A short proof on lifting of projection properties in Riesz spaces

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Abstract. Let L be an Archimedean Riesz space with a weak order unit u. A sufficient condition under which Dedekind [σ-]completeness of the principal ideal A_u can be lifted to L is given (Lemma). This yields a concise proof of two theorems of Luxemburg and Zaanen concerning projection properties of $C(X)$-spaces. Similar results are obtained for the Riesz spaces $B_n(T)$, $n = 1, 2, \dots$, of all functions of the nth Baire class on a metric space T.

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The purpose of this note is to give a short and concise proof of the following result established by Luxemburg and Zaanen ([3, Theorems 43.2 and 43.3]).

Theorem. *Let $C(X)$ and $C_b(X)$, respectively, denote the Riesz spaces of all real continuous and continuous and bounded, respectively, functions on a topological space X. Then the following conditions are equivalent.*

(i) *$C(X)$ has the* [*principal*] *projection property.*
(ii) *$C(X)$ is Dedekind* [σ-]*complete.*
(iii) *$C_b(X)$ has the* [*principal*] *projection property.*
(iv) *$C_b(X)$ is Dedekind* [σ-]*complete.*

As remarked in ([3, p. 283]), the only nontrivial implication is (iv) $\Rightarrow$ (ii). Our proof replaces a large part of the direct argument in [3] by an appeal to a lemma (see below), inspired by the classical proof of the Tietze extension theorem ([1, p. 158], the unbounded case).

Let S be a nonempty set. In the rest of the paper L denotes a Riesz subspace of the Riesz space $\mathbb{R}^S$ (pointwise ordering) containing the constant-one on S function e, and B_e denotes the set $\{f \in L : |f(s)| < 1, s \in S\}$. It is obvious that B_e is a (nonlinear) sublattice of A_e. The symbol $\circ$ denotes composition of functions.

Lemma. *If there exists a strictly increasing and continuous function ϕ from $\mathbb{R}$ onto $(-1, 1)$ such that both*

(a) *$\phi \circ f \in B_e$ for every $f \in L$, and*
(b) *$\phi^{-1} \circ g \in L$ for every $g \in B_e$,*

then L and B_e are order isomorphic as partially ordered sets. In particular, Dedekind [σ-]completeness of A_e implies Dedekind [σ-]completeness of L.

Examples. 1. If $L = C(X)$ then every strictly increasing, continuous and onto function $\phi : \mathbb{R} \to (-1, 1)$ fulfills both (a) and (b), and the same holds for the Riesz spaces $B_n(T)$, $n = 1, 2, \dots$, of all functions $T \to \mathbb{R}$ of the nth class on a metric space T.

2. If L consists of all continuous and piecewise functions on $[0, 1]$, then ϕ must be piecewise linear to fulfil the condition (a).

Proof of Lemma: By (a) and (b), L and B_e are order isomorphic as partially ordered sets (in the sense of the definition given in [3, p. 186]) via the mapping $\hat{\phi}(f) = \phi \circ f$, $f \in L$. Since, by ([3, Definitions 1.1 and 23.1]), Dedekind [σ-]completeness both is invariant under such isomorphisms and is heredited from A_e by B_e, the result follows. □

Proof of Theorem (the nontrivial implication (iv) $\Rightarrow$ (ii)): It follows by Lemma and Example 1. □

Remark. Since bounded functions of the nth Baire class $B_n^b(T)$, $n = 1, 2, \dots$, endowed with the sup-norm form AM-spaces with units ([2, Theorem 12.3.7]), the notions of the [principal] projection property and Dedekind [σ-]completeness coincide (by Theorem). Moreover, Lemma and Example 1 prove that $B_n^b(T)$ and $B_n(T)$ are Dedekind [σ-]complete simultaneously. These observations yield the result similar to that of Theorem when $C(X)$ is replaced by $B_n(T)$ and $C_b(X)$ by $B_n^b(T)$.

References

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