice of closed sets then every maximal $\mathcal{B}$-filter $\mathcal{F}$ has a well orderable filter base.*

Proof: MC $\rightarrow$ (1), (2), (3), (4), (5), (6), (7) and (8). We prove MC $\rightarrow$ (1). All the other implications can be proved similarly. Fix (X,T) and $\mathcal{F}$ as in (1). By Levy's Lemma, there exists a disjoint family $G = \{G_n : n \in k\}$, k a cardinal, of finite sets covering $\mathcal{F}$. Clearly, $\mathcal{B} = \{B_n = \bigcap G_n : n \in k\}$ is a well ordered base for $\mathcal{F}$. (As G covers $\mathcal{F}$ it follows that each member of $\mathcal{F}$ includes a member of G. Furthermore, as $\mathcal{F}$ is a filter, we see that $\mathcal{B} \subseteq \mathcal{F}$ and consequently $\mathcal{B}$ is a base.)

(2) $\rightarrow$ MC, (5) $\rightarrow$ MC. Fix $A = \{A_i : i \in k\}$ a disjoint family of infinite sets. Put $X = \bigcup A$ and let $\mathcal{F} = \{f \subseteq X : |A_i \backslash f| < \omega \text{ for all } i \in k\}$. Clearly, $\mathcal{F}$ is a filter of X and an open filter of X taken with the discrete topology. Let $\mathcal{B} = \{B_n : n \in \mu\}$ be a well ordered base for $\mathcal{F}$. For every $i \in k$ we let $n_i = min\{n \in \mu : |A_i \backslash B_n| \neq 0\}$. It is straightforward to verify that $D = \{D_i = A_i \backslash B_{n_i} : i \in k\}$ is a multiple choice for A finishing the proof of (2) $\rightarrow$ MC and (5) $\rightarrow$ MC.

(3) $\rightarrow$ MC, (4) $\rightarrow$ MC. Fix A as in (2) $\rightarrow$ MC and put on $X = \bigcup A$ the cofinite topology T. Let $\mathcal{F} = T \backslash \{\emptyset\}$. Clearly, $\mathcal{F}$ is an open ultrafilter of the dense-in-itself T_1 space (X,T). Let $\mathcal{B} = \{B_n : n \in \mu\}$ be a well ordered base for $\mathcal{F}$. It is straightforward to see that the set D as defined in (2) $\rightarrow$ MC is a multiple choice for A as required.

(6) $\rightarrow$ MC. Fix A an infinite set. It suffices, in view of Levy's Lemma to show that A can be covered by a well ordered family of finite sets. Put $X = [A]^{<\omega}$ (i.e. the set of all finite subsets of A) and let T be the topology on X in which basic neighborhoods of points $x \in X$ are all sets of the form

$$B(x,z) = \{x\} \cup \{y \in X : y \cap z = \emptyset\},\ z \in X,\ x \subseteq z.$$

Claim 1. (X,T) is a T_1 space. Indeed, fix $x, y \in X$, $x \neq y$. We consider the following cases.
(a) If $x \cap y = \emptyset$ or if $x \backslash y \neq \emptyset$ and $y \backslash x \neq \emptyset$, then $B(x, x \cup y)$ and $B(y, x \cup y)$ are neighborhoods of x and y avoiding y and x respectively.
(b) If $x \subseteq y$ then $B(x, y)$ and $B(y, y)$ are neighborhoods of x and y avoiding y and x respectively.
(c) If $y \subseteq x$ then case (b) applies.

For every $x \in X$, put

$$G_x = \{y \in X : x \subseteq y\}. \tag{3.1}$$

Claim 2. G_x is closed in (X,T). Fix $z \notin G_x$ then $B(z, x \cup z)$ is a neighborhood of z avoiding G_x. Hence G_x is closed as required.

Let $\mathcal{F}$ be the closed (necessarily free) filter which is generated by the collection $\mathcal{G} = \{G_x : x \in X\}$. Let $\mathcal{D} = \{D_i : i \in k\}$ be a well ordered filter base for $\mathcal{F}$. Without loss of generality we may assume that each D_i is included in a G_x.

For every $i \in k$, we let $z_i = \bigcup Z_i$, $Z_i = \{x \in X : D_i \subseteq G_x\}$.

Claim 3. $|z_i| < \omega$. It suffices to show that $|Z_i| < \omega$. Assume on the contrary that Z_i is infinite. Clearly z_i is an infinite set and any member of $D_i = \bigcap\{G_x : D_i \subseteq G_x\}$ includes z_i. Thus, $D_i = \emptyset$. On the other hand, as $D_i \subseteq D_i$ we have $D_i \neq \emptyset$. This contradiction establishes Claim 3.

Put $Z = \{z_i : i \in k\}$. Clearly Z is a well ordered cover of A consisting of finite sets. Since we can always disjointify Z, Levy's Lemma follows and the proof of (6) $\rightarrow$ MC is complete.

(1) $\rightarrow$ (7). This is straightforward.

(7) $\rightarrow$ MC. Fix A be an infinite set and let $X = [A]^{<\omega}$. N. Brunner ([3]) has shown that the collection $\mathcal{B}$ of all sets of the form $B(y,z) = \{x \in X : y \subseteq x \wedge x \cap z = \emptyset\}$, $y, z \in X$, $y \cap z = \emptyset$ generates a dense in-itself T_2 topology on X. Now, the sets G_x given in (3.1) are closed in X. (If $z \in X$, $z \notin G_x$, then $B(z, x \backslash z)$ is a neighborhood of z avoiding G_x.) We can finish the proof of (7) $\rightarrow$ MC as in (6) $\rightarrow$ MC.

(8) $\rightarrow$ MC. Let A, (X,T) and $\mathcal{G} = \{G_x : x \in X\}$ be as in (7) $\rightarrow$ MC. It is easy to see that $\mathcal{B} = \{\bigcup Q : Q \in [\mathcal{G}]^{<\omega}\}$ is a lattice of closed sets which is also a $\mathcal{B}$-filter. Thus $\mathcal{B}$ is a maximal $\mathcal{B}$-filter and by the hypothesis it has a well orderable filter base $\mathcal{D}$. Continue as in the proof of (6) $\rightarrow$ MC to write A as a well ordered union of finite sets finishing the proof of (8) $\rightarrow$ MC and of the theorem. □

In the next corollary we give a list of equivalents of AC. We would like to stress out the resemblance of (9) and (12).

Corollary 9. *The following are equivalent: AC.*
(9) *Every closed filter $\mathcal{F}$ in a T_1 space (X,T) extends to a maximal closed filter with a well orderable filter base (= CFE + (13)).*
(10) *For every set $A \neq \emptyset$, every filter $\mathcal{F} \subseteq \mathcal{P}(A)$ extends to an ultrafilter with a*

well orderable filter base (= *BPI* + (14)).
(11) *Every open filter $\mathcal{F}$ in a T_1 space (X,T) extends to a maximal open filter with a well orderable filter base* (= *every open filter $\mathcal{F}$ in a T_1 space extends to a maximal open filter* + (4)) (= *BPI* + (4)).
(12) *Every closed filter $\mathcal{F}$ in a T_1 space (X,T) has a well orderable base and is included in a maximal closed filter $\mathcal{G}$* (= (1) + *CFE*).

Proof: AC → (9), (11), (12), (9) → (10) and (11) → (10). These are straightforward.

(12) → AC. This follows from Proposition 1 and Theorem 8.

In order to complete the proof of the theorem it suffices to show that (10) → AC. Clearly, (10) implies BPI which in turn implies AC_{fin}, the axiom of choice for families of finite sets. In order to complete the proof, it suffices to show that (10) also implies MC. The proof of (6) → MC of Theorem 8 goes through even in case $\mathcal{D}$ is a base for some ultrafilter $\mathcal{H}$ extending the filter $\mathcal{F}$. Thus, (10) implies MC, finishing the proof of the corollary. □

4. Maximal closed filters in $\mathbf{T}_1$ spaces do not have countable characters

In this section we give an equivalent form of CMC.

Theorem 10. *CMC iff CCFC* (*i.e. no maximal closed free filter in a T_1 space has countable character*).

Proof: CMC → CCFC. Assume that $B = \{B_n : n \in \omega\}$ is a countable filter base for a free filter $\mathcal{G}$ of closed sets in the T_1 space (X,T). Without loss of generality we may assume that B is strictly descending. Put $A_n = B_n \backslash B_{n+1}$ for all $n \in \omega$ and $A = \{A_n : n \in \omega\}$. Let $\mathcal{F} = \{F_n : n \in \omega\}$ satisfy CMC for A. Then $\mathcal{F}$ is a locally finite family of closed sets in X. Indeed let $x \in X$. We consider the following two cases:
(i) $x \notin \bigcup \mathcal{F}$. Since $\mathcal{G}$ is a free filter there exists an $n \in \omega$ such that $x \notin B_n$. Then $W = (B_n)^c \backslash (F_0 \cup \cdots \cup F_{n-1})$ is a neighborhood of x which avoids every element of $\mathcal{F}$.
(ii) $x \in F_n$ for some $n \in \omega$. Then $W = (B_{n+1})^c \backslash (F_0 \cup \cdots \cup F_{n-1})$ is a neighborhood of x which meets only one element of $\mathcal{F}$, namely F_n.

Thus $\mathcal{F}$ is locally finite and consequently the family $\mathcal{H} = \{F_{2n} : n \in \omega\}$ is also locally finite. It is a well known fact (see [10]) that the union of a locally finite family of closed sets is a closed set, consequently, $c = \bigcup \mathcal{H}$ is a closed set meeting each but not including properly any member of B. Thus, the closed filter generated by $\{c\} \cup \mathcal{G}$ extends properly $\mathcal{G}$ meaning that $\mathcal{G}$ is not an ultrafilter.

CCFC → CMC. In view of Proposition 6 it suffices to show that CCFC → PCMC. Fix $B = \{B_n : n \in \omega\}$ a family of disjoint infinite sets. Topologize $X = \bigcup B$ by declaring basic neighborhoods of points $x \in X$, $x \in B_n$ to be all supersets of $\{x\}$ whose complements in $X_n = \bigcup \{B_m : m \leq n\}$ are finite. Clearly,

(1) *X is a T_1 space,*

(2) *if a set $U \neq X$ is closed in X, then for all $n \in \omega$, $|U \cap B_n| < \omega$ or $B_n \subseteq U$,*

(3) *$A = \{A_i = \bigcup\{B_n : n \geq i\} : i \in \omega\}$ is a descending family of closed (nowhere dense-except A_0) sets with empty intersection.*

Let $\mathcal{F}$ be the closed filter (necessarily free) which is generated by A. As A is countable, it follows from CCFC that $\mathcal{F}$ is not maximal. Thus, there exists a non empty closed set Q meeting non trivially but not including any member of A. Hence, there is a set $\bar{B} = \{B_{n_i} : i \in \omega\} \subseteq B$ such that $F_i = Q \cap B_{n_i} \neq \emptyset$. Since $Q \neq X$, we see that F_i is finite and consequently $F = \{F_i : i \in \omega\}$ is a multiple choice for $\bar{B}$ finishing the proof of the theorem. □

5. Independence results

Lemma 11. *MC (and consequently (1) through (8)) implies (14) but the converse is not true.*

Proof: By Theorem 8, we have that MC implies (14). On the other hand, A. Blass has shown in [1] that there exists a ZF model $(\mathcal{M}, \in)$ without free ultrafilters. Thus, in $\mathcal{M}$ (14) holds but AC, and consequently MC, fails. □

Lemma 12. (i) *MC does not imply CFE.*
(ii) *CAC does not imply CFE.*
(iii) *CMC does not imply* (14).
(iv) *CAC does not imply* (14).
(v) *CAC does not imply* (13).
(vi) *Neither* (14) *nor* (13) *imply CAC.*

Proof: (i) By Proposition 1, CFE + CMC $\leftrightarrow$ AC. Now, in model $\mathcal{N}2$ (the Second Fraenkel Model) in [5] MC and consequently CMC holds but AC fails. Therefore CFE fails in $\mathcal{N}2$.

(ii) By Proposition 1, CFE + CAC $\leftrightarrow$ AC. There are both Cohen models and permutational models $(M, \in)$ where CAC holds but AC fails. See, for example Solovay's model, Model $\mathcal{M}5(\aleph)$ in [5]. Thus, in $\mathcal{M}5(\aleph)$ CAC holds but CFE fails.

(iii) It is known, see model $\mathcal{N}38$ in [5], that CMC+BPI does not imply AC. ($\mathcal{N}38$ is a permutational model satisfying CMC, BPI and the negation of AC.) On the other hand, in view of Corollary 9, (14) + BPI does imply AC. Thus (14) fails in $\mathcal{N}38$.

(iv) CAC holds in $\mathcal{N}38$ but as we have seen in (iii), (14) fails.

(v) Clearly, (13) implies (14). On the other hand CAC, in view of (iv), holds in $\mathcal{N}38$ but (14) fails. Hence (13) fails also in $\mathcal{N}38$.

(vi) In the Second Fraenkel Model, Model $\mathcal{N}2$ in [5], MC and consequently (13) and (14) hold but CAC fails. □

Lemma 13. (i) *BPI does not imply neither* (14) *nor* (13).
(ii) *None of* (13) *and* (14) *implies BPI.*

Proof: (i) The statements BPI+(14) and BPI+(13) are, in view of Corollary 9, equivalent to AC. Now, in Cohen's original model, Model $\mathcal{M}1$ in [5], BPI is true but AC is false. Thus, in $\mathcal{M}1$, BPI is true but (14) and (13) are false.

(ii) As seen in (vi) of Lemma 12, (13) and (14) are true in model $\mathcal{N}2$. On the other hand, BPI is false in $\mathcal{N}2$ (BPI is equivalent to Form 14J (= the product of compact T_2 spaces is compact) in [5] and there exists a family in $\mathcal{N}2$ of compact T_2 spaces such that their Tychonoff product is not compact).

In Blass' model, in [1], (14) is true whereas BPI is false (see [12]). □

6. Summary

$$\begin{tikzcd}
\boxed{\begin{array}{c}\mathrm{AC}=\mathrm{CFE}_0=(9)=\\(10)=(11)=(12)\end{array}} \arrow[rr] \arrow[dr] \arrow[ddd] & & \boxed{\begin{array}{c}\mathrm{MC}=(1)=(2)=(3)\\(4)=(5)=(6)=(7)=(8)\end{array}} \arrow[dd] \\
& \boxed{\mathrm{BPI}} & \\
& & \boxed{\begin{array}{c}(14)\\\uparrow(\downarrow?)\\(13)\end{array}} \\
\boxed{\boxed{\mathrm{CFE}}\ (\to?)\ \boxed{\mathrm{CAC}}} \arrow[rr] & & \boxed{\begin{array}{c}\mathrm{CMC}\\\updownarrow\\\mathrm{PCMC}\\\updownarrow\\\mathrm{CCFC}\end{array}}
\end{tikzcd}$$

Questions. (i) Does CFE imply CMC?
(ii) Is CCFC2 provable in ZF-AC?
(iii) Does (13) imply MC?

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Department of Mathematics, University of the Aegean, Karlovassi 83200, Samos, Greece

E-mail: kker@aegean.gr
ltah@math.aegean.gr

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