Comment.Math.Univ.Carolin. 40,2 (1999)393–401

A formula for calculation of metric dimension of converging sequences

Ladislav Mišík Jr., Tibor Žáčik

Abstract. Converging sequences in metric space have Hausdorff dimension zero, but their metric dimension (limit capacity, entropy dimension, box-counting dimension, Hausdorff dimension, Kolmogorov dimension, Minkowski dimension, Bouligand dimension, respectively) can be positive. Dimensions of such sequences are calculated using a different approach for each type.

In this paper, a rather simple formula for (lower, upper) metric dimension of any sequence given by a differentiable convex function, is derived.

Keywords: metric dimension, limit capacity, entropy dimension, box-counting dimension, Hausdorff dimension, Kolmogorov dimension, Minkowski dimension, Bouligand dimension, converging sequences, convex sequences, differentiable function

Classification: 54F50, 40A05

Let X be a totally bounded metric space and let ε be a positive real number. Denote by $N(\varepsilon, X)$ the least number of closed balls with diameter ε covering X. For simplicity we use also the notation $N(\varepsilon)$ when dealing with a fixed metric space.

The following characteristic of the space X

$$\underline{\dim}\, X = \liminf_{r>0} \frac{\log N(r,X)}{-\log r}$$

was first defined in [P–S] and it is called the *lower metric dimension* (limit capacity, entropy dimension, box-counting dimension, Hausdorff dimension, Kolmogorov dimension, Minkowski dimension, Bouligand dimension, respectively) of the space X. Similarly,

$$\overline{\dim}\, X = \limsup_{r>0} \frac{\log N(r,X)}{-\log r}$$

is known as the *upper metric dimension* (limit capacity, entropy dimension, box-counting dimension, Hausdorff dimension, Kolmogorov dimension, Minkowski dimension, Bouligand dimension, respectively). In case $\underline{\dim}\, X = \overline{\dim}\, X$ we say that for X there exists metric dimension.

Partially supported by Grant Agency VEGA 1/4064/97 and 95/5305/471.

For a countable set, these dimensions can be positive, whereas the well known Hausdorff dimension of a countable set is zero. Hence the metric dimension, which is always equal to or greater than the Hausdorff dimension, provides a more sensitive characteristic for some sets. The only method of calculation of (upper) metric dimension of sequences, according to authors knowledge, was developed in [H, Theorem 3.1]. The result was based on the paper [B–T] and it allows to calculate the metric dimension of compact nowhere dense subsets of the set of real numbers by means of convergence of infinite series. Some examples of calculation of metric dimension can be found in [K–A].

The aim of our paper is to give an analytic formula for the calculation of the metric dimension of converging sequences by means of limits. Notice that the significance of converging sequences in the theory of metric dimension is documented by the result of [M–Z] which says that in each compact metric space there exists a converging sequence having the same upper metric dimension as the whole space does.

In this paper we consider the class of decreasing sequences of positive real numbers converging to zero. For such sequences $\{a_n\}$ we will write $A = \{a_n\}$ instead of $A = \bigcup_{n\in\mathbb{N}} \{a_n\} \cup \{\lim_{n\to\infty} a_n\} = \bigcup_{n\in\mathbb{N}} \{a_n\} \cup \{0\}$. All logarithms in the paper are taken to the base greater than one.

Definition 1. By a *convex sequence* (c-sequence, in short) we mean a decreasing sequence of positive real numbers converging to zero, which is convex as a function defined on the set of all positive integers.

This means that a sequence $\{a_n\}_{n=1}^{\infty}$ is a c-sequence iff for all $n \in \mathbb{N}$

$$a_n - a_{n+1} > a_{n+1} - a_{n+2}\,.$$

In general, the function $N(\cdot\,, X)$ fulfills the condition

$$\lim_{r\to r_0^+} N(r,X) + 1 \le \lim_{r\to r_0^-} N(r,X) \le 2\cdot \lim_{r\to r_0^+} N(r,X),$$

for a set $X \subset \mathbb{R}$ and for any point r_0 of its discontinuity ([M–Z, Theorem 3]).

In the case of c-sequences a more precise estimation holds.

Proposition 1. *Let $A = \{a_n\}$ be a c-sequence. Then for any point r_0 of discontinuity of the function $N(\cdot\,, A)$,*

$$\lim_{r\to r_0^-} N(r,A) = \lim_{r\to r_0^+} N(r,A) + 1.$$

Proof: Let r_0 be a point of discontinuity of the function $N(\cdot\,, A)$. Note that a covering of A by the least number of closed intervals of lengths r_0 can be chosen such that right endpoints of all intervals in a covering are members of the sequence $\{a_n\}$. Thus take $\mathcal{C} = \{C_i\}_{i=1}^{n} = \{\langle p_i, a_{k_i}\rangle\}_{i=1}^{n}$, a minimal covering of A. Now consider the system of intervals $\mathcal{C}' = \{C_i'\}_{i=1}^n = \{\langle p_i - d_i, a_{k_i+1}\rangle\}_{i=1}^n$, where $d_i = a_{k_i} - a_{k_i+1}$. As differences between successive members of A are decreasing, the system $\mathcal{C}'$ possesses the following properties:

(i) $\mathcal{C}'$ covers A except a_1;

(ii) both C_n and C_n' contain an infinite number of members of A. On the other hand, if C_i contains m_i members $\{a_{k_i}, \dots, a_{k_i+m_i-1}\}$ of A then, C_i' contains at least m_i members $\{a_{k_i+1}, \dots, a_{k_i+m_i}\}$ of A;

(iii) $a_{k_i+m_i}$ is greater than the left endpoint of the interval C_i', $i = 1, \dots, n-1$, i.e., $a_{k_i+m_i} > p_i - d_i$. The endpoint of C_n' is a negative number, say h_n.

Denote by h_i the positive number $a_{k_i+m_i} - (p_i - d_i)$ for all $i = 1, \dots, n-1$ and choose a positive ε, $0 < \varepsilon < \min\{h_i \colon 1 \le i \le n-1\}$.

The system $\mathcal{C}'' = \{\langle p_i - d_i + \varepsilon, a_{k_i+1}\rangle\}_{i=1}^n$ covers A except a_1 as well. Thus the system $\mathcal{C}'' \cup \langle a_1 - r_0 + \varepsilon, a_1\rangle$ is a covering of A by $n+1$ intervals of diameter $r_0 - \varepsilon$, which yields the requested equality. $\square$

Many examples of converging to zero sequences are given by differentiable convex functions. For example, $\{a_n\} = \{\frac{1}{n^k}\}$ is derived from the function $f(x) = x^{-k}$ by the relation $a_n = f(n)$. Similarly, the sequences $\{\frac{1}{k^n}\}$, $\{\frac{1}{\log n}\}$ can be obtained from the functions $f(x) = k^{-x}$ and $f(x) = (\log x)^{-1}$, respectively. The resulting sequences are c-sequences.

The opposite is also true. For any c-sequence $\{a_n\}$ there exists a differentiable convex decreasing function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$, with limit zero at infinity, such that $a_n = f(n)$. Any such C^1-function will be called an *associated function* to this sequence.

The construction of an associated function to a convex sequence is shown in the following example.

Example. Let $\{a_n\}$ be a convex sequence. Put $d_1 = 2(a_2 - a_1)$ and define $d_n = \frac{a_{n+1} - a_{n-1}}{2}$ for each $n = 2, 3, 4 \dots$. Then evidently $d_n < a_{n+1} - a_n < d_{n+1} < 0$. Further, let $p_1 = \frac{a_3 - 2a_2 + a_1}{2a_1 - 2a_2}$ and $p_n = \frac{a_{n+2} - 2a_{n+1} + a_n}{a_{n+1} - 2a_n + a_{n-1}}$ for $n > 1$. Define $\varphi_n(x) = (d_{n+1} - d_n)(x - n)^{p_n} + d_n$ for $x \in \langle n, n+1)$ and for each $n = 1, 2, 3 \dots$. Let $f_n(x) = a_n + \int_n^x \varphi_n(t)\, dt$. Finally, define the function f in pieces by $f(x) = f_n(x)$ for each $n = 1, 2, 3 \dots$ and all $x \in \langle n, n+1)$. The straightforward computation yields that f is an associated function to the sequence $\{a_n\}$.

Lemma 1. *Let $A = \{a_n\}$ be a c-sequence and let $r < a_1 - a_2$ be a positive number. Let f be any associated function to A. Then there is a unique integer n_r and a unique real number $x \in (n_r, n_r + 2)$, such that*

$$a_{n_r+1} - a_{n_r+2} \le r < a_{n_r} - a_{n_r+1}\,, \tag{1}$$

and

$$r = -f'(x).$$

Proof: The existence of an index n_r fulfilling (1) is obvious. The Mean Value Theorem implies the existence of a $u \in (n_r, n_r+1)$ and $v \in (n_r+1, n_r+2)$ such that $a_{n_r+1} - a_{n_r} = f'(u)$ and $a_{n_r+2} - a_{n_r+1} = f'(v)$. By (1)

$$-f'(v) = a_{n_r+1} - a_{n_r+2} \leq r < a_{n_r} - a_{n_r+1} = -f'(u).$$

From the Darboux property of derivatives there is an $x \in (u, v\rangle \subset (n_r, n_r+2)$ such that $r = -f'(x)$. Due to monotonicity of f' this x is unique. □

Remark. As the function $-f'$ is positive and decreasing, Lemma 1 provides a one-to-one correspondence between positive real numbers r tending to zero and positive real numbers x tending to infinity.

Lemma 2. *Let $A = \{a_n\}$ be a c-sequence and let $r < a_1 - a_2$ be a positive number. Let n_r be the integer from Lemma* 1. *Then*

$$n_r + 1 + \frac{a_{n_r+2}}{2r} \leq N(r) \leq n_r + 2 + \frac{a_{n_r+2}}{r}\,. \tag{2}$$

Proof: Fix $i > n_r$ and cover the sequence A by intervals of length r as follows. $a_1, \dots, a_i$ can be covered by i intervals and the rest of the sequence A by $\left[\frac{a_{i+1}}{r}\right]+1$ intervals. Thus $N(r) \leq i + \frac{a_{i+1}}{r} + 1$, so the right inequality is proved.

On the other hand, to cover the first n_r+1 terms of A we need n_r+1 intervals of length r. Any covering of the rest of A consists of some number of intervals of the length r, and some number of gaps between these intervals. The number of gaps is less than the number of intervals. Since A is a c-sequence, the definition of n_r yields that the length of each gap is less than r. Hence the sum of lengths of covering intervals is greater than one half of the length of the interval $\langle 0, a_{n_r+2}\rangle$. Therefore the number of intervals needed to cover $a_{n_r+2}, a_{n_r+3}, \dots$ is greater than $\frac{a_{n_r+2}}{2r}$, so the total number of intervals to cover the whole sequence A is greater than $n_r + 1 + \frac{a_{n_r+2}}{2r}$. □

Lemma 3. *Let $A = \{a_n\}$ be a c-sequence and let f be any associated function to A. Then for each $x > 1$,*

$$x - 2 + \frac{f(x)}{-2f'(x)} \leq N\bigl(-f'(x)\bigr) \leq x + 2 + \frac{f(x)}{-f'(x)}\,.$$

Proof: Since f is an associated function to A, we can write $f(n)$ instead of a_n, $n = 1, 2, \dots$. Substituting $r = -f'(x)$ in (2) gives

$$n_r + 1 + \frac{f(n_r+2)}{-2f'(x)} \leq N\bigl(-f'(x)\bigr) \leq n_r + 2 + \frac{f(n_r+2)}{-f'(x)}\,. \tag{3}$$

The Mean Value Theorem guarantees the existence of a number $t \in (x, n_r+2)$ such that

$$f(n_r+2) = f(x) + f'(t)(n_r + 2 - x).$$

Substitute this to (3) to get

$$n_r + 1 + \frac{f(x) + f'(t)(n_r{+}2{-}x)}{-2f'(x)} \le N\big(-f'(x)\big) \le n_r + 2 + \frac{f(x) + f'(t)(n_r{+}2{-}x)}{-f'(x)}\,,$$

or

$$\begin{aligned}(4)\quad n_r + 1 + \frac{f(x)}{-2f'(x)} - \frac{-f'(t)}{-2f'(x)}(n_r + 2 - x) &\le N\big(-f'(x)\big)\\ &\le n_r + 2 + \frac{f(x)}{-f'(x)} - \frac{-f'(t)}{-f'(x)}(n_r + 2 - x).\end{aligned}$$

As f is a decreasing convex function, the function f' is negative and increasing. Therefore

$$0 < \frac{f'(t)}{f'(x)} < 1.$$

By Lemma 1, $x \in (n_r, n_r + 2)$, hence $n_r + 2 - x < 2$ and so

$$\frac{-f'(t)}{-2f'(x)}(n_r + 2 - x) < 1.$$

So the left side of the first inequality in (4) can be minorized

$$\begin{aligned}(5)\quad n_r + 1 + \frac{f(x)}{-2f'(x)} &- \frac{-f'(t)}{-2f'(x)}(n_r + 2 - x)\\ &= n_r + \frac{f(x)}{-2f'(x)} + \Big(1 - \frac{-f'(t)}{-2f'(x)}(n_r + 2 - x)\Big)\\ &> x - 2 + \frac{f(x)}{-2f'(x)}\,.\end{aligned}$$

Similarly, as $\frac{-f'(t)}{-f'(x)}(n_r + 2 - x)$ is a positive number, the right side of the second inequality in (4) can be majorized

$$(6)\qquad n_r + 2 + \frac{f(x)}{-f'(x)} - \frac{-f'(t)}{-f'(x)}(n_r + 2 - x) < x + 2 + \frac{f(x)}{-f'(x)}\,.$$

Using (5) and (6) in (4), we obtain the assertion of the lemma. □

Theorem. *Let $A = \{a_n\}$ be a c-sequence with an associated function f. Then*

$$\underline{\dim}\, A = \liminf_{x\to\infty} \frac{\log\Big(x + \frac{f(x)}{-f'(x)}\Big)}{-\log\big(-f'(x)\big)}\,,$$

and

$$\overline{\dim}\,A = \limsup_{x\to\infty} \frac{\log\left(x + \frac{f(x)}{-f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)}\,.$$

Proof: Evidently,

$$x - 2 + \frac{f(x)}{-2f'(x)} < x + 2 + \frac{f(x)}{-f'(x)} < 2\left(x - 2 + \frac{f(x)}{-2f'(x)}\right),$$

for $x > 6$. Taking logarithms and dividing by positive number $-\log\bigl(-f'(x)\bigr)$ gives

$$\frac{\log\left(x-2+\frac{f(x)}{-2f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} \le \frac{\log\left(x+2+\frac{f(x)}{-f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} \le \frac{\log 2}{-\log\bigl(-f'(x)\bigr)} + \frac{\log\left(x-2+\frac{f(x)}{-2f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)}\,.$$

Since $\lim\limits_{x\to\infty} \frac{\log 2}{-\log(-f'(x))} = 0$, the lim inf of the outer terms are equal and therefore

$$\liminf_{x\to\infty} \frac{\log\left(x - 2 + \frac{f(x)}{-2f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} = \liminf_{x\to\infty} \frac{\log\left(x + 2 + \frac{f(x)}{-f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)}\,.$$

Obviously,

$$\frac{\log\left(x - 2 + \frac{f(x)}{-2f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} \le \frac{\log\left(x + \frac{f(x)}{-f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} \le \frac{\log\left(x + 2 + \frac{f(x)}{-f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)},$$

and, from the assertion of Lemma 3,

$$\frac{\log\left(x - 2 + \frac{f(x)}{-2f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} \le \frac{\log\bigl(N(-f'(x))\bigr)}{-\log\bigl(-f'(x)\bigr)} \le \frac{\log\left(x + 2 + \frac{f(x)}{-f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)}\,.$$

Hence

$$\underline{\dim}\,A = \liminf_{x\to\infty} \frac{\log\bigl(N(-f'(x))\bigr)}{-\log\bigl(-f'(x)\bigr)} = \liminf_{x\to\infty} \frac{\log\left(x + \frac{f(x)}{-f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)},$$

by Sandwich Theorem.

The same result is true for $\overline{\dim}\,A$, considering lim sup instead of lim inf. □

The following Corollary 1 is an immediate consequence of Theorem.

Corollary 1. *Let $A = \{a_n\}$ be a c-sequence with an associated function f. Then*

$$\overline{\dim}\, A = \underline{\dim}\, A = \dim A = \lim_{x\to\infty} \frac{\log\left(x - \frac{f(x)}{f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)}\,,$$

provided the last limit exists.

The following consequence of Corollary 1 will be useful.

Corollary 2. *Let $A = \{a_n\}$ be a c-sequence with an associated function f. Then*

$$\overline{\dim}\, A = \underline{\dim}\, A = \dim A = \max\left\{\lim_{x\to\infty} \frac{\log x}{-\log\bigl(-f'(x)\bigr)}\,, 1 - \lim_{x\to\infty} \frac{\log\bigl(f(x)\bigr)}{\log\bigl(-f'(x)\bigr)}\right\},$$

provided both limits exist.

Proof:

$$\max\left\{x\,, \frac{f(x)}{-f'(x)}\right\} \le x - \frac{f(x)}{f'(x)} \le 2\max\left\{x\,, \frac{f(x)}{-f'(x)}\right\}$$

as both x and $\frac{f(x)}{-f'(x)}$ are positive. Consequently

$$\max\left\{\frac{\log x}{-\log\bigl(-f'(x)\bigr)}\,, \frac{\log\left(\frac{f(x)}{-f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)}\right\}$$

$$\le \frac{\log\left(x - \frac{f(x)}{f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} \le \frac{\log 2}{-\log\bigl(-f'(x)\bigr)} + \max\left\{\frac{\log x}{-\log\bigl(-f'(x)\bigr)}\,, \frac{\log\left(\frac{f(x)}{-f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)}\right\}$$

for sufficiently large x.

Take limits in the last inequalities, apply the Sandwich Theorem and notice that

$$\frac{\log\left(\frac{f(x)}{-f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} = 1 - \frac{\log\bigl(f(x)\bigr)}{\log\bigl(-f'(x)\bigr)}$$

to finish the proof. □

Examples

1) Let $\{a_n\}_{n=1}^{\infty} = \{1/n^{\alpha}\}_{n=1}^{\infty}$, $\alpha > 1$, with an associated function $f(x) = x^{-\alpha}$. Then

$$\dim\{1/n^{\alpha}\} = \lim_{x\to\infty} \frac{\log\left(x - \frac{f(x)}{f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} = \lim_{x\to\infty} \frac{\log\left(x + \frac{x^{-\alpha}}{\alpha x^{-\alpha-1}}\right)}{-\log(\alpha x^{-\alpha-1})}$$

$$= \lim_{x\to\infty} \frac{\log\left(x + \frac{x}{\alpha}\right)}{-\log\alpha + (\alpha+1)\log x} = \frac{1}{\alpha+1}\,.$$

2) Let $\{a_n\}_{n=1}^{\infty} = \{r^n\}_{n=1}^{\infty}$, $0 < r < 1$, with an associated function $f(x) = r^x$. Then

$$\dim\{r^n\} = \lim_{x\to\infty} \frac{\log\left(x - \frac{f(x)}{f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} = \lim_{x\to\infty} \frac{\log\left(x - \frac{r^x}{r^x \log r}\right)}{-\log(-r^x \log r)}$$

$$= \lim_{x\to\infty} \frac{\log\left(x - \frac{1}{\log r}\right)}{-\log(-\log r) - x\log r} = 0.$$

3) Let $\{a_n\}_{n=1}^{\infty} = \{1/\log n\}_{n=1}^{\infty}$, with an associated function $f(x) = (\log x)^{-1}$. Then

$$\dim\{1/\log n\} = \lim_{x\to\infty} \frac{\log\left(x - \frac{f(x)}{f'(x)}\right)}{-\log\bigl(-f'(x)\bigr)} = \lim_{x\to\infty} \frac{\log(x + x\log x)}{\log\bigl(x(\log^2 x)\bigr)}$$

$$= \lim_{x\to\infty} \frac{\log x + \log(1 + \log x)}{\log x + 2\log(\log x)} = 1.$$

4) Let $\{a_n\}_{n=1}^{\infty} = \{\frac{\pi}{2} - \arctan n\}_{n=1}^{\infty}$, with an associated function $f(x) = \frac{\pi}{2} - \arctan x$. Let us apply Corollary 2. Calculate

$$\lim_{x\to\infty} \frac{\log x}{-\log\left(-f'(x)\right)} = \lim_{x\to\infty} \frac{\log x}{-\log\left(\frac{1}{1+x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{\log x}{\log\left(1+x^2\right)} = \frac{1}{2},$$

and

$$1 + \lim_{x\to\infty} \frac{\log\bigl(f(x)\bigr)}{-\log\bigl(-f'(x)\bigr)} = 1 + \lim_{x\to\infty} \frac{\log\left(\frac{\pi}{2} - \arctan x\right)}{\log\left(1+x^2\right)}$$

$$= 1 + \lim_{x\to\infty} \frac{\frac{-1}{\left(\frac{\pi}{2} - \arctan x\right)(1+x^2)}}{\frac{2x}{1+x^2}} = 1 - \lim_{x\to\infty} \frac{\frac{1}{2x}}{\frac{\pi}{2} - \arctan x}$$

$$= 1 - \lim_{x\to\infty} \frac{\frac{-1}{2x^2}}{\frac{-1}{1+x^2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Thus

$$\dim\left\{\frac{\pi}{2} - \arctan n\right\}_{n=1}^{\infty} = \frac{1}{2}.$$

5) $\{a_n\}_{n=1}^{\infty} = \{1 - e^{-\frac{1}{n}}\}_{n=1}^{\infty}$, with an associated function $f(x) = 1 - e^{-\frac{1}{x}}$. Calculate

$$\lim_{x\to\infty} \frac{\log x}{-\log\left(-f'(x)\right)} = \lim_{x\to\infty} \frac{\log x}{-\log\left(\frac{e^{\frac{-1}{x}}}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{\log x}{\log\bigl(x^2 e^{\frac{1}{x}}\bigr)} = \lim_{x\to\infty} \frac{\log x}{\frac{1}{x} + \log x^2} = \frac{1}{2},$$

and

$$
\begin{aligned}
1+\lim_{x\to\infty}\frac{\log\bigl(f(x)\bigr)}{-\log\bigl(-f'(x)\bigr)} &= 1+\lim_{x\to\infty}\frac{\log\left(1-e^{\frac{-1}{x}}\right)}{\log\left(x^2e^{\frac{1}{x}}\right)}\\
&= 1+\lim_{x\to\infty}\frac{\frac{\frac{-e^{\frac{-1}{x}}}{x^2}}{1-e^{\frac{-1}{x}}}}{\frac{2xe^{\frac{1}{x}}+x^2\frac{-e^{\frac{1}{x}}}{x^2}}{x^2e^{\frac{1}{x}}}} = 1+\lim_{x\to\infty}\frac{\frac{-1}{2x-1}}{e^{\frac{1}{x}}-1}\\
&= 1+\lim_{x\to\infty}\frac{\frac{2}{(2x-1)^2}}{\frac{-e^{\frac{1}{x}}}{x^2}} = 1-\frac{1}{2}=\frac{1}{2}\,.
\end{aligned}
$$

Thus

$$\dim\bigl\{1-e^{-\frac{1}{n}}\bigr\}_{n=1}^{\infty}=\frac{1}{2}\,.$$

References

[M–Z] Mišík L., Žáčik T., *On some properties of the metric dimension*, Comment. Math. Univ. Carolinae **31.4** (1990), 781–791.

[P–S] Pontryagin L.S., Snirelman L.G., *Sur une propriete metrique de la dimension*, Annals of Math. **33** (1932), 156–162, Appendix to the Russian translation of "Dimension Theory" by W. Hurewitcz and H. Wallman, Izdat. Inostr. Lit. Moscow, 1948.

[H] Hawkes J., *Hausdorff measure, entropy and the independents of small sets*, Proc. London Math. Soc. (3) **28** (1974), 700–724.

[B–T] Besicovitch A.S., Taylor S.J., *On the complementary intervals of a linear closed sets of zero Lebesgue measure*, J. London Math. Soc. **29** (1954), 449–459.

[K–A] Koçak Ş., Azcan H., *Fractal dimensions of some sequences of real numbers*, Doğa – Tr. J. of Mathematics **17** (1993), 298–304.

Department of Mathematics, Faculty of Civil Engineering, Slovak Technical University, Radlinského 11, 813 68 Bratislava, Slovak Republic

E-mail: misik@vox.svf.stuba.sk

Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovak Republic

E-mail: zacik@savba.sk

(*Received* June 20, 1997, *revised* November 30, 1998)