A remark on associative copulas

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Abstract. A method for producing associative copulas from a binary operation and a convex function on an interval is described.

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Let I denote the unit interval $[0,1]$. Copulas are cumulative distribution functions on I^2 with uniform marginals; more precisely, a *copula* is a function $C(x,y)$ on I^2 that satisfies

(1) (Boundary Conditions)

$$C(x,0) = C(0,y) = 0,\ C(x,1) = x \ \text{ and } \ C(1,y) = y \ \text{ for all } \ x,y \in I,$$

and

(2) (Monotonicity)

$$C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) + C(x_1,y_1) \geq 0,$$

if $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$.

For φ, a continuous, strictly decreasing function from I to $[0,\infty]$ such that $\varphi(1) = 0$, we define the *pseudo-inverse* of φ to be the function $\varphi^{[-1]} : [0,\infty] \to I$ defined by

$$\varphi^{[-1]}(x) = \begin{cases} \varphi^{-1}(x), & \text{if } \ 0 \leq x \leq \varphi(0), \\ 0, & \text{if } \ \varphi(0) \leq x \leq \infty. \end{cases} \tag{1}$$

We say that C is an *Archimedean* copula with additive generator φ provided that it is a copula and that there exists a function φ of the type described here such that

$$C(x,y) = \varphi^{[-1]}(\varphi(x) + \varphi(y)). \tag{2}$$

To quote from [2], "These copulas find a wide range of applications for a number of reasons: (1) The ease with which they can be constructed; (2) The great variety of families of copulas which belong to this class; and (3) The many nice properties possessed by the members of this class." One of the most salient of these properties is that C is associative, that is,

$$C(x, C(y,z)) = C(C(x,y),z).$$

We have the following characterization of Archimedean copulas:

Theorem 1. *Let φ be a continuous, strictly decreasing function from I to $[0,\infty]$ such that $\varphi(1)=0$. Then the function C defined by* (2) *is a copula if and only if φ is convex.*

Proof of this theorem can be found in [1] and [2]. Discussion of related Archimedean binary operations can be found in [3].

We show this theorem can be generalized in a simple and elegant fashion in which, instead of dealing with pseudo-inverses, we extend the notion of convexity.

Let $\oplus$ be a continuous associative operation in $[0,a]$, $a \in [0,\infty]$, such that $t \oplus 0 = 0 \oplus t = t$ and $t \oplus a = a \oplus t = a$ for all $t \in [0,a]$.

Example 2. Let $a=\infty$ and $\oplus$ by the ordinary addition extended to $[0,\infty]$ in the obvious way. Clearly, the above conditions are satisfied.

Example 3. Let $a \in [0,\infty]$ be arbitrary and let $\oplus$ be defined by $s \oplus t = \max(s,t)$. Again, it is easy to check that the above conditions are satisfied.

Example 4. Let $a \in [0,\infty]$ be arbitrary and let $\oplus$ be defined by $s \oplus t = \min(s+t,a)$. Simple argument shows that the above conditions are satisfied.

A function $\psi : [0,a] \to \mathbb{R}$ is called $\oplus$-*convex* if

$$\psi(r \oplus t) - \psi(r) \le \psi(s \oplus t) - \psi(s) \tag{3}$$

for every $r \le s$ and any t.

Lemma 5. *If $\oplus$ is ordinary addition and ψ is continuous, then ψ satisfies* (3) *if and only if ψ is convex.*

Lemma 6. *If $s \oplus t = \max(s,t)$, then ψ satisfies is $\oplus$-convex if and only if ψ is decreasing.*

Proof: In order to show that (3) implies that ψ is decreasing it suffices to take $t=s$.

Now consider $r \le s$. We consider three cases. If $t \le r$, then (3) becomes

$$\psi(r) - \psi(r) \le \psi(s) - \psi(s),$$

which is always true. If $r \le t \le s$, then (3) reduces to

$$\psi(t) - \psi(r) \le \psi(s) - \psi(s) = 0,$$

which is true since ψ is decreasing. Finally, if $s \le t$, then (3) becomes

$$\psi(t) - \psi(r) \le \psi(t) - \psi(s),$$

or

$$\psi(s) \le \psi(r),$$

which is again true since ψ is decreasing. $\square$

Now we prove the main theorem of this note.

Theorem 7. *Let $\oplus$ be a continuous associative operation in $[0,a]$, $a \in [0,\infty]$, such that $t\oplus 0 = 0\oplus t = t$ and $t\oplus a = a\oplus t = a$ for all $t \in [0,a]$. Let $\varphi : [0,1] \to [0,a]$ be a strictly decreasing continuous surjection. Define*

$$C(x,y) = \varphi^{-1}\left(\varphi(x) \oplus \varphi(y)\right). \tag{4}$$

Then
(a) $C(0,z) = C(z,0) = 0$ *and* $C(1,z) = C(z,1) = z$ *for all* $z \in [0,1]$,
(b) C *is associative,*
(c) C *is a copula if and only if* φ^{-1} *is* $\oplus$*-convex.*

Parts (a) and (b) are easy. Before we prove part (c) we prove the following lemma.

Lemma 8. *Let $\oplus$, φ, and C be as in Theorem 7. Then C is monotonic if and only if*

$$C(u_2,v) - C(u_1,v) \le u_2 - u_1 \quad \text{whenever} \quad u_1 \le u_2. \tag{5}$$

Proof: Since every copula satisfies (5), it suffices to prove that (5) implies that C is monotonic.

Assume (5) and consider $v_1 \le v_2$. Since φ and $\oplus$ are continuous and $\varphi(v_2) \le \varphi(v_1)$, there exists $t \in [0,1]$ such that

$$\varphi(v_2) \oplus \varphi(t) = \varphi(v_1).$$

Hence

$$\begin{aligned}
C(u_2,v_1) &- C(u_1,v_1) \\
&= \varphi^{-1}\left(\varphi(u_2) \oplus \varphi(v_1)\right) - \varphi^{-1}\left(\varphi(u_1) \oplus \varphi(v_1)\right) \\
&= \varphi^{-1}\left(\varphi(u_2) \oplus (\varphi(v_2) \oplus \varphi(t))\right) - \varphi^{-1}\left(\varphi(u_1) \oplus (\varphi(v_2) \oplus \varphi(t))\right) \\
&= \varphi^{-1}\left((\varphi(u_2) \oplus \varphi(v_2)) \oplus \varphi(t)\right) - \varphi^{-1}\left((\varphi(u_1) \oplus \varphi(v_2)) \oplus \varphi(t)\right) \\
&= C(C(u_2,v_2),t) - C(C(u_1,v_2),t) \\
&\le C(u_2,v_2) - C(u_1,v_2),
\end{aligned}$$

which proves that C is monotonic. □

Proof of part (c) in Theorem 7: Suppose C is monotonic. Let $r \le s$ and $t > 0$. Let $u_1 = \varphi^{-1}(s)$, $u_2 = \varphi^{-1}(r)$, and $v = \varphi^{-1}(t)$. Since $u_1 \le u_2$, by Lemma 8, we have (5) and consequently

$$\varphi^{-1}(r \oplus t) - \varphi^{-1}(s \oplus t) \le \varphi^{-1}(r) - \varphi^{-1}(s),$$

which proves that φ^{-1} is $\oplus$-convex.

Now suppose φ^{-1} is $\oplus$-convex. Let $u_1 \le u_2$ and v be arbitrary. Define $r = \varphi(u_2)$, $s = \varphi(u_1)$, and $t = \varphi(v)$. Then (3) implies (5), which proves that C is monotonic by Lemma 8. □

The following simple theorem shows that every associative copula can be obtained in the way described in Theorem 7.

Theorem 9. *For every associative copula C there exist $\oplus$ and φ as in Theorem 7, with $a = 1$, such that*

$$C(x, y) = \varphi^{-1}\left(\varphi(x) \oplus \varphi(y)\right).$$

Proof: It suffices to define

$$r \oplus s = 1 - C(1 - r, 1 - s)$$

and

$$\varphi(t) = 1 - t.$$

□

The copula defined by (2) is commutative, because $+$ is a commutative operation. Since, in Theorem 7, commutativity plays no role, one may expect that by using a noncommutative $\oplus$ we can construct noncommutative associative copulas. However, this is not possible. In [1] the following theorem is proved.

Theorem 10. *Let $T : I^2 \to I$ be a continuous mapping such that*

$$T(x, 0) = T(0, y) = 0 \quad \text{and} \quad T(x, 1) = T(1, x) = x \quad \text{for all} \quad x \in I,$$

and

$$T(T(x, y), z) = T(x, T(y, z)) \quad \text{for all} \quad x, y, z \in I.$$

Then

$$T(x, y) = T(y, x) \quad \text{for all} \quad x \in I.$$

From this result we easily obtain the following property of the operation $\oplus$.

Theorem 11. *A continuous associative operation $\oplus$ in $[0, a]$, $a \in [0, \infty]$, such that $t \oplus 0 = 0 \oplus t = t$ and $t \oplus a = a \oplus t = a$ for all $t \in [0, a]$, is commutative.*

Proof: If $a < \infty$, then define

$$T(x, y) = 1 - \frac{1}{a}\left((a - ax) \oplus (a - ay)\right).$$

If $a = \infty$, then define

$$T(x, y) = \frac{1}{1 + \frac{1-x}{x} \oplus \frac{1-y}{y}}.$$

In both cases T satisfies the assumptions of Theorem 10, and thus $T(x, y) = T(y, x)$ for all $x, y \in I$. Now commutativity of $\oplus$ follows easily. □

Note that although every decreasing function is $\oplus$-convex with respect to the operation $\oplus$ defined in Example 3, we do not get a variety of copulas this way. Indeed,

$$\varphi^{-1}\left(\max(\varphi(x), \varphi(y))\right) = \min(x, y),$$

whenever φ is decreasing.

References

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