Primes, coprimes and multiplicative elements

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Abstract. The purpose of this paper is to study conditions under which the restriction of a certain Galois connection on a complete lattice yields an isomorphism from a set of prime elements to a set of coprime elements. An important part of our study involves the set on which the way-below relation is multiplicative.

Keywords: complete lattices, completely distributive lattices, Galois connection, multiplicative elements, way-below relation

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1. Introduction and preliminaries

In a finite lattice the set of coprime elements is order isomorphic to the set of prime elements under the correspondence that sends an element x to the join of the set of elements not greater than or equal to x. The associated inverse sends y to the meet of the set of elements not less than or equal to y. These two unary operations regarded on the whole lattice form a Galois connection and play an important role in the theory of completely distributive lattices ([5], [3]). We refer to these operations as "Raney's mappings". In this paper we investigate when Raney's mappings yield an isomorphism between certain subsets of prime and coprime elements. Our results and techniques are strictly lattice theoretic but it is possible to use a topological point of view, which we will pursue in a future paper.

An important part of our investigation is the notion of a multiplicative element. Specifically, an element x in a complete lattice L is *multiplicative* if the set of all elements y such that x is way-below y is either empty or a lattice filter. If every element of L is multiplicative then the way-below relation on L is said to be multiplicative, (see [2]). If L is a distributive continuous lattice, then there are several interesting characterizations for when the way-below relation is multiplicative on L, (see [2]). For some recent work in this direction we refer the reader to [6].

After introducing some notation in the remainder of this section, we present in Section 2 results regarding Raney's mappings and isolate a situation in which a restriction of these maps provides an isomorphism between certain subsets of prime and coprime elements. This leads to a connection with the set of elements where the way-below relation is multiplicative. The section concludes with examples that show that this isomorphism cannot in general be extended. Section 3 is devoted to topics such as continuous lattices, completely distributive lattices, and the meaning of the way-below relation being multiplicative.

In order to keep this paper essentially self-contained we present here the needed definitions. When terminology is not standard we will follow the conventions used in [2]. Throughout this paper L will denote a complete lattice and L^{op} will denote L with the order reversed. We begin with the way-below relation, for $x, y \in L$ we say x is *way-below* y, denoted $x \ll y$ if for every directed set $A \subseteq L$, $y \leq \sup A$ implies that there is an $a \in A$ such that $x \leq a$. Elements satisfying $x \ll x$ are called *isolated from below* or *compact.* Further for any $x \in L$, let $\uparrow x = \{y \in L \,|\, x \leq y\}$, $\downarrow x = \{y \in L \,|\, y \leq x\}$, $\twoheaddownarrow x = \{y \in L \,|\, y \ll x\}$ and $\twoheaduparrow x = \{y \in L \,|\, x \ll y\}$.

We now give a formal definition of Raney's mappings. For $x \in L$ let

$$R_v(x) = x_v = \bigvee (L \setminus \uparrow x) \quad \text{and} \quad R^u(x) = x^u = \bigwedge (L \setminus \downarrow x)\,.$$

As mentioned above, the pair (R_v, R^u) is a Galois connection on L ([2]). Consequently, R_v preserves arbitrary joins and R^u preserves arbitrary meets.

A lattice L is a complete Heyting algebra if L is a complete lattice in which the following distributive property holds:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y \,|\, y \in Y\}, \text{ for all } x \in L,\, Y \subseteq L.$$

A complete lattice L is *completely distributive* if

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = \bigvee_{f \in M} \bigwedge_{j \in J} x_{j,f(j)},$$

where M is the set of all functions f on J such that $f(j) \in K(j)$. Further, a *continuous* lattice L is a complete lattice with the property that $x = \bigvee \twoheaddownarrow x$, for every $x \in L$. We note that a distributive continuous lattice is the same as a complete Heyting algebra. Furthermore, every completely distributive lattice is continuous but the converse does not hold. The element $p \in L$ is *join irreducible* if $a \vee b = p$ implies $a = p$ or $b = p$ and q is *completely join irreducible* if, for any nonempty subset $S \subseteq L$, $\bigvee S = p$ implies $p \in S$. *Meet irreducible* and *completely meet irreducible* are defined dually. An element $p \in L$ is *prime* if $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$ and $q \in L$ is *coprime* if q is prime in L^{op}, that is, L with the order reversed. Note that every prime is meet irreducible and every coprime is join irreducible, and in a distributive lattice the converse also holds. A completely distributive lattice L is a *ring of sets* if the set of completely join irreducible elements is a join dense subset of L. This is equivalent to the fact that the set of completely meet irreducible elements is a meet dense subset of L.

We will use the following notation for subsets of importance to our study:

$$\begin{aligned}
S^u &= \{x^u \mid x \in S \subseteq L\}\\
S_v &= \{x_v \mid x \in S \subseteq L\}\\
C(L) &= \{q \in L \setminus \{0\} \mid q \text{ is a coprime}\}\\
P(L) &= \{p \in L \setminus \{1\} \mid p \text{ is a prime}\}\\
C^u(L) &= \begin{cases} (C(L) \cap L^u) \setminus \{1\} \text{ if 1 is not completely join irreducible}\\ (C(L) \cap L^u) \cup \{1\} \text{ if 1 is completely join irreducible}\end{cases}\\
P_v(L) &= \begin{cases} (P(L) \cap L_v) \setminus \{0\} \text{ if 0 is not completely meet irreducible}\\ (P(L) \cap L_v) \cup \{0\} \text{ if 0 is completely meet irreducible}\end{cases}\\
M(L) &= \{x \in L \mid \Uparrow x \text{ is a filter or empty}\}.
\end{aligned}$$

The set $M(L)$ is the set of multiplicative elements of L and $x \in M(L)$ is equivalent to $x \ll a$ and $x \ll b$ implies $x \ll a \wedge b$. Further note that $\{0,1\} \subseteq M(L)$.

2. Results on complete lattices

In this section we study the connection between $M(L)$ and properties of Raney's mappings R_v and R^u.

Lemma 2.1. *Let L be a complete lattice.*

(i) *If $p = q_v$ for some $q \in C(L)$, then $L \setminus \downarrow p = \Uparrow q$.*
(ii) *If $q = p^u$ for some $p \in P(L)$, then $L \setminus \uparrow q = \Uparrow_{L^{op}} p$.*

PROOF: (i). Let $p = \bigvee (L \setminus \uparrow q)$. Since q is coprime, $L \setminus \uparrow q$ is an ideal. Note that $q \not\ll p$. Hence, if $q \ll x$, then $x \not\leq p$. Thus $\Uparrow q \subseteq L \setminus \downarrow p$.

To show the opposite containment, note that $p = \bigvee (L \setminus \uparrow q)$ implies that $L \setminus \uparrow q \subseteq \downarrow p$. So $\uparrow q \supseteq L \setminus \downarrow p$. Let $x \in L \setminus \downarrow p$ and suppose that $x \leq \bigvee D$ for some directed set D. Then there exists $d \in D$ such that $d \not\leq p$ (otherwise $x \leq \bigvee D \leq p$). So $d \in L \setminus \downarrow p \subseteq \uparrow q$, i.e., $q \leq d$. Thus $q \ll x$ and we have $L \setminus \downarrow p \subseteq \Uparrow q$.

(ii). Since $\Uparrow_{L^{op}} p$ means that we take the way-below relation in L^{op}, thus this follows from a dual argument. □

Note that under the assumptions of item (i) we have that $p^u = \bigwedge (L \setminus \downarrow p) = \bigwedge \Uparrow q$ and thus $q = p^u$ if and only if $q = \bigwedge \Uparrow q$ furthermore this is the same as $q_v = \bigvee (L \setminus \Uparrow q)$. Similarly, under the assumptions of (ii) $p = q_v$ if and only if $p = \bigvee \Uparrow_{L^{op}} p$ which is the same as $p^u = \bigwedge (L \setminus \Uparrow_{L^{op}} p)$.

Lemma 2.2. *Let L be a complete lattice.*

(i) *If $q \in C(L)$ then $q \in M(L)$ if and only if $q_v \in P(L) \cup \{1\}$.*
(ii) *If $p \in P(L)$ then $p \in M(L^{op})$ if and only if $p^u \in C(L) \cup \{0\}$.*

PROOF: (i) ($\Rightarrow$). Let $q \in C(L)$ and $p = q_v$. By Lemma 2.1 $\Uparrow q = L \setminus \downarrow p$. Since $q \in M(L)$, $\Uparrow q$ is a filter or empty, which implies that $p \in P(L) \cup \{1\}$.

($\Leftarrow$). If $p \in P(L) \cup \{1\}$, then $\Uparrow q = L \setminus \downarrow p$ is a filter or empty, hence $q \in M(L)$. Dually (ii) follows. $\square$

The above lemmas are the key to the following theorem:

Theorem 2.3. *Let L be a complete lattice. The restriction $\widehat{R}_v$ of R_v to $C^u(L) \cap M(L)$ is an isomorphism of $C^u(L) \cap M(L)$ onto $P_v(L) \cap M(L^{op})$ whose inverse is $\widehat{R}^u$, the restriction of R^u to $P_v(L) \cap M(L^{op})$.*

PROOF: Note that $\widehat{R}_v$ and $\widehat{R}^u$ are injective. Now let $q \in C^u(L) \cap M(L)$, then by Lemma 2.2 $q_v \in P_v(L) \cup \{1\}$ and it suffices to show that $p = q_v \neq 1$ and $p \in M(L^{op})$. Suppose $p = 1$, then $q = (q_v)^u = p^u = 1 \in C^u(L)$. This implies that 1 is a complete join irreducible and hence that $1 > 1_v = \bigvee\{x \mid x < 1\}$ contradicting that $1_v = q_v = p = 1$. Further, $p^u = (q_v)^u = q \in C^u(L)$ and thus by Lemma 2.2(i) $p \in M(L^{op})$. Thus $\widehat{R}_v$ is an injective map from $C^u(L) \cap M(L)$ to $P_v(L) \cap M(L^{op})$. By a dual argument we can establish the corresponding result for $\widehat{R}^u$. Since (R_v, R^u) forms a Galois connection on L, we have that $\widehat{R}^u$ is the inverse of $\widehat{R}_v$, and thus the proof is complete. $\square$

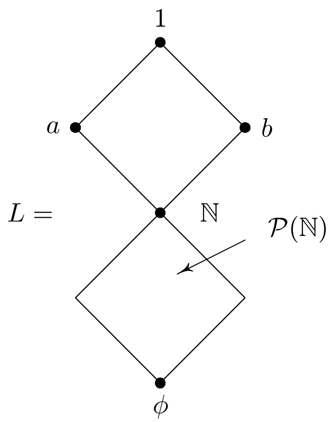
Note that if L is a complete Heyting algebra, then the set of completely join irreducible elements is contained in $C^u(L) \cap M(L)$ and the set of completely meet irreducible elements is contained in $P_v(L) \cap M(L^{op})$. Furthermore, these sets are isomorphic under Raney's mappings.

The following corollary is immediate.

Corollary 2.4. *If L is a complete lattice and the way-below relations on L and L^{op} are multiplicative, then the restriction of R_v to $C^u(L)$ is an order isomorphism from $C^u(L)$ onto $P_v(L)$ with R^u as the associated inverse.*

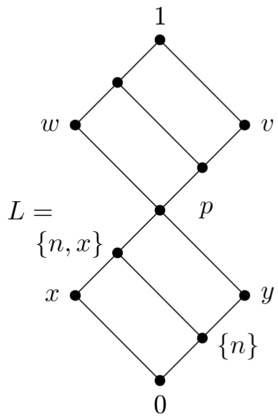
The interesting aspect of Corollary 2.4 is that the converse does not hold, not even in the case when L, and hence L^{op}, is a ring of sets.

Example 2.5. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the natural numbers. Let $\mathcal{P}(\mathbb{N})$ be the power set of $\mathbb{N}$ and $L = \mathcal{P}(\mathbb{N}) \cup \{a, b, 1\}$ where a and b are not comparable, $a, b \geq \{n\}$ for all $n \in \mathbb{N}$ and 1 is the largest element of L.



Here we have $\mathbb{N} \ll a$, $\mathbb{N} \ll b$ but $\mathbb{N} \not\ll a \wedge b$ and hence the way-below relation is not multiplicative. It is clear, however, that R_v is an isomorphism from $C^u(L) =$ the atoms of $\mathcal{P}(\mathbb{N}) \cup \{a, b\}$ to $P_v(L) =$ the coatoms of $\mathcal{P}(\mathbb{N}) \cup \{a, b\}$ whose inverse is R^u.

Example 2.6. Even in view of the above example, one might expect that the associated isomorphisms R_v and R^u extend to isomorphisms between the coprime and prime elements of L. This example shows that even if a pair of isomorphisms exists between $C(L)$ and $P(L)$, they need not be extensions of R_v and R^u, respectively, not even if L is a ring of sets.



x is an atom
v is a coatom
w covers p, p covers y
the remaining intervals are isomorphic to $\mathbb{Z} \cup \{-\infty, \infty\}$

Observe that:

$0^u = 0$,
$1^u = 1$,
$w^u = p$,
$t \in [x, p) \Rightarrow t^u \in [0, y]$,
$t \in [0, y] \Rightarrow t^u = x$,
$L^u = \{[0, y), x, p, w, [p, v), 1\}$,
$C^u(L) = \{(0, y), x, (p, v), w\}$,
$P_v(L) = \{(x, p), y, (w, 1), v\}$,
$y \in C(L)$ and $y_v = \bigvee(L \setminus \uparrow y) = p \notin P(L)$.

This indicates that Theorem 2.3 is as far as we can go to use the R_v and R^u mappings as candidates for isomorphisms between $C(L)$ and $P(L)$ (unlike in the finite case).

3. Results for completely distributive lattices

The previous section showed that if there are "enough" multiplicative elements then the R_v and R^u mappings are isomorphisms between $C^u(L)$ and $P_v(L)$ and vice versa. We begin this section by investigating conditions that provide enough multiplicative elements, in particular we characterize those completely distributive lattices for which the way below relation is multiplicative on all of L and L^{op}, thus providing us with a class of lattices for which these isomorphisms exist.

We first recall that a subset B of L is a *basis* for a lattice L if $0 \in B$, B is closed under finite joins and $x = \bigvee(\downarrow x \cap B)$ for all x in L. This is Definition III.4.1 in [2]. Further note that if L is completely distributive then B is a basis for L if and only if B is closed under finite joins and is join dense in L, [2, I.3.42].

The crucial result for this section is contained in the following theorem.

Theorem 3.1. *Let L be a complete lattice.*
(i) *If L is continuous then the following are equivalent:*

  (a) *the way-below relation is multiplicative on L,*
  (b) *there is a basis B for L such that $B \subseteq M(L)$ and B is a sublattice of L,*
  (c) *there is a basis B for L which satisfies $b_1 \wedge b_2 \in M(L)$ for all $b_1, b_2 \in B$.*

(ii) *If L^{op} is continuous then the following are equivalent:*

  (a) *the way-below relation is multiplicative on L^{op},*
  (b) *there is a basis B for L^{op} such that $B \subseteq M(L^{op})$ and B is a sublattice of L^{op},*
  (c) *there is a basis B for L^{op} which satisfies $b_1 \vee b_2 \in M(L^{op})$ for all $b_1, b_2 \in B$.*

Proof: (i) (a) $\Rightarrow$ (b). Take $B = L$. (b) $\Rightarrow$ (c) obvious. (c) $\Rightarrow$(a). Let $a \in L$ with $a \ll x$ and $a \ll y$. By [2, Proposition III.4.2, p. 168], there exist $b_1, b_2 \in B$ such that $a \le b_1 \ll x$ and $a \le b_2 \ll y$. Thus $a \le b_1 \wedge b_2$ and $b_1 \wedge b_2 \ll x$ as well as $b_1 \wedge b_2 \ll y$. Since $b_1 \wedge b_2 \in M(L)$ it follows that $b_1 \wedge b_2 \ll x \wedge y$ and hence $a \ll x \wedge y$.
Item (ii) follows by a dual argument. □

The preceding theorem says that the way-below relation is multiplicative on L if and only if $M(L)$ contains a sublattice of L which is a basis for L. In particular, the way-below relation on an algebraic lattice is multiplicative if and only if $K(L)$, the set of compact elements of L, is a sublattice of L, that is, L is arithmetic. This can be found in [2, Proposition I.4.7, p. 86] and thus Theorem 3.1 is an extension of this result.

Our next result is a similar fact about completely distributive lattices, and is based on the fact that, for any completely distributive lattice L, the set $C(L)$ is a join dense subset of L.

Theorem 3.2. *Let L be a completely distributive lattice.*

  (i) *The way-below relation is multiplicative on L if and only if $q_1 \wedge q_2 \in M(L)$ for any $q_1, q_2 \in C(L)$.*
  (ii) *The way-below relation is multiplicative on L^{op} if and only if $p_1 \vee p_2 \in M(L^{op})$ for any $p_1, p_2 \in P(L)$.*

Proof: (i) ($\Rightarrow$). This direction is obvious.

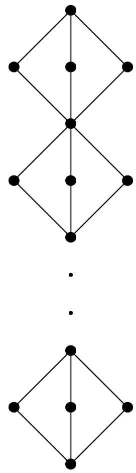
($\Leftarrow$). Let $q \in C(L)$ and set $p = q_v$. Then $L \backslash \downarrow p = \Uparrow q$ by Lemma 2.1(i). Since p is prime, $\Uparrow q$ is a filter or empty, hence $C(L) \subseteq M(L)$. Let $B = \{x \in L \mid x =$

$q_1 \vee \ldots \vee q_r$ for $q_i \in C(L)\}$. Then $B \cup \{0\}$ is a basis for L and $B \cup \{0\} \subseteq M(L)$. Furthermore, $b_1 \wedge b_2 \in M(L)$ for any $b_1, b_2 \in B \cup \{0\}$ and by Theorem 3.1(i) the way-below relation is multiplicative on L.

(ii) This part follows using (ii) of Theorem 3.1. □

The next example shows that distributivity is crucial in the above theorem.

Example 3.3. Following [2, p. 249], we take L to be

$L =$ 

Then $q_1 \wedge q_2 \in M(L)$ for any $q_1, q_2 \in C(L)$ is true vacuously since $C(L)$ is empty, but $M(L) \neq L$.

An interesting problem is to see whether Theorem 3.2 holds for distributive continuous lattices (that is for complete Heyting algebras). For example, if $L = \{(0,0)\} \cup (0,1] \times (0,1]$, then L is distributive and continuous but not completely distributive. $C(L) \subseteq M(L)$ again holds vacuously and $M(L) = L$. Indeed, all characterizations of the multiplicativity of the way-below relation given in [2] are for distributive continuous lattices.

The next lemma is an analog of Lemma 2.2 for completely distributive lattices:

Lemma 3.4. *Let L be a completely distributive lattice.*

(i) $L_v \subseteq P(L) \cup \{0,1\}$ *if and only if* $P_v(L) \cup \{1\}$ *is closed under arbitrary, nonempty joins and* $C(L) \subseteq M(L)$.

(ii) $L^u \subseteq C(L) \cup \{0,1\}$ *if and only if* $C^u(L) \cup \{0\}$ *is closed under arbitrary, nonempty meets and* $P(L) \subseteq M(L^{op})$.

Proof: $(\Rightarrow)$. Let $q \in C(L)$, then $p = q_v \in L_v$. If $p = 0$ then $0^u = p^u = (q_v)^u \geq q > 0$ and $0 \in P(L)$ (since it is completely meet irreducible), and thus $q_v \in P(L) \cup \{1\}$ and Lemma 2.2 yields $C(L) \subseteq M(L)$. Let $\emptyset \neq S \subseteq P_v(L) \cup \{1\}$.

Since, if $1 \in S$ then $\bigvee S = 1 \in P_v(L) \cup \{1\}$, we may assume that $1 \notin S$ and hence $S \subseteq P_v(L)$. This in turn implies that $s = (s^u)_v$ for each $s \in S$. Since the R_v map preserves joins and $L_v \subseteq P(L) \cup \{0,1\}$ we obtain

$$\bigvee S = \bigvee\{(s^u)_v \mid s \in S\} = \Big(\bigvee\{s^u \mid s \in S\}\Big)_v \in L_v \subseteq P(L) \cup \{0,1\}.$$

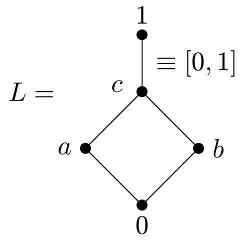
But if $\bigvee S = 0$ then $0 \in S$, since $S \neq \emptyset$ and thus $0 \in P(L) \cup \{1\}$. Hence $\bigvee S \in P_v(L) \cup \{1\}$.

($\Leftarrow$). By Lemma 2.2(i) we have that $R_v(C(L)) \subseteq P(L) \cup \{1\}$. Since L is completely distributive we have $x = \bigvee\{q \leq x \,|\, q \in C(L)\}$ for all $x \in L$. Since R_v preserves joins, we have $x_v = \bigvee\{q_v \,|\, q \leq x \text{ and } q \in C(L)\}$. But since $C(L) \subseteq M(L)$ we have q_v is a prime or 1 and thus in $P_v(L) \cup \{1\}$. Since this set is closed under joins we have $L_v \subseteq P(L) \cup \{0,1\}$.

(ii). This again follows dually. □

If L is completely distributive then so is L^{op} and one might expect that (i) and (ii) of Lemma 3.4 are equivalent. The next example shows that this is not the case:

Example 3.5. If L is the vertical sum of 2^2 and the interval $(0,1]$, as in the picture below, then L is completely distributive and satisfies (i) of Lemma 3.4, but L^{op} does not satisfy (ii).



This motivates our final theorem which shows what happens when we combine the two items in Lemma 3.4.

Theorem 3.6. *Let L be a completely distributive lattice. Then the following are equivalent:*

(i) *the way-below relations on L and L^{op} are multiplicative, $P_v(L) \cup \{1\}$ is closed under arbitrary, nonempty joins, and $C^u(L) \cup \{0\}$ is closed under arbitrary, nonempty meets,*

(ii) *$L_v \subseteq P(L) \cup \{0,1\}$ and $L^u \subseteq C(L) \cup \{0,1\}$.*

Proof: (i) $\Rightarrow$ (ii). Apply Lemma 3.4(i) and (ii).

(ii) $\Rightarrow$ (i). By Lemma 2.2(i) it follows that $C(L) \subseteq M(L)$ and that $P_v(L) \cup \{1\}$ is closed under arbitrary joins. Recall that L^u is join dense in L ([3], [5]) and that we are assuming $L^u \subseteq C(L) \cup \{0,1\}$, hence $L^u \subseteq M(L)$. Since the R^u map preserves meets we have that $q_1 \wedge q_2 \in L^u$ for any $q_1, q_2 \in L^u$. By Theorem 3.2(i) (using L^u instead of $C(L)$) we get that the way-below relation is multiplicative on L. The remaining parts of (i) now follow from a dual argument. □

The part of (i) in Theorem 3.6 requiring $L \subseteq M(L)$ and $L^{op} \subseteq M(L^{op})$ can be weakened to the following condition: there exists a join dense subset B and a meet dense subset B' of L such that $B \subseteq M(L)$ and $B' \subseteq M(L^{op})$. If we take $B = C(L)$ and $B' = P(L)$, then Theorem 3.6 is exactly the combination of items (i) and (ii) in Lemma 3.4. If L is a ring of sets, then, automatically, $M(L)$ contains a join dense subset of L and $M(L^{op})$ contains a meet dense subset of L. Consequently, item (ii) in Theorem 3.6 is equivalent to: (i)$'$ $P_v(L) \cup \{1\}$ is closed under arbitrary nonempty joins and $C^u(L) \cup \{0\}$ is closed under arbitrary nonempty meets. Moreover, we do not have an example of a completely distributive lattice L where (i)$'$ and (ii) are not equivalent.

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