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Lattice points in super spheres

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Abstract. In this article we consider the number $R_{k,p}(x)$ of lattice points in p-dimensional super spheres with even power $k \geq 4$. We give an asymptotic expansion of the d-fold anti-derivative of $R_{k,p}(x)$ for sufficiently large d. From this we deduce a new estimation for the error term in the asymptotic representation of $R_{k,p}(x)$ for $p < k < 2p-4$.

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1. Introduction

Let k, p be integers with $k \geq 3$, $p \geq 2$. Let $R_{k,p}(x)$ denote the number of lattice points inside the super sphere

$$|t_1|^k + |t_2|^k + \ldots + |t_p|^k \leq x.$$

This means

$$R_{k,p}(x) = \#\{(n_1, n_2, \ldots, n_p) \in \mathbb{Z}^p : |n_1|^k + |n_2|^k + \ldots + |n_p|^k \leq x\}.$$

The asymptotic representation

$$R_{k,p}(x) = V_{k,p} x^{\frac{p}{k}} + H_{k,p,1}(x) + P_{k,p}(x)$$

contains two main terms. The first main term is given by the volume of the super sphere, such that

$$V_{k,p} = \left(\frac{2}{k}\right)^p \frac{\Gamma^p(\frac{1}{k})}{\Gamma(1+\frac{p}{k})}\,.$$

The second main term comes from points of the boundary, where the Gaussian curvature vanishes. It can be developed into an infinite series over generalized Bessel functions. These functions are defined by (see [3])

$$(1) \qquad J_\nu^{(k)}(x) = \frac{2}{\sqrt{\pi}\Gamma(\nu+1-\frac{1}{k})} \left(\frac{x}{kn}\right)^{\frac{k\nu}{2}} \int_0^1 (1-t^k)^{\nu-\frac{1}{k}} \cos xt\, dt$$

for $k = 2, 3, \dots$, $x \in \mathbb{R}$, $\nu \in \mathbb{C}$, $Re(\nu) > 1/k - 1$. Further, let for real ν

$$\psi_\nu^{(k)}(x) = 2\sqrt{\pi}\Gamma(\nu + 1 - \frac{1}{k}) \sum_{n=1}^{\infty} \left(\frac{x}{\pi n}\right)^{\frac{k\nu}{2}} J_\nu^{(k)}(2\pi n x), \tag{2}$$

where the infinite series is absolutely convergent for $\nu > 1/k$. Then

$$H_{k,p,1}(x) = pV_{k,p-1}\psi_{p/k}^{(k)}(x^{\frac{1}{k}}), \tag{3}$$

and the second main term has the precise order

$$x^{\frac{p-1}{k}(1-\frac{1}{k})}.$$

For the estimation of the remainder $P_{k,p}(x)$ we use the notation

$$P_{k,p}(x) \ll x^{\lambda_{k,p}}. \tag{4}$$

The simplest non-trivial estimation is given by (see [3])

$$\lambda_{k,p} = \frac{p-2}{k} + \frac{2}{k(p+1)} \qquad \text{for} \quad k \le p+1, \tag{5}$$

$$\lambda_{k,p} = \frac{p-2}{k} + \frac{2(k-p+2)}{3k^2} \qquad \text{for} \quad k > p+1. \tag{6}$$

Since

$$\frac{p-1}{k}(1 - \frac{1}{k}) \le \frac{p-2}{k} + \frac{2}{k(p+1)} \quad \text{for} \quad k \le p+1,$$

the second main term is meaningless in case (5). But in case (6) it is significant. Otherwise we have

$$\frac{p-2}{k} + \frac{2(k-p+2)}{3k^2} > \frac{p-2}{k} + \frac{2}{k(p+1)} \quad \text{for} \quad p > 2,$$

which leaves much to be desired.

Some improvements of the estimates (5) and (6) were given in special cases by E. Krätzel [3], R. Schmidt-Röh [6], W. Müller and W.G. Nowak [5], G. Kuba [4] and S. Hoeppner and E. Krätzel [2]. So we have

$$\begin{aligned} \lambda_{k,2} &= \frac{46}{73}k + \varepsilon && (\varepsilon > 0), \\ \lambda_{k,p} &= \frac{p-2}{k} + \frac{12}{k(7p+4)} && \text{for} \quad k \le p+1, \quad 3 \le p \le 7, \\ \lambda_{k,p} &= \frac{p-2}{k} + \frac{5}{k(3p+1)} && \text{for} \quad l \le p+1, \quad p \ge 8, \end{aligned} \tag{7}$$

which were proved by G. Kuba for $p = 2$, the author for $p = 3$ and R. Schmidt-Röh for $p > 3$. Furthermore, one can find some minor improvements of (6) in [2]. But all these results show again that the quality of estimations in case of $k > p+1$ is not so good as for $k \le p+1$. One aim of this paper is to prove an estimation like (7) which holds true even in the interval $p+1 < k < 2p-4$. However, this is within reach only for even k.

Furthermore, assuming that k is an even integer, R.E. Wild [8] proved

$$\int_0^x R_{k,2}(t)\,dt = \frac{k}{k+2} V_{k,2} x^{\frac{2}{k}+1} + H^{(1)}_{k,2,1}(x) + \Delta^{(1)}_{k,2}(x)$$

with

$$H^{(1)}_{k,2,1}(x) = -c_1 x^{1-\frac{1}{k^2}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n x^{\frac{1}{k}} - \frac{\pi}{2k})}{n^{2+\frac{1}{k}}} + O\left(x^{1-\frac{1}{k}-\frac{1}{k^2}}\right),$$

$$\Delta^{(1)}_{k,2}(x) = -c_2 x^{1-\frac{1}{2k}} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\cos(2\pi N_2 x^{\frac{1}{k}} - \frac{\pi}{4})}{(n_1 n_2)^{\frac{k-2}{2k-2}} N_2^{\frac{3}{2}+\frac{1}{k-1}}} + \delta_k(x),$$

where

$$c_1 = \frac{8}{\pi^2}\left(\frac{k}{2\pi}\right)^{\frac{1}{k}} \Gamma(\frac{1}{k}), \qquad c_2 = \frac{4k}{\pi^2\sqrt{k-1}},$$

$$N_2 = \left(n_1^{\frac{k}{k-1}} + n_2^{\frac{k}{k-1}}\right)^{1-\frac{1}{k}}.$$

The remainder $\delta_k(x)$ is estimated by

$$\delta_k(x) \ll x^{1-\frac{1}{k}}.$$

L. Schnabel [7] proved Wild's result also for odd k. Moreover, he proved the asymptotic representation

$$\delta_k(x) = c_3 x^{1-\frac{1}{k}-\frac{1}{k^2}} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x^{\frac{1}{k}} - \frac{\pi}{2k})}{n^{3+\frac{1}{k}}},$$

$$c_3 = \frac{k-1}{\pi^3}\left(\frac{k}{2\pi}\right)^{\frac{1}{k}} \Gamma(2+\frac{1}{k}).$$

A further aim of this paper is to generalize the result of R.E. Wild to dimension $p > 2$. However, we must again assume that k is an even integer. We give an asymptotic expansion for the integral

$$\int_0^x (x-t)^{d-1} R_{k,p}(t)\,dt,$$

where d is supposed to be sufficiently large. From this we deduce better estimations for the error term in case of $4 < p < k < 2p-4$.

2. Statement of results

We consider the d-fold anti-derivative

$$R^{(d)}_{k,p}(x) = \frac{1}{(d-1)!} \int_0^x (x-t)^{d-1} R_{k,p}(t)\, dt$$

of $R_{k,p}(x)$ for $d = 1, 2, \ldots$ and put

$$R^{(0)}_{k,p}(x) = R_{k,p}(x).$$

The definition of $R^{(d)}_{k,p}(x)$ makes sense also for $p = 1$. So we will use it also for $p = 1$. Further we put

$$V^{(d)}_{k,p} = \left(\frac{2}{k}\right)^p \frac{\Gamma^p(\frac{1}{k})}{\Gamma(d+1+\frac{p}{k})}, \qquad V^{(0)}_{k,p} = V_{k,p}.$$

We use for abbreviation the following notations for $r \geq 1$:

$$d_{p,r} = d + \frac{p-r}{k},$$

$$\vec{t_r} = (t_1, t_2, \ldots, t_r), \qquad \vec{n_r} = (n_1, n_2, \ldots, n_r), \qquad n_j \in \mathbb{N},$$

$$\vec{n_r}\vec{t_r} = \sum_{\nu=1}^{r} n_\nu t_\nu,$$

$$\sum_{\vec{n_r}} = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty}$$

Moreover, we define

$$F^{(d)}_{k,p,r}(x) = \int \cdots \int \left(x - t_1^k - \ldots - t_r^k\right)^{d_{p,r}} \cos(2\pi \vec{n_r}\vec{t_r})\, dt_1 \cdots dt_r \tag{8}$$

for $p \geq 1$, $r = 1, 2, \ldots$, $d \geq 0$ and with the integration condition

$$t_1^k + \ldots + t_r^k \leq x.$$

Theorem 1. *If k is an even integer and if d is sufficiently large the representation*

$$R^{(d)}_{k,p}(x) = V^{(d)}_{k,p} x^{d+\frac{p}{k}} + \sum_{r=1}^{p-1} H^{(d)}_{k,p,r}(x) + \Delta^{(d)}_{k,p}(x) \tag{9}$$

holds true with

$$H^{(d)}_{k,p,r}(x) = 2^r \binom{p}{r} V^{(d)}_{k,p-r} \sum_{\vec{n_r}} F^{(d)}_{k,p,r}(x; \vec{n_r}) \tag{10}$$

for $r=1,2,\dots,p-1,p$, *where*

$$\Delta_{k,p}^{(d)}(x)=H_{k,p,p}^{(d)}(x).$$

Furthermore, the representation

$$H_{k,p,r}^{(d)}(x)=\binom{p}{r}V_{k,p-r}^{(d)}\frac{\Gamma(d_{p,r}+1)}{\Gamma(\frac{p-r}{k})}\int_0^x(x-t)^{\frac{p-r}{k}-1}\Delta_{k,r}^{(d)}(t)\,dt \tag{11}$$

holds true for $r=1,\dots,p-1$.

The next theorem shows that the infinite series for $\Delta_{k,p}^{(d)}(x)$ is absolutely convergent for $d>(p-1)/2$. Therefore, (9) holds true for such values of d. But we shall see that the infinite series (10) may be absolutely convergent for $r<p$ and for some smaller values of d.

Further, we use the abbreviations for $r=1,2,\dots,p$:

$$a_{p,r}=\frac{4}{k}\left(\frac{2\pi}{k(k-1)}\right)^{\frac{r-1}{2}}\left(\frac{k}{2\pi}\right)^{d_{p,r}+\frac{r+1}{2}}\Gamma(d_{p,r}+1),$$

$$N_r=\left(\sum_{\nu=1}^{r}n_\nu^{\frac{k}{k-1}}\right)^{1-\frac{1}{k}}.$$

Theorem 2. *For even* $k\ge 4$, $d\ge 0$ *and* $r=1$ *we have the identity*

$$H_{k,p,1}^{(d)}(x)=pV_{k,p-1}^{(d)}\psi_{d+p/k}^{(k)}(x^{\frac{1}{k}}), \tag{12}$$

where the ψ*-function is defined by* (2), *and the asymptotic representation as* $x\to\infty$

$$H_{k,p,1}^{(d)}(x)=x^{d_{p,1}(1-\frac{1}{k})}\left\{\sum_{n=1}^{\infty}\frac{\cos(2\pi nx^{\frac{1}{k}}-\frac{\pi}{2}(d_{p,1}+1))}{n^{d_{p,1}+1}}+O\left(x^{-\frac{1}{k}}\right)\right\}. \tag{13}$$

For even $k\ge 4$, $d\ge 0$ *and* $r=2,3\dots,p$ *we have as* $x\to\infty$

$$\begin{aligned}F_{k,p,r}^{(d)}(x)&=a_{p,r}x^{d_{p,r}(1-\frac{1}{k})+\frac{r-1}{2k}}\times\\&\times\frac{\cos(2\pi N_r x^{\frac{1}{k}}-\frac{\pi}{2}(d_{p,r}+1+\frac{r-1}{2}))}{(n_1\cdots n_r)^{\frac{k-2}{2k-2}}N_r^{d_{p,r}+\frac{r}{2k-2}+\frac{1}{2}}}\left\{1+O\left(x^{-\frac{1}{2k}}\right)\right\},\end{aligned} \tag{14}$$

where the O*-term does not depend on* $n_1,n_2,\dots,n_r$.

Replacing the arithmetic mean N_r by the geometric mean, it is seen from (14) that the infinite series (10) are absolutely convergent if

$$\frac{k-2}{2k-2}+\frac{1}{r}\left(d_{p,r}+\frac{r}{2k-2}+\frac{1}{2}\right)>1.$$

This is the case if $d_{p,r}>(r-1)/2$. Hence, we obtain the following corollary.

Corollary. *The infinite series* (10) *are absolutely convergent if*

$$d_{p,r} = d + \frac{p-r}{k} > \frac{r-1}{2}\,.$$

Especially, all infinite series are absolutely convergent if $d > (p-1)/2$, *and the series for* $d = 0$ *are absolutely convergent if*

$$r < \frac{k+2p}{k+2}\,.$$

From Theorems 1 and 2 we can find upper bounds for the remainder $P_{k,p}$. If we use in the infinite series (10) only the trivial estimation with $|\cos(\dots)| \le 1$ we obtain for $\lambda_{k,p}$ in (4)

$$\lambda_{k,p} = \frac{p-2}{k} + \frac{2}{k(p+1)} \quad \text{for } \ 2p \ge k. \tag{15}$$

This is better than (6) for $p+1 < k \le 2p$. However, applying a simple theorem from theory of exponential sums, we can improve this result.

Theorem 3. *If* $k \ge 4$ *is an even integer, the asymptotic representation*

$$R_{k,p}(x) = V_{k,p}x^{\frac{p}{k}} + pV_{k,p-1}\psi^{(k)}_{p/k}(x^{\frac{1}{k}}) + P_{k,p}(x) \tag{16}$$

holds true with (4) *and*

$$\lambda_{k,p} = \frac{p-2}{k} + \frac{5}{k(3p+1)} \quad \textit{for } \ 2p > k+4. \tag{17}$$

Again, the second main term is meaningless for $k \le p$ *and significant for* $k \ge p+1 > 5$.

3. Proof of Theorem 1

We give only a formal proof of Theorem 1, because the convergence properties follow from Theorem 2.

Applying the p-dimensional Poisson sum formula we obtain

$$\begin{aligned} R^{(d)}_{k,p}(x) &= \frac{1}{d!} \sum_{n_1^k + \dots n_p^k \le x} (x - n_1^k - \dots n_p^k)^d \\ &= \frac{1}{d!} \sum_{n_1=-\infty}^{+\infty} \cdots \sum_{n_p=-\infty}^{+\infty} \int \cdots \int (x - t_1^k \dots - t_p^k)^d e^{2\pi i \vec{n_p}\vec{t_p}}\, dt_1 \dots dt_p \end{aligned}$$

with the integration condition

$$t_1^k + \ldots + t_p^k \le x.$$

Now it is easily seen that the term with $n_1 = n_2 = \ldots = n_p = 0$ gives the first main term in (9). The terms with exact r variables $n_\nu \neq 0$ and $p - r$ variables $n_\mu = 0$ lead to the terms $H_{k,p,r}^{(d)}(x)$ with $r = 1, 2, \ldots, p-1, p$. There are $\binom{p}{r}$ r-fold infinite series, where the variables are now denoted by $n_1, n_2, \ldots, n_r$. We write for abbreviation

$$D = R_{k,p}^{(d)}(x) - V_{k,p}^{(d)} x^{d+\frac{p}{k}}.$$

Since k is an even integer we get, by (8) and (10), with the same integration condition as above

$$D = \sum_{r=1}^{p} \binom{p}{r} \frac{1}{d!} \sum_{n_1=-\infty}^{+\infty} \cdots \sum_{n_p=-\infty}^{+\infty} \int \cdots \int (x - t_1^k - \ldots - t_p^k)^d e^{2\pi i \vec{n_r} \vec{t_r}}\, dt_1 \ldots dt_p.$$

Now we integrate over $t_{r+1}, \ldots, t_p$. Then we obtain, by the integration condition

$$t_1^k + \ldots t_r^k \le x,$$

$$\begin{aligned} D &= \sum_{r=1}^{p} 2^r \binom{p}{r} V_{k,p-r}^{(d)} \sum_{\vec{n_r}} \int \cdots \int \left(x - t_1^k - \ldots t_r^k\right)^{d_{p,r}} \cos(2\pi \vec{n_r} \vec{t_r})\, dt_1 \ldots dt_r \\ &= \sum_{r=1}^{p} 2^r \binom{p}{r} V_{k,p-r}^{(d)} \sum_{\vec{n_r}} F_{k,p,r}^{(d)}(x; \vec{n_r}) \\ &= \sum_{r=1}^{p} H_{k,p,r}^{(d)}(x). \end{aligned}$$

This gives (10).

The representation (11) follows from (8) and (10) at once, since we can write

$$\begin{aligned} F_{k,p,r}^{(d)}(x) = \frac{\Gamma(d_{p,r}+1)}{d!\Gamma(\frac{p-r}{k})} \int_0^x (x-t)^{\frac{p-r}{k}-1}\, dt \times \\ \times \int \cdots \int \left(t - t_1^k - \ldots - t_r^k\right)^d \cos(2\pi \vec{n_r} \vec{t_r})\, dt_1 \cdots dt_r \end{aligned}$$

with the integration condition

$$t_1^k + \ldots + t_r^k \le t. \qquad \square$$

4. Proof of Theorem 2

Proof of (12): We get from (8) and (10)

$$H^{(d)}_{k,p,1}(x) = 2pV^{(d)}_{k,p-1} \sum_{n=1}^{\infty} F^{(d)}_{k,p,1}(x;n),$$

$$F^{(d)}_{k,p,1}(x;n) = 2x^{d+\frac{p}{k}} \int_0^1 \left(1-t^k\right)^{d_{p,1}} \cos(2\pi n x^{\frac{1}{k}} t)\, dt.$$

Now we obtain (12) by definition (1) of the generalized Bessel functions and by definition (3) of the ψ-function. □

Proof of (13): If $(p-1)/k$ is an integer we get (13) by partial integrating at once.

If $(p-1)/k \notin \mathbb{N}$ we may use the asymptotic expansions of the generalized Bessel functions in [3]. But we want to have here a somewhat better error term than there. Hence, we apply the following lemma, which can be found in the book of E.T. Copson [1, Chapter 3, Section 12]:

Lemma 1. *Let $\Phi(t)$ be N times continuously differentiable in $\alpha \le t \le \beta$. Let $0 < \lambda < 1$, $0 < \mu < 1$. Then, as $y \to +\infty$,*

$$\begin{aligned} &\int_\alpha^\beta e^{iyt}(t-\alpha)^{\lambda-1}(\beta-t)^{\mu-1}\Phi(t)\, dt \\ &= \sum_{m=0}^{M-1} \frac{\Gamma(m+\lambda)}{m!\, y^{m+\lambda}} e^{iy\alpha + \frac{\pi i}{2}(m+\lambda)} \frac{d^m}{d\alpha^m}((\beta-\alpha)^{\mu-1}\Phi(\alpha)) \\ &+ \sum_{m=0}^{M-1} \frac{\Gamma(m+\mu)}{m!\, y^{m+\mu}} e^{iy\beta + \frac{\pi i}{2}(m-\mu)} \frac{d^m}{d\beta^m}((\beta-\alpha)^{\lambda-1}\Phi(\beta)) + O\left(\frac{1}{y^M}\right). \end{aligned} \tag{18}$$

We write

$$F^{(d)}_{k,p,1}(x;n) = x^{d+\frac{p}{k}} \int_{-1}^{+1} (1-t^k)^{d+\frac{p-1}{2}} e^{2\pi i n x^{1/k} t}\, dt$$

and apply this lemma with

$$\alpha = -1, \qquad \beta = +1, \qquad \lambda = \mu = \frac{p-1}{k} - \left[\frac{p-1}{k}\right], \qquad y = 2\pi n x^{1/k},$$

$$\Phi(t) = (1-t^2)^{d+1+[\frac{p-1}{k}]} \left(\frac{1-t^k}{1-t^2}\right)^{d+\frac{p-1}{k}},$$

$$M = d+3+\left[\frac{p-1}{k}\right].$$

Then the terms with $0 \leq m \leq d + [\frac{p-1}{k}]$ vanish in both sums in (18). The both non-vanishing terms with $m = d + 1 + [\frac{p-1}{k}]$ are of order

$$y^{-(d+1+\frac{p-1}{k})}.$$

They yield the leading term in (13). The next terms with $m = d + 2 + [\frac{p-1}{k}]$ are of order

$$y^{-(d+2+\frac{p-1}{k})}.$$

The error term in (18) is of order

$$y^{-M} = y^{-(d+3+[\frac{p-1}{k}])} \ll y^{-(d+2+\frac{p-1}{k})}.$$

Thus we get the error term in (13). □

Proof of (14): We begin with two lemmas.

Lemma 2. *Let $r = 2, 3, \ldots, p$, $\vec{t_{r-1}} = (t_1, t_2, \ldots, t_{r-1})$. Let the function $P = P(v, \vec{t_{r-1}})$ be given by*

$$P = \sum_{j=1}^{r-1} n_j^{\frac{k}{k-1}} t_j^k + n_r^{\frac{k}{k-1}} \left(\left(\frac{N_r}{n_r} \right)^{\frac{k}{k-1}} v - \sum_{j=1}^{r-1} \left(\frac{n_j}{n_r} \right)^{\frac{k}{k-1}} t_j \right)^k, \tag{19}$$

and let $f_r(v)$ be defined by

$$f_r(v) = \int \cdots \int \left(N_r^{\frac{k}{k-1}} - P \right)^{d_{p,r}} dt_1 \cdots dt_{r-1} \tag{20}$$

with the integration condition

$$v^k N_r^{\frac{k}{k-1}} \leq P \leq N_r^{\frac{k}{k-1}}.$$

Then the function $F = F_{k,p,r}^{(d)}(x; \vec{n_r})$, defined by (8), can be represented by the integral

$$F = \left(\frac{n_1 \cdots n_{r-1}}{N_r^{p+k(d-1)}} \right)^{\frac{1}{k-1}} \frac{x^{d+\frac{p}{k}}}{n_r} \int_{-1}^{+1} f_r(v) \cos(2\pi x^{\frac{1}{k}} v) \, dv. \tag{21}$$

□

Proof: We substitute $t_j \to x^{1/k} t_j$ for $j = 1, 2, \ldots, r-1$ and obtain

$$F = x^{d+\frac{p}{k}} \int \cdots \int \left(1 - t_1^k - \ldots - t_r^k \right)^{d_{p,r}} \cos(2\pi x^{\frac{1}{k}} \vec{n_r} \vec{t_r}) \, dt_1 \cdots dt_r$$

with the summation condition

$$t_1^k + \ldots + t_r^k \le 1.$$

We see, by means of Hölder's inequality,

$$\vec{n_r}\vec{t_r} \le \sum_{j=1}^{r} n_j |t_j| \le N_r \left(\sum_{j=1}^{r} t_j^k \right)^{\frac{1}{k}} \le N_r,$$

and we have equality if

$$t_j = \left(\frac{n_j}{N_r} \right)^{\frac{1}{k-1}} \qquad \text{for} \quad j = 1, 2, \ldots, r.$$

Analogously we see

$$\vec{n_r}\vec{t_r} \ge -N_r$$

with corresponding equality. Thus we make the substitution

$$t_j \to \left(\frac{n_j}{N_r} \right)^{\frac{1}{k-1}} t_j \qquad \text{for} \quad j = 1, 2, \ldots, r,$$

and then we put

$$\sum_{j=1}^{r} n_j^{\frac{k}{k-1}} t_j = N_r^{\frac{k}{k-1}} v$$

such that $-1 \le v \le +1$. Then we obtain (21) with (19) and (20). □

A remark to the integration condition in (20): The upper bound of P is clear. Further, it is easily seen that $P(v, \vec{t_{r-1}})$ has its minimum for $\vec{t_{r-1}} = (v, v, \ldots, v)$. In this point we have

$$P = v^k N_r^{\frac{k}{k-1}}$$

and so we find the lower bound of P.

Lemma 3. *Let the function $g_r(v)$ for $r = 2, 3, \ldots, p$ be defined by*

$$g_r(v) = c_{p,r} N_r^{d_{p,r} \frac{k}{k-1}} \left(\frac{n_r N_r^{r-2}}{n_1 \cdots n_{r-1}} \right)^{\frac{k}{2k-2}} \left(1 - v^k \right)^{d_{p,r} + \frac{r-1}{2}} \tag{22}$$

with

$$c_{p,r} = \left(\frac{2\pi}{k(k-1)} \right)^{\frac{r-1}{2}} \frac{\Gamma(d_{p,r} + 1)}{\Gamma(d_{p,r} + \frac{r+1}{2})} .$$

Then the asymptotic representation

$$f_r(v) \sim g_r(v) \quad \text{as} \quad v^2 \to 1 \tag{23}$$

holds true.

PROOF: With the integral representation (20) in the form

$$f_r(v) = \int \cdots \int \left((1-v^k) N_r^{\frac{k}{k-1}} - (P - v^k N_r^{\frac{k}{k-1}}) \right)^{d_{p,r}} \, dt_1 \dots dt_{r-1},$$

where the integration condition is given by

$$0 \le P - v^k N_r^{\frac{k}{k-1}} \le (1-v^k) N_r^{\frac{k}{k-1}}.$$

Further, we put $t_j = v + u_j$ in (19) such that P has its minimum at the point $(u_1, u_2, \dots, u_{r-1}) = (0, 0, \dots, 0)$. So we obtain for (19)

$$P - v^k N_r^{\frac{k}{k-1}} = \sum_{\nu=2}^{k} \binom{k}{\nu} v^{k-\nu} \left\{ \sum_{j=1}^{r-1} n_j^{\frac{k}{k-1}} u_j^\nu + n_r^{\frac{k}{k-1}} \left(-\sum_{j=1}^{r-1} \left(\frac{n_j}{n_r} \right)^{\frac{k}{k-1}} u_j \right)^\nu \right\}.$$

Moreover, we make the substitution

$$u_j = \left(\left(\frac{N_r}{n_j} \right)^{\frac{k}{k-1}} (1-v^k) \right)^{\frac{1}{2}} y_j, \qquad j = 1, 2, \dots, r-1,$$

and we define a function

$$Q = Q(v, \vec{y_{r-1}}) = \frac{P - v^k N^{\frac{k}{k-1}}}{1-v^k} N^{\frac{k}{k-1}}.$$

Then $Q(v, \vec{y_{r-1}})$ and $f_r(v)$ are represented by

$$\begin{aligned} Q = \sum_{\nu=1}^{k} \binom{k}{\nu} v^{k-\nu} & \left\{ \sum_{j=1}^{r-1} \left(\left(\frac{n_j}{N_r} \right)^{\frac{k}{k-1}} (1-v^k) \right)^{\frac{\nu}{2}-1} y_j^\nu + \right. \\ & \left. + \left(\frac{n_r}{N_r} \right)^{\frac{k}{k-1}} \left(-\sum_{j=1}^{r-1} \left(\frac{n_j N_r}{n_r^2} \right)^{\frac{k}{2k-2}} y_j \right)^\nu \left(1 - v^k \right)^{\frac{\nu}{2}-1} \right\}, \end{aligned} \tag{24}$$

$$\begin{aligned} f_r(v) = N_r^{d_{p,r} \frac{k}{k-1}} & \left(\frac{N_r^{r-1}}{n_1 \cdots n_{r-1}} \right)^{\frac{k}{2k-2}} \left(1 - v^k \right)^{d_{p,r} + \frac{r-1}{2}} \times \\ & \times \int \cdots \int_{Q \le 1} (1 - Q(v, \vec{y_{r-1}}))^{d_{p,r}} \, dy_1 \cdots dy_{r-1}. \end{aligned} \tag{25}$$

Clearly, the limit values of

$$f_r(v)\left(1-v^k\right)^{-d_{p,r}-\frac{r-1}{2}} \qquad \text{as} \qquad v \to \pm 1$$

exist such that

$$f_r(v) \sim N_r^{d_{p,r}\frac{k}{k-1}} \left(\frac{N_r^{r-1}}{n_1 \dots n_{r-1}}\right)^{\frac{k}{2k-2}} \left(1-v^k\right)^{d_{p,r}+\frac{r-1}{2}} \times$$
$$\times \int \dots \int_{0 \le P_2 \le 1} (1-P_2(\vec{y_{r-1}}))^{d_{p,r}}\ dy_1 \cdots dy_{r-1}$$

with

$$P_2 = P_2(\vec{y_{r-1}}) = \binom{k}{2}\left\{\sum_{j=1}^{r-1} y_j^2 + \left(\frac{n_r}{N_r}\right)^{\frac{k}{k-1}} \left(\sum_{j=1}^{r-1}\left(\frac{n_j N_r}{n_r^2}\right)^{\frac{k}{2k-2}} y_j\right)^2\right\}.$$

It remains to compute the integral, which is easy. Then we obtain the asymptotic representation (23) with the function (22). □

Now we are going to prove the asymptotic representation (14) by means of Lemma 1. First of all we remark that we may assume $n_1, n_2, \dots, n_{r-1} \le n_r$ without loss of generality in Lemma 2. Then

$$n_r \le N_r \le r^{1-\frac{1}{k}} n_r.$$

So it follows that the coefficients of y_j in (24) are bounded. Hence, the integral in (25) is bounded, and we obtain

$$f_r(v)\left(1-v^k\right)^{-d_{p,r}-\frac{r-1}{2}} \ll N_r^{d_{p,r}\frac{k}{k-1}} \left(\frac{N_r^{r-1}}{n_1 \cdots n_{r-1}}\right)^{\frac{k}{2k-2}}$$
$$\ll N_r^{d_{p,r}\frac{k}{k-1}} \left(\frac{n_r N_r^{r-2}}{n_1 \cdots n_{r-1}}\right)^{\frac{k}{2k-2}}.$$

This estimation now holds in the whole interval $-1 \le v \le +1$. Moreover we may write

$$f_r(v) = \left(1-v^k\right)^{d_{p,r}+\frac{r-1}{2}} \left(\sum_{\nu=0}^{q} c_\nu \left(1-v^k\right)^{\frac{\nu}{2}} + h_{r,q}(v)\right), \tag{26}$$

where q is some positive integer. All the coefficients c_ν and the derivatives of the remainder function $h_{r,q}(v)$ up to the order $[d_{p,r} + \frac{r+q}{2}]$ have the same order with respect to $n_1, n_2, \dots, n_r$ as above.

Now we write

$$f_r(v) = g_r(v) + (f_r(v) - g_r(v)).$$

In order to obtain a first approximation of $F^{(d)}_{k,p,r}(x;\vec{n_r})$ we replace $f_r(v)$ by $g_r(v)$ in (21) and we consider the function

$$G = G^{(d)}_{k,p,r}(x;\vec{n_r}) = \left(\frac{n_1\cdots n_{r-1}}{N_r^{p+k(d-1)}}\right)^{\frac{1}{k-1}} \frac{x^{d+\frac{p}{k}}}{n_r}\int_{-1}^{+1} g_r(v)\cos(2\pi N_r x^{\frac{1}{k}} v)\,dv.$$

Then, by (22) and replacing v by t,

$$G = c_{p,r}\left(\frac{N_r^r}{n_1\cdots n_r}\right)^{\frac{k-2}{2k-2}} x^{d+\frac{p}{k}}\int_{-1}^{+1}(1-t_k)^{d_{p,r}+\frac{r-1}{2}}\cos(2\pi N_r x^{\frac{1}{k}} t)\,dt.$$

We apply Lemma 1 with

$$\alpha = -1, \qquad \beta = +1, \qquad \lambda = \mu = \frac{p-r}{k} + \frac{r-1}{2} - \left[\frac{p-r}{k} + \frac{r-1}{2}\right],$$

$$y = 2\pi N_r x^{\frac{1}{k}},$$

$$\Phi(t) = (1-t^2)^{d+1+[\frac{p-r}{k}+\frac{r-1}{2}]}\left(\frac{1-t^k}{1-t^2}\right)^{d_{p,r}+\frac{r-1}{2}},$$

$$M = d + 3 + \left[\frac{p-r}{k} + \frac{r-1}{2}\right].$$

Then the terms with $0 \le m \le d + [\frac{p-r}{k} + \frac{r-1}{2}]$ vanish in both sums in (18). The both non-vanishing terms with $m = d + 1 + [\frac{p-r}{k} + \frac{r-1}{2}]$ are of order

$$y^{-(d_{p,r}+\frac{r+1}{2})}.$$

Analogously to the proof of (13) the both next terms and the error term are of order

$$y^{-(d_{p,r}+1+\frac{r+1}{2})}.$$

Thus we get (14), even with the better error term $O(x^{-1/k})$, provided that

$$\frac{p-r}{k} + \frac{r-1}{2} \notin \mathbb{N}.$$

Otherwise, we obtain the same result by partial integrating.

Now it is clear from (22) and (26) that

$$f_r(v) - g_r(v) \sim c_1\left(1 - v^k\right)^{d_{p,r}+\frac{r-1}{2}+\frac{1}{2}} \qquad \text{as} \qquad v^k \to 1$$

with a possibly non-vanishing coefficient c_1. This gives the error term in (14), and the proof is complete.

5. Proof of Theorem 3

Let the function $F(t)$ be integrable on every interval $0 \le t \le x$. Let $F^{(r)}(x)$ be defined by

$$F^{(r)}(x) = \frac{1}{(r-1)!} \int_0^x (x-t)^{r-1} F(t)\, dt$$

for $r \in \mathbb{N}$. Let $D_y^{(r)}(H(x))$ denote the r-fold iterated difference operator with respect to a function $H(x)$. That is

$$D_y^{(r)}(H(x)) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} H(x+jy). \tag{27}$$

Then it is easily seen that

$$D_y^{(r)}(F^{(r)}(x)) = \int\limits_{x}^{x+y} dt_1 \int\limits_{t_1}^{t_+y} dt_2 \ldots \int\limits_{t_{r-1}}^{t_{r-1}+y} F(t_r)\, dt_r. \tag{28}$$

We use this for the function $R_{k,p}^{(d)}(x)$ with $r = d$, and we shall now constantly assume that $d = [\frac{p-1}{2}] + 1$. Then all the infinite series in (10) are absolutely convergent. We get

$$D_y^{(d)}(R_{k,p}^{(d)}(x)) \ge y^d R_{k,p}(x), \tag{29}$$

$$(-1)^d D_{-y}^{(d)}(R_{k,p}^{(d)}(x)) \le y^d R_{k,p}(x) \tag{30}$$

for $x > dy > 0$. We consider only the inequality (29), since the corresponding results for (30) in the opposite direction follow analogously. We obtain from (9) and (29)

$$R_{k,p}(x) \le y^{-d} D_y^{(d)} \left(V_{k,p}^{(d)} x^{d+\frac{p}{k}} \right) + y^{-d} \sum_{r=1}^{p} D_y^{(d)}(H_{k,p,r}^{(d)}(x)). \tag{31}$$

Since

$$V_{k,p}^{(d)} x^{d+\frac{p}{k}} = \frac{1}{(d-1)!} V_{k,p} \int_0^x (x-t)^{d-1} t^{\frac{p}{k}}\, dt$$

we obtain from (28)

$$\begin{aligned} y^{-d} D_y^{(d)} \left(V_{k,p}^{(d)} x^{d+\frac{p}{k}} \right) &\le V_{k,p} (x+dy)^{\frac{p}{k}} \\ &\le V_{k,p} x^{\frac{p}{k}} + O\left(x^{\frac{p}{k}-1} y \right). \end{aligned} \tag{32}$$

In order to estimate the differences of $H_{k,p,r}^{(d)}(x)$ in (31) we must distinguish between $1 \le r < (k+2p)/(k+2)$ and $r \ge (k+2p)/(k+2)$. In the first case the infinite series in (10) are all absolutely convergent for $d = 0$. Hence, by (10), (14) and (28),

$$y^{-d} D_y^{(d)}(H_{k,p,r}^{(d)}(x)) \ll x^{\frac{p-r}{k}(1-\frac{1}{k})+\frac{r-1}{2k}} \quad \text{for } 1 \le r < \frac{k+2p}{k+2}. \tag{33}$$

Later on, we feel us compelled to consider the case $r = 1$ once more.

For $(k+2p)/(k+2) \le r \le p$ we divide the sum (10) into two parts. We write

$$\begin{aligned} H_{k,p,r}^{(d)}(x) &= 2^r \binom{p}{r} V_{k,p-r}^{(d)}(S_1 + S_2), \\ S_1 &= \sum_{N_r \le z_r} F_{k,p,r}^{(d)}(x; \vec{n_r}), \\ S_2 &= \sum_{N_r > z_r} F_{k,p,r}^{(d)}(x; \vec{n_r}) \end{aligned}$$

with a suitable parameter $z_r > 1$. We apply the difference operator with the representation (27) for the sum S_1 and with the representation (28) for the sum S_2. Then

$$\begin{aligned} y^{-d} D_y^{(d)}(H_{k,p,r}^{(d)}(x)) \ll &\max_{x \le w \le x+dy} \left| \sum_{N_r \le z_r} F_{k,p,r}^{(d)}(w; \vec{n_r}) \right| \\ &+ y^{-d} \max_{x \le w \le x+dy} \left| \sum_{N_r > z_r} F_{k,p,r}^{(d)}(w; \vec{n_r}) \right|. \end{aligned} \tag{34}$$

Furthermore, we arrange the values $n_1, n_2, \dots, n_r$ with respect to their quantity. Let us assume without loss of generality that $n_1, n_2, \dots, n_{r-1}$ are less than n_r. With a view of (14) we consider the exponential sum

$$\sum_{M_1 \le n_r \le M_2} e^{2\pi i N_r w^{1/k}}$$

with $M < M_1 < M_2 \le 2M$ for fixed $n_1, n_2, \dots, n_{r-1}$. We apply a special case of Theorem 2.6 from [3, p. 34]: Let $f(t)$ be a real function with continuous derivatives up to the third order for $M \le t \le M' \le 2M$ and let

$$0 < \lambda_3 \ll |f^{(3)}| \ll \lambda_3.$$

Then

$$\sum_{M < m \le M'} e^{2\pi i f(m)} \ll M\lambda_3^{\frac{1}{6}} + M^{\frac{1}{2}}\lambda_3^{-\frac{1}{6}}.$$

We use this theorem with

$$f(t)=\left(n_1^{\frac{k}{k-1}}+\ldots+n_{r-1}^{\frac{k}{k-1}}+t^{\frac{k}{k-1}}\right)^{1-\frac{1}{k}}w^{\frac{1}{k}}.$$

Then

$$\lambda_3=\frac{1}{M^2}w^{\frac{1}{k}}\asymp\frac{1}{M^2}x^{\frac{1}{k}}$$

and

$$\sum_{M_1\le n_r\le M_2}e^{2\pi iN_rw^{1/k}}\ll M^{\frac{2}{3}}x^{-\frac{1}{6k}}.$$

Note that $n_r\asymp M,\quad N_r\asymp M$. Hence, we get

$$\sum_{n_1\ll M}\cdots\sum_{n_{r-1}\ll M}\sum_{n_r\asymp M}\frac{\cos\left(2\pi N_rw^{1/k}-\frac{\pi}{2}\left(d_p+\frac{r+1}{2}\right)\right)}{(n_1\cdots n_r)^{\frac{k-2}{2k-2}}\,N_r^{d_{p,r}+\frac{r}{2k-2}+\frac{1}{2}}}\left\{1+O\left(x^{-\frac{1}{2k}}\right)\right\}$$
$$\ll M^{-d_{p,r}+\frac{r-1}{2}}\left\{M^{-\frac{1}{3}}x^{\frac{1}{6k}}+M^{-\frac{1}{6}}x^{-\frac{1}{6k}}+x^{-\frac{1}{2k}}\right\}.$$

We use this result for the first term in (34) with $M=z_r2^{-m}$ and for the second term with $M=z_r2^{m}$. Then let m run through the sequence $m=0,1,\ldots$. Therefore, we obtain from (14) and (34)

$$y^{-d}D_y^{(d)}(H_{k,p,r}^{(d)}(x))\ll x^{\frac{p-r}{k}(1-\frac{1}{k})+\frac{r-1}{2k}}\sum_{m=0}^{\infty}\left(1+\left(\frac{x^{1-\frac{1}{k}}}{yz_r2^m}\right)^d\right)(A_m+A_{-m})$$

with

$$A_m=\left(z_r2^{-m}\right)^{-\frac{p-r}{k}+\frac{r-1}{2}}\left\{\left(z_r2^{-m}\right)^{-\frac{1}{3}}x^{\frac{1}{6k}}+\left(z_r2^{-m}\right)^{-\frac{1}{6}}x^{-\frac{1}{6k}}+x^{-\frac{1}{2k}}\right\}.$$

It is easily seen that the first sum makes only sense if

$$\frac{r-1}{2}>\frac{p-r}{k}+\frac{1}{3}.$$

Thus, we assume this inequality for the present. Then

$$y^{-d}D_y^{(d)}(H_{k,p,r}^{(d)}(x))\ll x^{\frac{p-r}{k}(1-\frac{1}{k}+\frac{r-1}{2k})}\left(1+\left(\frac{x^{1-\frac{1}{k}}}{yz_r}\right)^d\right)\times$$
$$\times z_r^{-\frac{p-r}{k}+\frac{r-1}{2}}\left(z_r^{-\frac{1}{3}}x^{\frac{1}{6k}}+z_r^{-\frac{1}{6}}x^{-\frac{1}{6k}}+x^{-\frac{1}{2k}}\right)$$

Now we put

$$z_r = \frac{1}{y} x^{1-\frac{1}{k}}$$

provided that

$$y < x^{1-\frac{1}{k}}.$$

Then we get

$$y^{-d} D_y^{(d)}(H_{k,p,r}^{(d)}(x)) \ll x^{\frac{r-1}{2}} y^{\frac{p-r}{k} - \frac{r-1}{2}} \left(y^{\frac{1}{3}} x^{\frac{1}{2k} - \frac{1}{3}} + y^{\frac{1}{6}} x^{-\frac{1}{6}} + x^{-\frac{1}{2k}} \right).$$

Further, suppose that

$$y^{\frac{1}{k}+\frac{1}{2}} < x^{\frac{1}{2}}.$$

Then the right-hand side of this inequality is monotonically increasing with respect to r. In order to find an optimal y we balance the first order term with $r = p$ against the term $O(x^{p/k-1})$ in (32). That is, we put

$$y = x^{1-\frac{1}{k}\frac{6p-3}{3p+1}}.$$

Then the above inequalities are satisfied, even for $12p \geq 5k + 6$, and we obtain

$$y^{-d} D_y^{(d)}(H_{k,p,r}^{(d)}(x)) \ll x^{\frac{p-2}{k} + \frac{5}{k(3p+1)}} = x^{\lambda_{p,k}} \quad \text{for} \quad \frac{r-1}{2} > \frac{p-r}{k} + \frac{1}{3}. \tag{35}$$

Now we consider the case

$$\frac{r-1}{2} \leq \frac{p-r}{k} + \frac{1}{3}\,. \tag{36}$$

Since we assume that $k < 2p - 4$, we obtain

$$2 < \frac{k+2p}{k+2} \leq r \leq \frac{k+2p}{k+2} + \frac{2k}{3k+6}\,.$$

This inequality can be fulfilled for at most one $r > 2$. Then we estimate the exponential sum trivially and we get analogously for this r

$$y^{-d} D_y^{(d)}(H_{k,p,r}^{(d)}(x)) \ll x^{\frac{p-r}{k}(1-\frac{1}{k}) + \frac{r-1}{2k}} \left(1 + \left(\frac{x^{1-\frac{1}{k}}}{y z_r} \right)^d \right) z_r^{-\frac{p-r}{k} + \frac{r-1}{2}}$$

provided that $(r-1)/2 < (p-r)/k$. In case of equality $(r-1)/2 = (p-r)/k$ we let m in the first sum of (34) only run in the interval $0 \leq m \ll \log z_r$. Thus we get an additional factor $\log z_r$. We use the same values for z_r and y. Then

$$y^{-d} D_y^{(d)}(H_{k,p,r}^{(d)}(x)) \ll x^{\omega_{k,p}} \log x$$

with

$$\omega_{k,p} = \frac{p-r}{k}(1-\frac{1}{k}) + \frac{r-1}{2k} + (\frac{r-1}{2} - \frac{p-r}{k})\frac{3p-4}{k(3p+1)}.$$

We find, by (36),

$$\omega_{k,p} \le \frac{p-r}{k} + \frac{2p-1}{k(3p+1)} < \lambda_{p,k} \quad \text{for} \ \ r > 2.$$

Hence, (35) holds true for all $r \ge (k+2p)/(k+2)$, and the above factor $\log x$ is meaningless.

The estimations (33) are less than the estimation (35) for all $r \ge 1$ if $k \le p$. Therefore, we obtain from (31), (32), (33), (35)

$$R_{k,p}(x) \le V_{k,p} x^{\frac{p}{k}} + O\left(x^{\lambda_{p,k}}\right) \quad \text{for} \ \ k \le p. \tag{37}$$

Now let $k > p$. Then the estimations (33) are less than the estimations (35) only for $r \ge 2$. So we conclude for the time being

$$R_{k,p}(x) \le V_{k,p} x^{\frac{p}{k}} + y^{-d} D_y^{(d)}(H_{k,p,1}^{(d)}(x)) + O\left(x^{\lambda_{p,k}}\right), \tag{38}$$

of course with the same y as above. Now we consider (27) with $r = d$ and $H(x) = H_{k,p,1}^{(d)}(x)$. In the Taylor-expansion of (27) with respect to y the first $d-1$ terms vanish. Therefore, it exists a ϑ with $0 < \vartheta < 1$ such that

$$\begin{aligned} y^{-d} D_y^{(d)}(H_{k,p,1}^{(d)}(x)) &= \sum_{j=0}^{d} (-1)^{d-j} \binom{d}{j} j^d \frac{1}{d!} H_{k,p,1}(x + j\vartheta y) \\ &= H_{k,p,1}(x) + \sum_{j=0}^{d} (-1)^{d-j} \binom{d}{j} \frac{j^d}{d!} S_j \end{aligned} \tag{39}$$

with

$$S_j = H_{k,p,1}(x + j\vartheta y) - H_{k,p,1}(x)$$

and, by (3),

$$T_j = \frac{1}{pV_{k,p-1}} S_j = \psi_{p/k}^{(k)}\left((x + j\vartheta y)^{\frac{1}{k}}\right) - \psi_{p/k}^{(k)}(x^{\frac{1}{k}}).$$

Further, we obtain from [3, Lemma 3.12, p. 147],

$$\begin{aligned} T_j = 2 \sum_{0 \le n^k \le x + j\vartheta y} \left(x + j\vartheta y - n^k\right)^{\frac{p-1}{k}} - 2 \sum_{0 \le n^k \le x} \left(x - n^k\right)^{\frac{p-1}{k}} \\ - \frac{2\Gamma(\frac{p-1}{k}+1)\Gamma(\frac{1}{k})}{k\Gamma(\frac{p}{k}+1)} \left((x + j\vartheta y)^{\frac{p}{k}} - x^{\frac{p}{k}}\right), \end{aligned}$$

where the terms with $n=0$ get a factor $1/2$. Since $y = o(x^{1-\frac{1}{k}})$ we have between $x^{1/k}$ and $(x+j\vartheta y)^{1/k}$ and also between $(x-y)^{1/k}$ and $x^{1/k}$ at most one natural number. Hence

$$\begin{aligned} T_j &\ll \sum_{0 \le n^k \le x-y} \left(\left(x + j\vartheta y - n^k\right)^{\frac{p-1}{k}} - \left(x - n^k\right)^{\frac{p-1}{k}} \right) y^{\frac{p-1}{k}} + x^{\frac{p}{k}-1} y \\ &\ll \sum_{0 \le n^k \le x-y} y \left(x - n^k\right)^{\frac{p-1}{k}-1} y^{\frac{p-1}{k}} + x^{\frac{p}{k}-1} y \\ &\ll y^{\frac{p-1}{k}} + x^{\frac{p}{k}} y \\ &\ll x^{\lambda_{p,k}} \end{aligned}$$

what is the case with the above y and because of $k < 2p-4$. Hence, we obtain from (3), (38), (39) and the last estimation

$$R_{k,p}(x) \le V_{k,p} x^{\frac{p}{k}} + p V_{k,p} \psi_{p/k}(x^{\frac{1}{k}}) + O(x^{\lambda_{p,k}}) \tag{40}$$

for $k > p$ and $k < 2p-4$.

Applying the inequality (30) we obtain inequalities of type (37) and (40), but in the other direction. This completes the proof of Theorem 3. □

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