Productivity of coreflective classes of topological groups

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To George Strecker — inspiring mathematician and wonderful friend.

Abstract. Every nontrivial countably productive coreflective subcategory of topological linear spaces is κ-productive for a large cardinal κ (see [10]). Unlike that case, in uniform spaces for every infinite regular cardinal κ, there are coreflective subcategories that are κ-productive and not κ^+-productive (see [8]). From certain points of view, the category of topological groups lies in between those categories above and we shall show that the corresponding results on productivity of coreflective subcategories are also "in between": for some coreflections the results analogous to those in topological linear spaces are true, for others the results analogous to those for uniform spaces hold.

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A nonzero cardinal κ is called *measurable* if there exists a nontrivial κ-additive two-valued measure on κ being zero on points (κ-additivity of μ means that $\mu(\bigcup_I A_i) = \sum_I \mu(A_i)$ for every disjoint family $\{A_i\}_I$ with $|I| < \kappa$). The first measurable cardinal $\mathfrak{m}_0$ equals to ω_0 and the second one $\mathfrak{m}_1$ is the Ulam measurable cardinal (all cardinals less than $\mathfrak{m}_1$ are called Ulam nonmeasurable).

The first cardinal that admits a nontrivial $\mathbb{R}$-valued ω_1-additive measure is denoted by $\mathfrak{m}_{\mathbb{R}}$. It is known (see [13]) that the consistencies of the theories $\{ZFC + (\exists \mathfrak{m}_1)\}, \{ZFC + (\exists \mathfrak{m}_{\mathbb{R}})\}$ and $\{ZFC + (\exists \mathfrak{m}_{\mathbb{R}} \leq 2^\omega)\}$ are equivalent. Also, either $\mathfrak{m}_{\mathbb{R}} \leq 2^\omega$ or $\mathfrak{m}_{\mathbb{R}} = \mathfrak{m}_1$. The second case occurs, e.g., if MA holds or, more generally, if $\mathbb{R}$ satisfies the Baire category theorem for less than 2^ω open dense sets — see [4].

By $\mathfrak{s}$ (*sequential cardinal*) we denote the first cardinal such that there exists a noncontinuous sequentially continuous map $2^{\mathfrak{s}} \to \mathbb{R}$; it is shown in [9] that $\mathfrak{s}$ is the first cardinal such that there exists a noncontinuous uniformly sequentially continuous map $2^{\mathfrak{s}} \to \mathbb{R}$, or the first cardinal such that there exists a noncontinuous sequentially continuous homomorphism from $\mathbb{Z}^{\mathfrak{s}}$ (or from $\mathbb{Z}_2^{\mathfrak{s}}$) into a topological group, or a noncontinuous sequentially continuous pseudonorm on $\mathbb{Z}^{\mathfrak{s}}$ (or on $\mathbb{Z}_2^{\mathfrak{s}}$). It was proved by Balcar (see [2]) that $\mathfrak{s}$ is the first cardinal admitting a Maharam submeasure (a sequentially continuous nontrivial submeasure — see Section 2 for

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more details). Certainly, $\mathfrak{s} \leq \mathfrak{m}_{\mathbb{R}}$, and the famous Keisler-Tarski problem asks whether the equality holds. Under some set-theoretical assumptions (like CH, MA) one has $\mathfrak{s} = \mathfrak{m}_{\mathbb{R}} = \mathfrak{m}_1$. In any case, $\mathfrak{s}$ is a large cardinal.

We say that a subcategory $\mathcal{C}$ of a category $\mathcal{K}$ is *κ-productive* if every product (in $\mathcal{K}$) of less than κ members of $\mathcal{C}$ belongs to $\mathcal{C}$; finite (or countable) productivity is another expression for ω- (or ω_1- , resp.) productivity. A class $\mathcal{C}$ is called *exactly κ-productive* if it is κ-productive and not κ^+-productive (such a cardinal κ must be regular). In topological vector spaces it means that the class is κ-productive and that no product of at least κ many spaces from $\mathcal{C}$ having nontrivial separated modifications, belongs to $\mathcal{C}$. We shall see later that in our case of topological groups, the situation is not so simple.

1. Coreflectivity in topological groups

In this paper we shall consider productivity of coreflective subcategories $\mathcal{C}$ of topological groups. The coreflectivity is considered with respect to a surreflective subcategory $\mathcal{K}$ of TopGr (i.e., the reflection maps are onto maps, or equivalently, $\mathcal{K}$ is productive and hereditary in TopGr); **we shall always suppose that** $\mathbb{Z} \in \mathcal{K}$. Basic examples of $\mathcal{K}$ are TopGr, $\mathsf{TopAbGr}$ or their separated reflections. In the last section we shall deal with yet another example.

Unlike topological and similar non-algebraic structures, descriptions of coreflective subcategories of (topological) groups are more complicated. We shall concentrate on two kinds only, namely the monocoreflective and bicoreflective subcategories. Realize that monomorphisms in $\mathcal{K}$ are exactly one-to-one continuous homomorphisms. By a quotient in $\mathcal{K}$ we mean a homomorphism h onto, the range of which is endowed with the finest topology in $\mathcal{K}$ making h continuous.

Proposition 1. *A subcategory $\mathcal{C}$ of $\mathcal{K}$ is monocoreflective in $\mathcal{K}$ iff it is closed under quotients and sums in $\mathcal{K}$.*

Proof: If $\mathcal{C}$ is monocoreflective then it is (as every coreflective subcategory) closed under sums. Let $f : C \to X$ be a quotient in $\mathcal{K}$ and $C \in \mathcal{C}$. Since X is an image of a member of $\mathcal{C}$, a coreflection $c : cX \to X$ in $\mathcal{C}$ must be onto and, thus, c is continuous and a group isomorphism. Since f is quotient, c is quotient, too. Consequently, cX is isomorphic to X in $\mathcal{K}$.

Suppose that $\mathcal{C}$ is closed under sums and quotients in $\mathcal{K}$. For a given X take a representative set of all pairs $\{(C, g) : C \in \mathcal{C}, g$ is a monomorphism of C into $X\}$ and construct the sum $h : S \to X$ of such pairs in $\mathcal{K}$. Take the quotient of $h : S \to X$ in TopGr, say $h' : S \to Y$ (thus $h = ih'$ for some monomorphism $i : Y \to X$), and consider the reflection $g : Y \to rY$ of Y in $\mathcal{K}$. Since i factors via g, the map g must be a group isomorphism, thus we may regard rY as the same group Y endowed with a coarser topology. Consequently, gh' is a quotient in $\mathcal{K}$, therefore $rY \in \mathcal{C}$, and $rY \to X$ is a requested coreflection of X in $\mathcal{C}$. $\square$

Corollary 1. *Every subcategory of $\mathcal{K}$ has a monocoreflective hull in $\mathcal{K}$.*

The monocoreflective hull in $\mathcal{K}$ of a subcategory $\mathcal{S}$ is formed by all quotients of sums in $\mathcal{K}$ of objects from $\mathcal{S}$. The obtained class is, clearly, closed under quotients. It suffices to show that a sum of quotients is a quotient of a sum, which follows from general results (associativity) of inductive (final) generation.

Of course, every coreflective subcategory is closed under sums, which implies that it is also closed under finite products if we work in $\mathsf{TopAbGr}$ since the finite sums in $\mathcal{K}$ and finite products coincide there. In TopGr, every finite product is a nice quotient of a finite sum and that property transfers to $\mathcal{K}$ by reflections and so we have:

Corollary 2. *Every monocoreflective subcategory of $\mathcal{K}$ is finitely productive.*

The example following Corollary 4 shows that coreflective subcategories of TopGr need not be finitely productive.

Since the discrete group $\mathbb{Z}$ is a separator in $\mathcal{K}$, it is easy to prove the following characterization of bicoreflective subcategories (bicoreflective in this paper means that the coreflection maps are group isomorphisms).

Proposition 2. *A coreflective subcategory $\mathcal{C}$ of $\mathcal{K}$ is bicoreflective in $\mathcal{K}$ iff it contains $\mathbb{Z}$.*

Corollary 3. *A subcategory $\mathcal{C}$ of $\mathcal{K}$ is bicoreflective in $\mathcal{K}$ iff it is closed under quotients and sums in $\mathcal{K}$ and contains $\mathbb{Z}$.*

Corollary 4. *Every subcategory of $\mathcal{K}$ has a bicoreflective hull in $\mathcal{K}$.*

The preceding procedure implies that AbGr contains no nontrivial coreflective subcategory containing $\mathbb{Z}$. This phenomenon is well known (see e.g. [11] for a more general result). Of course, $\mathsf{TopAbGr}$ contains many nontrivial bicoreflective subcategories. The next example shows a variability of coreflective classes of groups.

Example. Denote by $\mathcal{C}_1$ all the sums of the group $\mathbb{Z}_2$, by $\mathcal{C}_2$ the monocoreflective hull of $\mathbb{Z}_2$, and by $\mathcal{C}_3$ the bicoreflective hull of $\mathbb{Z}_2$, everything in TopGr. Clearly, $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3$. We shall show that $\mathcal{C}_1$ is coreflective (thus it is the coreflective hull of $\mathbb{Z}_2$ in TopGr) and not finitely productive. All three classes are different.

For any topological group G the sum $h : S \to G$ of all the nonzero homomorphisms $h_i : \mathbb{Z}_2 \to G$ has, clearly, the factorization property: for any sum $g : X \to G$ of some homomorphisms $g_j : \mathbb{Z}_2 \to G$ there exists a homomorphism $f : X \to S$ such that $g = h \circ f$. It suffices to show that such an f is unique. We may suppose that $X = \mathbb{Z}_2$ and that we have a nonzero homomorphism $f : X \to S$ with $h \circ f = 0$. The only possibility for f is the canonical embedding of $\mathbb{Z}_2$ into the sum S and, thus, its composition with h must coincide with some h_i that was nonzero. This contradiction proves the uniqueness of the above factorization.

It is clear that $\mathcal{C}_1$ is not productive; for instance, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to any sum of the groups $\mathbb{Z}_2$ in TopGr. Therefore, $\mathcal{C}_1$ differs from the finitely productive class $\mathcal{C}_2$. To show that $\mathcal{C}_2 \neq \mathcal{C}_3$, it suffices to realize that $\mathbb{Z}$ cannot belong to $\mathcal{C}_2$ since every homomorphism of $\mathbb{Z}_2$ into $\mathbb{Z}$ is zero. □

In the next sections we shall first look at bicoreflective subcategories of topological groups, where the results on productivity of coreflective subcategories are analogous to those in topological linear spaces. Then we shall show that monocoreflective subcategories of topological groups behave, as for products, similarly like coreflective subcategories of uniform spaces. In the last section we improve the results from Section 2 for special topological groups (e.g., for compact groups).

2. Productivity of bicoreflective subcategories of topological groups

The next result corresponds to the one for topological linear spaces, where productivity of coreflective subcategories can be checked by means of powers of scalars (see [5], [6] and [14]).

Theorem 1. *A bicoreflective subcategory $\mathcal{C}$ of $\mathcal{K}$ is κ-productive iff $\mathbb{Z}^\lambda \in \mathcal{C}$ for every $\lambda < \kappa$.*

PROOF: We must prove that if, for some $\lambda \geq \omega$, the power $\mathbb{Z}^\lambda \in \mathcal{C}$ and $\{X_\alpha : \alpha < \lambda\} \subset \mathcal{C}$, then $\Pi_\lambda X_\alpha \in \mathcal{C}$. This task is equivalent to showing that every continuous homomorphism f on $c\Pi_\lambda X_\alpha$ into any member M of $\mathcal{K}$ is continuous on $\Pi_\lambda X_\alpha$. Since every topological group is initially generated by homomorphisms into metric right-invariant groups, we may assume that M is such a group endowed with a right-invariant metric d (its neutral element will be denoted by e).

First we shall assume that $\lambda = \omega$, i.e., we shall show that $\mathcal{C}$ is countably productive provided $\mathbb{Z}^\omega \in \mathcal{C}$. To prove that, it suffices to show that our $f : \Pi_{n\in\omega} X_n \to M$ can be uniformly approximated by continuous maps. For every $\varepsilon > 0$ we shall find $k \in \omega$ such that $d(f(x), e) < \varepsilon$ whenever the point x from ΠX_n has the first k coordinates equal to the neutral elements — then one may take a continuous homomorphism g as the composition of f with the canonical embedding of $\Pi_{n\leq k} X_n$ into $\Pi_\omega X_n$ and with the projection $\Pi_\omega X_n \to \Pi_{n<k} X_n$ (the composition is continuous because the canonical embedding factorizes via $c(\Pi_\omega X_n)$); it is clear that $d(f(y), g(y)) < \varepsilon$ for every $y \in \Pi_\omega X_n$.

Suppose that there exists an $\varepsilon > 0$ such that for every k we can find $x_k \in \Pi X_n$ having the first k coordinates equal to the neutral elements and such that $d(f(x_k), e) \geq \varepsilon$. Denote by $x_{k,m}$ the point of $\Pi_\omega X_n$ having the i-th coordinate equal to that of x_k provided $i < m$ and equal to the neutral elements otherwise. Clearly, $\{x_{k,m}\} \to x_k$ in $\Pi_\omega X_n$ when $m \to \infty$; we want to show that the convergence holds also in $c(\Pi_\omega X_n)$. Indeed, taking the continuous homomorphisms $\phi_m : \mathbb{Z} \to X_m$ with $\phi(1) = \mathrm{pr}_m(x_k)$ and their product $\phi : \mathbb{Z}^\omega \to \Pi_\omega X_m$ (that factorizes via $c(\Pi_\omega X_m)$), we see that ϕ maps the points y_m having only the first m coordinates nonzero and equal to 1, to $x_{k,m}$, and the point y having all the coordinates equal to 1, to x_k. Since $y_m \to y$ in $\mathbb{Z}^\omega$, our assertion follows. Consequently, for every k we can find a point $z_k = x_{k,m_k}$ having a finite support, such that $d(f(z_k), e) > \varepsilon/2$. Taking a convenient subsequence, we may assume that the supports S_k of z_k's are disjoint. Then we may define another continuous product homomorphism $\psi = \Pi\psi_k : \mathbb{Z}^\omega \to \Pi_k \Pi_{S_k} X_n$ where $\psi_k(1)$ equals to the restriction of z_k to its support S_k. The points in $\mathbb{Z}^\omega$ having just one coordinate nonzero, converge to 0 and, thus, the points z_k converge to the neutral element in $c(\Pi_\omega X_n)$. Consequently, $f(z_k)$ converges to e in M, which contradicts our assumption.

Suppose now that $\lambda > \omega$. We shall show that our f depends on a countable set $J \subset \lambda$, i.e., that it factorizes via $\Pi_J X_\alpha$ as $g\,\mathrm{pr}_J$; since $\Pi_J X_\alpha$ belongs to $\mathcal{C}$, the map g is continuous on $\Pi_J X_\alpha$ and, thus, f is continuous on $\Pi_\lambda X_\alpha$.

Denote by J the set $\{j \in \lambda : \exists x_j : f(x_j) \neq e, \mathrm{pr}_{\lambda\setminus\{j\}} x_j = e\}$. For every $j \in J$ we have a homomorphism $h_j : \mathbb{Z} \to X_j$ defined by $h_j(1) = \mathrm{pr}_j x_j$ (x_j is the point from the definition of the set J), and for the other elements of λ let the homomorphism h_j be zero. The corresponding product map $\mathbb{Z}^\lambda \to \Pi_\lambda X_\alpha$ of h_j's factorizes via $c\Pi_\lambda X_\alpha$ (because $\mathbb{Z}^\lambda \in \mathcal{C}$) and composed with f forms a continuous homomorphism of $\mathbb{Z}^\lambda$ into M. Such a continuous homomorphism depends on countably many coordinates (since, e.g., it is uniformly continuous), and the smallest such a set of coordinates equals to J. So, $|J| \leq \omega$.

Take now a point $x \in \Pi_\lambda X_\alpha$ such that $\mathrm{pr}_J x = e$. We want to show that $f(x) = e$. Take the product homomorphism $g : \mathbb{Z}^\lambda \to \Pi_\lambda X_\alpha$ mapping the α's coordinate of $\{1\}$ to the α's coordinate of x. Then the characteristic functions ξ_F of finite sets $F \subset \lambda$ converge to the point $\{1\} \in \mathbb{Z}^\lambda$, $f(g(\xi_F))$ is the neutral element for every finite set F (because of the definition of J), and $g(\{1\}) = x$. Consequently, $f(x) = e$ and f depends on J. $\square$

The case $\lambda = \omega$ can be simplified for $\mathcal{K} \subset \mathsf{TopAbGr}$ (we may use a product homomorphism $h : \Pi_{n=1}^\infty \mathbb{Z}^{n-1} \to \Pi_{n=1}^\infty X_n$, where we regard the $\mathbb{Z}^{n-1}$ as the sum of $\mathbb{Z}$'s (see the second paragraph of the proof of Theorem 2)).

Corollary 5. *A bicoreflective subcategory $\mathcal{C}$ of $\mathcal{K}$ is κ-productive provided it contains a κ-productive bicoreflective subcategory of $\mathcal{K}$.*

In topological linear spaces, every countably productive coreflective subcategory is κ-productive, where κ is a sequential cardinal (or Ulam measurable cardinal if we deal with locally convex spaces). So, for instance, there is no exactly ω_1-productive coreflective subcategory of topological linear spaces or of locally convex spaces. We shall now prove that the situation for bicoreflective subcategories of topological groups is completely analogous.

For the purpose explained in the preceding paragraph, we shall define *higher* sequential cardinals. Our approach will not go via generalization of sequentially continuous maps but via a measure approach, which is enabled by a recent result of Balcar that the sequential cardinal $\mathfrak{s}$ admits a nontrivial Maharam submeasure. For that reason, our large cardinals will be called submeasurable. We shall then prove the following result:

Theorem 2. *A bicoreflective subcategory of $\mathcal{K}$ is either productive or it is exactly κ-productive for some submeasurable cardinal κ.*

We recall that a *submeasure* μ on κ is a real-valued mapping defined on all subsets of κ having the following properties:

$\mu(\emptyset) = 0$,
$\mu(A) \le \mu(B)$ for $A \subset B \subset \kappa$,
$\mu(A \cup B) \le \mu(A) + \mu(B)$ for $A, B \subset \kappa$.

A mapping f between topological spaces is said to be *κ-continuous* if it preserves limits of well-ordered nets of lengths less than κ (it is easy to show that a submeasure is κ-continuous iff $\mu(A_\alpha) \to 0$ for $\{A_\alpha\}_{\alpha<\lambda} \searrow 0$, where $\lambda < \kappa$).

Definition 1. A cardinal κ is said to be *submeasurable* if there exists a nonzero κ-continuous submeasure on κ having zero values at the points.

Clearly, the first submeasurable cardinal equals ω. The above mentioned result from Balcar says that the second submeasurable cardinal equals the Mazur-Noble sequential cardinal $\mathfrak{s}$ defined in the introduction.

One can find basic properties of submeasurable cardinals in [3]. We shall here state some of them which may clarify some situations or which will be needed in the sequel. Recall that a pseudonorm p on a group G is a nonnegative real-valued function satisfying $p(e) = 0$, $p(x^{-1}) = p(x)$, $p(xy) \le p(x) + p(y)$ for the neutral element e of G and for arbitrary $x, y \in G$; the topology of a topological group G is generated by all continuous pseudonorms on G.

(1) Every submeasure on κ is a pseudonorm on $\mathbb{Z}_2^\kappa$.

(2) If p is a pseudonorm on $\mathbb{Z}_2^\kappa$ then $\mu(A) = \sup\{p(B) : B \subset A\}$ is the smallest submeasure on κ greater than p. This operation preserves discontinuity and κ-continuity.

(3) If a submeasure μ is κ-continuous, then it is κ-subadditive (i.e., $\mu(\bigcup_I A_i) \le \sum_I \mu(A_i)$ for $|I| < \kappa$). The converse assertion does not hold even for bounded submeasures.

(4) A cardinal κ is submeasurable iff there exists a noncontinuous sequentially continuous pseudonorm on $\mathbb{Z}_2^\kappa$ (or on $\mathbb{Z}^\kappa$, resp.) having κ-additive null sets (i.e., the submeasure from (2) is κ-additive on its null sets).

(5) A cardinal κ is submeasurable iff there exists a noncontinuous κ-continuous homomorphism on $\mathbb{Z}_2^\kappa$ (or on $\mathbb{Z}^\kappa$, resp.) into a topological group (iff there exists a noncontinuous κ-continuous real-valued map on 2^κ).

Proof of Theorem 2: Take the first cardinal κ such that $\mathbb{Z}^\kappa$ does not belong to our bicoreflective subcategory $\mathcal{C}$ of $\mathcal{K}$. If such a cardinal does not exist then $\mathcal{C}$ is productive. If it exists, we shall show that it is submeasurable by the above property (4). Indeed, there is a pseudonorm p on $\mathbb{Z}^\kappa$ that is not continuous but it is continuous on the coreflection of $\mathbb{Z}^\kappa$ in $\mathcal{C}$. We must prove that p is sequentially continuous and has κ-additive null sets. We may assume that $\kappa > \omega$.

Take a sequence $\{z_n\}$ in $\mathbb{Z}^\kappa$ converging to zero. We shall construct a continuous homomorphism $\phi : \mathbb{Z}^\omega \to \mathbb{Z}^\kappa$ and a sequence $\{u_n\}$ in $\mathbb{Z}^\omega$ converging to zero such that $\phi(u_n) = z_n$ for each n. For each n define $A_n = \{\alpha \in \kappa : \mathrm{pr}_\alpha(z_n) \ne 0, \mathrm{pr}_\alpha(z_k) = 0 \ \forall k > n\}$. The sets A_n are disjoint. Define now a homomorphism $\phi_n : \mathbb{Z}^n \to \mathbb{Z}^{A_n}$ by $\phi(\{r_i\}) = \sum r_i \mathrm{pr}_{A_n}(z_i)$. Let ϕ be the countable product of all ϕ_n. The sequence $\{u_n\}$ is defined as follows: the restriction of the point u_k from $\Pi_\omega \mathbb{Z}^n$ to $\mathbb{Z}^n$ has at most one nonzero coordinate, namely on the k-th place with the value $\phi_n(u_k) = \mathrm{pr}_{A_n}(z_k)$. Since ϕ factors via the coreflection of $\mathbb{Z}^\kappa$ in $\mathcal{C}$, z_n converges to zero in the coreflection and the p-image must converge to zero in $\mathbb{R}$. Thus p is sequentially continuous.

It remains to show the κ-additivity of null sets. Take disjoint subsets $A_\alpha, \alpha \in \lambda$ of κ for some $\lambda < \kappa$, such that $p(x) = 0$ for every x with support in A_α. Take now $z \in \mathbb{Z}^\kappa$ with support in $\bigcup_\lambda A_\alpha$. Define a homomorphism $\phi_\alpha : \mathbb{Z} \to \mathbb{Z}^{A_\alpha}$ by $\phi_\alpha(1) = \mathrm{pr}_{A_\alpha}(z)$. The composition ψ of the product ϕ of all $\phi_\alpha, \alpha < \lambda$ with the embedding of $\mathbb{Z}^{\cup A_\alpha}$ into $\mathbb{Z}^\lambda$ is continuous and factors via the coreflection of $\mathbb{Z}^\kappa$ in $\mathcal{C}$, so the composition of p with ψ is continuous. The point x of $\mathbb{Z}^{\cup A_\alpha}$ having all the coordinates equal to 1 is mapped by ψ to z, and it is the limit of the points $x_\beta, \beta < \lambda$ having all the coordinates up to β equal to 1 and to 0 for the remaining coordinates. The point $\psi(x_\beta)$ has support contained in $\bigcup_{\alpha<\beta} A_\alpha$. Using transfinite induction (we know from the preceding paragraph that $p(x_\omega) = 0$) we get that all $p(x_\beta) = 0$ and, thus, also the limit $p(z)$ must be zero. □

We shall now show that there are bicoreflective classes $\mathcal{C}$ that are not countably productive but contain any prescribed product of some nice topological groups. Thus we cannot define "exact κ-productivity" as in the case of topological linear spaces.

Example. Let $\mathcal{C}$ be the bicoreflective hull of $\{\mathbb{Z}, \mathbb{Z}_2^\kappa, \mathbb{R}^\kappa\}$ in $\mathsf{TopAbGr}$. Then $\mathcal{C}$ contains the κ-power of a torsion group and the κ-power of a divisible group but not the countable power of $\mathbb{Z}$ (since every homomorphism into $\mathbb{Z}$ from a torsion or a divisible group is zero). □

3. Productivity of monocoreflective subcategories of topological groups

The situation for monocoreflective subcategories of topological groups differs drastically from that for bicoreflective subcategories. We shall express our result in the next theorem.

Theorem 3. *For every infinite regular cardinal κ there exists a monocoreflective subcategory of* $\mathsf{TopAbGr}$ *that is exactly κ-productive.*

Proof: The assertion is trivial for $\kappa = \omega$ — it suffices to take all discrete Abelian groups as the coreflective subcategory.

In [12] Shelah constructed an Abelian group S of cardinality κ having trivial continuous endomorphisms only, for every uncountable regular cardinal κ (i.e., the zero one, the identity and its finite sums). Define T to be the subgroup of S^κ generated by the set $\{\{x_\alpha\} \in S^\kappa : \mathrm{card}\,\{x_\alpha\} < \kappa\}$. Similarly as in Herrlich's paper [7] we can show that the only continuous homomorphisms from powers of S to S are the zero morphism, the projections and their sums. It follows that every continuous homomorphic image of S^λ, for $\lambda < \kappa$, in S^κ is contained in T. Consequently, the monocoreflective hull $\mathcal{C}$ of $\{S^\lambda : \lambda < \kappa\}$ does not contain S^κ.

It remains to prove that $\mathcal{C}$ is κ-productive. Take some $\lambda < \kappa$ and $X_\alpha \in \mathcal{C}$ for $\alpha \in \lambda$. We want to prove that $\Pi_\lambda X_\alpha \in \mathcal{C}$. We know that every X_α is a quotient in $\mathsf{TopAbGr}$ of some $\psi_\alpha : \sum_{I_\alpha} S^{\lambda_i} \to X_\alpha$ for some $\lambda_i < \kappa$ and some sets I_α — we may and shall assume that those last sets coincide with a set I. Denote by m the coreflection monomorphism $c(\Pi_\lambda X_\alpha) \to \Pi_\lambda X_\alpha$.

At first we shall show that m is surjective, i.e., that every $x \in \Pi_\lambda X_\alpha$ is an image of some $y \in S^\lambda$ under a continuous homomorphism. For each $\alpha < \lambda$ we can find $z_\alpha \in \sum_I S^{\lambda_i}$ with $\psi_\alpha(z_\alpha) = \mathrm{pr}_\alpha(x)$. Denote by J_α the support of z_α and by ψ'_α the restriction of ψ_α to $\sum_{J_\alpha} S^{\lambda_i} = \Pi_{J_\alpha} S^{\lambda_\alpha}$ for some $\lambda_\alpha < \kappa$. Then $\psi = \Pi_\lambda \psi'_\alpha : S^{\lambda'} = \Pi_\lambda \Pi_{J_\alpha} S^{\lambda_\alpha} \to \Pi_\lambda X_\alpha$, $\lambda' < \kappa$, and the image of ψ contains x.

Now we prove that m is a homeomorphism, i.e., that every homomorphism $f : \Pi_\lambda X_\alpha \to (M, d)$, a right-invariant metric group, is continuous provided it is continuous on the coreflection $c(\Pi_\lambda X_\alpha)$. We shall proceed in a similar way as in the proof of Theorem 2.

At first we suppose $\lambda = \omega$ and prove that f can be uniformly approximated by continuous maps. Assume that there exists a positive ε such that for every $k \in \omega$ we can find $x_k \in \Pi_\omega X_n$ having the first k coordinates equal to the neutral elements and such that $d(f(x_k), e) > \varepsilon$. It follows from the preceding part that $\mathrm{pr}_i(x_k)$ is an image of a point $y_{k,i} \in S^{\lambda_{k,i}}$. We may assume that all the $\lambda_{k,i}$ coincide with some $\mu < \kappa$. Moreover, we assume that $y_{k,i} = 0$ whenever $\mathrm{pr}_i(x_k) = 0$. Thus the points $y_k = \{y_{k,i}\}_i \in (S^\mu)^\omega$ converge to 0. Consequently, their continuous images x_k must converge to 0, too, which contradicts our assumption $d(f(x_k), e) > \varepsilon$.

Thus for every $\varepsilon > 0$ there exists some $k \in \omega$ such that $d(f(x), f(y)) \le \varepsilon$ whenever the first k coordinates of x, y coincide. Defining $g(x) = f(\mathrm{pr}_{\le k}(x))$ we get a continuous function on $\Pi_\omega X_n$ such that $d(f(x), g(x)) \le \varepsilon$ for every $x \in \Pi_\omega X_n$.

Coming back to arbitrary $\lambda < \kappa$, it suffices to show that our map f depends on countably many coordinates. Define the set $J = \{\alpha :$ there exists some x_α with $\mathrm{pr}_{\lambda - \{\alpha\}}(x_\alpha) = 0, f(x_\alpha) \ne 0\}$. The set J must be countable since otherwise $d(f(x_\alpha), 0) \ge \varepsilon$ for some positive ε and countably many indices $\alpha \in T$, and every x_α is an image of some y_α under a continuous homomorphism $\psi_\alpha : S^{\lambda_\alpha} \to X_\alpha$ — if we embed every $y_\alpha, \alpha \in T$, canonically into $\Pi_T S^{\lambda_\alpha}$, then $\{y_\alpha\}_T \to 0$ and, thus, $f(x_\alpha) = f(\Pi_T \psi_\alpha(y_\alpha)) \to 0$, too, which contradicts our assumption.

It remains to prove that f depends on J. Take some $x \in \Pi_\lambda X_\alpha$ with $\mathrm{pr}_J(x) = 0$. There are $y_\alpha \in S^\mu, \mu < \kappa$, and continuous homomorphisms $\psi_\alpha : S^\mu \to X_\alpha$ mapping y_α into $\mathrm{pr}_\alpha(x)$ (we may suppose that $y_\alpha = 0$ for $\alpha \in J$). For finite $C \subset \lambda \setminus J$ let $z_{C,\alpha} = y_\alpha$ if $\alpha \in C$, $z_\alpha = 0$ otherwise. Then the net $\{\{z_{C,\alpha}\}_\alpha\}_C$ converges to $\{y_\alpha\}$, thus the continuous image $f(\Pi \psi_\alpha(z_C))$ converges to $f(\Pi \psi_\alpha(y_\alpha)) = f(x)$. Since $\psi_\alpha(z_C) = \mathrm{pr}_\alpha(x)$ for $\alpha \in C$ and C is finite and lies outside J, $f(\Pi \psi_\alpha(z_C)) = 0$. Consequently, $f(x) = 0$, which is what we had to prove. $\square$

4. Special coreflective subcategories of topological groups

As in topological linear spaces, we can find a reflective subcategory $\mathcal{K}_0$ of TopGr such that substituting this subcategory for $\mathcal{K}$ in the above theorems, we can use measurable cardinals instead of the submeasurable ones.

In [9] the following result is proved: *Every sequentially continuous homomorphism on a product of less than* $\mathfrak{m}_1$ *sequential groups into a compact group is continuous.* It follows from its proof that the class of ranges (here compact groups) can be extended to the class $\mathcal{K}_0$ of topological groups G such that every its Abelian subgroup is projectively generated by topological Abelian groups having the following property:

($\star$) for every sequence $\{x_n\}$ of nonzero elements there exist an infinite set $S \subset \mathbb{N}$ and a sequence $\{k_n\}$ of integers such that $\sum_S k_n x_n$ does not converge.

$\mathcal{K}_0$ is a surreflective subcategory of TopGr. **In the next $\mathcal{K}$ is a surreflective subcategory of $\mathcal{K}_0$ containing** $\mathbb{Z}$. Compact groups belong to $\mathcal{K}_0$ because every its Abelian subgroup is contained in a compact Abelian group and that is projectively generated by the group $\mathbb{R}/\mathbb{Z}$ having ($\star$).

We can now prove the following strengthening of Theorem 2 for our new class $\mathcal{K}_0$.

Theorem 4. *A bicoreflective subcategory $\mathcal{C}$ of $\mathcal{K}$ is either productive or it is exactly κ-productive for some measurable cardinal κ.*

PROOF: Let κ be the first cardinal with $\mathbb{Z}^\kappa \notin \mathcal{C}$. We shall prove that κ is measurable; we may assume that $\kappa > \omega$. There exists a continuous homomorphism $f : c(\mathbb{Z}^\kappa) \to G$ that is not continuous on $\mathbb{Z}^\kappa$, for some $G \in \mathcal{K}_0$ (it follows from the definition of $\mathcal{K}_0$ that we can find G to be Abelian having the property ($\star$)). The set $J = \{\alpha \in \kappa : f(x^\alpha) \neq e$ for some x^α having a one point support $\{\alpha\}\}$ is finite. Indeed, otherwise we can take an infinite sequence $\{x^{\alpha_n}\}$ from the definition of J and find integers k_n such that the series $\sum k_n f(x^{\alpha_n})$ does not converge in G; we shall get a contradiction by showing that $f : \mathbb{Z}^\kappa \to G$ preserves convergence of our sums (since the series $\sum k_n x^{\alpha_n}$ converges in $\mathbb{Z}^\kappa$). To show that, it suffices to construct a convenient continuous homomorphism $g : \mathbb{Z}^\omega \to \mathbb{Z}^\kappa$ (realize that g factors via $c(\mathbb{Z}^\kappa)$). The map g is a composition of the product of maps $\mathbb{Z} \to \mathbb{Z}$ mapping 1 into the value of x^{α_n} at α_n, for $n \in \omega$, and of the natural embedding of $\mathbb{Z}^{\{\alpha_n\}}$ into $\mathbb{Z}^\kappa$.

The map f cannot depend on J because in that case f would be continuous on $\mathbb{Z}^\kappa$. Therefore there exists a point $x \in \mathbb{Z}^\kappa$ with support contained in $\kappa \setminus J$ and such that $f(x) \neq 0$ — we shall say that a subset of κ has the property (p) if there exists a point x with its support contained in the subset and with $f(x) \neq 0$ (thus $\kappa \setminus J$ has (p)). We assert that there is a set $K \subset \kappa \setminus J$ having (p) and containing no two disjoint subsets having both (p). Indeed, otherwise we could construct a disjoint sequence $\{K_n\}$ of subsets of $\kappa \setminus J$ having (p), hence we could find x_n with support contained in K_n and $f(x_n) \neq 0$. Then again $\sum k_n x_n$ exists in $\mathbb{Z}^\kappa$ for any choice of integers k_n, which is not true for $\sum k_n f(x_n)$.

We can now define for $A \subset K$ a value $\mu(A)$ to be 1 or 0 depending whether A does or does not have the property (p). By the property from the last paragraph, μ is a nontrivial finitely additive two-valued measure having zero values at singletons. It remains to show that μ is κ-additive (that also gives $|K| = \kappa$ and, hence, κ is measurable). Take a disjoint system $\{A_\alpha : \alpha < \lambda\}, \lambda < \kappa$, of subsets of K with $\mu(A_\alpha) = 0$; we must prove that $\mu(\bigcup_\lambda A_\alpha) = 0$. Every $x \in \mathbb{Z}^\kappa$ with support in $\bigcup_\lambda A_\alpha$ is the limit of restrictions x_α of x to $\bigcup_{\beta<\alpha} A_\beta$. Since such a net $\{x_\alpha\}_{\alpha<\lambda}$ is an image of a converging net in $\mathbb{Z}^\lambda$ along a continuous homomorphism $\mathbb{Z}^\lambda \to \mathbb{Z}^\kappa$ (that factors via $c(\mathbb{Z}^\kappa)$), also the net $\{f(x_\alpha)\}$ converges to $f(x)$ in G. When we proceed by transfinite induction, we have $f(x_\alpha) = 0$ for all α and, consequently, $f(x) = 0$, which was to be proved. $\square$

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