Cauchy-Neumann problem for a class of nondiagonal parabolic systems with quadratic nonlinearities II. Local and global solvability results

A. Arkhipova

Abstract. We prove local in time solvability of the nonlinear initial-boundary problem to nonlinear nondiagonal parabolic systems of equations (multidimensional case). No growth restrictions are assumed on generating the system functions.

In the case of two spatial variables we construct the global in time solution to the Cauchy-Neumann problem for a class of nondiagonal parabolic systems. The solution is smooth almost everywhere and has an at most finite number of singular points.

Keywords: boundary value problem, nonlinear parabolic systems, solvability

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This article is a continuation of the author's work [9]. Here we prove two independent results. These are local and global in time solvability theorems for a nonlinear initial boundary-value problem to nondiagonal parabolic systems.

In §1 (Theorem 1) local classical solvability is stated for general situations, that is, we do not assume any structural restriction and growth conditions on forming system and boundary condition functions. A related result for *quasilinear* parabolic systems under the Dirichlet and Neumann boundary conditions was proved in [1], [2].

Global in time weak solvability of the Cauchy-Neumann problem for parabolic systems studied in [9] is proved in §2 (Theorem 2). We consider a variational structure of an elliptic operator and consider only the case of two spatial variables. These systems have a *nondiagonal* main matrix and *quadratic* nonlinearity in the gradient. Note that the global solvability result is essentially based on the extendibility theorem (Theorem 1, [9]) and the local solvability theorem (Theorem 1 of the present paper).

This investigation is a generalization of the author's results [3], [4] where global in time weak solvability of the Cauchy-Dirichlet problem was stated for the same class of parabolic systems.

Here we make use of the notation of the Part I of the paper (see [9]).

1. Local in time classical solvability

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with sufficiently smooth boundary $\partial\Omega$. For a fixed $T_1 > 0$ and $Q = \Omega \times (0, T_1)$ we consider a solution $u : Q \to \mathbb{R}^N$,

$u = (u^1, \dots, u^N)$, $N > 1$, of the parabolic system

(1.1) $u_t^k - A_{kl}^{\alpha\beta}(z, u, u_x)u^l_{x_\beta x_\alpha} + b^k(z, u, u_x) = 0, \quad z = (x, t) \in Q,\ k = 1, \dots, N.$

The function u satisfies the initial condition

$$u\big|_{t=0} = 0, \tag{1.2}$$

and nonlinear boundary condition

$$\begin{gathered}\Phi^k(z, u, u_x) + \psi_{kl}^{\alpha}(z, u)u^l_{x_\alpha} + g^k(z, u)\big|_\Gamma = 0,\\ \Gamma = \partial\Omega \times (0, T_1), \quad k \le N.\end{gathered} \tag{1.3}$$

We define the sets

$$\mathcal{M} = \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN}, \quad \mathcal{N} = \overline{\Gamma} \times \mathbb{R}^N \times \mathbb{R}^{nN}, \quad \mathcal{N}^0 = \overline{\Gamma} \times \mathbb{R}^N, \tag{1.4}$$

and suppose that the functions $A_{kl}^{\alpha\beta}$, b^k, Φ^k, ψ_{kl}^{α}, g^k are smooth enough on $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{N}_0$, respectively. (More exactly, see conditions $\mathbb{A}_1, \dots, \mathbb{A}_5$ below.)

Suppose that the matrix $A = \{A_{kl}^{\alpha\beta}\}_{k,l \le N}^{\alpha,\beta \le n}$ satisfies on $\mathcal{M}$ the condition

$$A_{kl}^{\alpha\beta}(z, u, p)\xi_\alpha^k \xi_\beta^l \ge \nu|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^{nN}, \quad \nu = \text{const} > 0. \tag{1.5}$$

We introduce the functions

$$\Phi^k(z, u, p) = \int_0^1 \frac{\partial \Phi^k(z, u, sp)}{\partial p_\beta^l}\, ds \cdot p_\beta^l + \Phi^k(z, u, 0) \equiv \varkappa_{kl}^{\beta}(z, u, p)p_\beta^l + \Phi^k(z, u, 0)$$

and suppose that

$$\Big(\varkappa_{kl}^{\beta}(z, u, p) + \psi_{kl}^{\beta}(z, u)\Big)\cos(\mathbf{n}, x_\beta)\eta^k\eta^l \ge \nu_0|\eta|^2, \qquad \forall\, \eta \in \mathbb{R}^N,\ \nu_0 = \text{const} > 0, \tag{1.6}$$

where $\mathbf{n} = \mathbf{n}(x)$ is the outward normal vector to Ω at a point $x \in \partial\Omega$, $(z, u, p) \in \mathcal{N}$.

We rewrite (1.3) in the form

$$\Big(\varkappa_{kl}^{\beta}(z, u, u_x) + \psi_{kl}^{\beta}(z, u)\Big)\, u^l_{x_\beta} + G^k(z, u)\big|_\Gamma = 0, \quad k \le N, \tag{1.7}$$

where $G^k(x, t, u) = g^k(x, t, u) + \Phi^k(x, t, u, 0)$.

The compatibility condition is written in the following form

$$G^k(x, 0, 0) = 0, \quad x \in \partial\Omega, \quad k = 1, \dots, N. \tag{1.8}$$

We intend to prove the existence of a smooth solution to (1.1)–(1.3) (or (1.1), (1.2), (1.7)) on some interval $[0, T_0)$, where $T_0 \leq T_1$.

Note that in the case $A = A(z, u)$ and condition (1.7) in the form

$$A_{kl}^{\alpha\beta}(z,u)u_{x_\beta}^l \cos(\mathbf{n}, x_\alpha) + G^k(z,u)\big|_\Gamma = 0,$$

local in time classical solvability of (1.1), (1.2) follows from [1], [2]. To prove the existence of a solution we use the contraction method.

We introduce the following notation

$$\langle v\rangle_{x,Q}^{(\alpha)} = \sup_{\substack{(x,t),(x',t)\in\bar{Q}\\ x\neq x'}} \frac{|v(x,t)-v(x',t)|}{|x-x'|^\alpha}, \quad \langle v\rangle_{t,Q}^{(\beta)} = \sup_{\substack{(x,t),(x,t')\in\bar{Q}\\ t\neq t'}} \frac{|v(x,t)-v(x,t')|}{|t-t'|^\beta},$$

$$[v]_Q^{(\alpha)} = \langle v\rangle_{x,Q}^{(\alpha)} + \langle v\rangle_{t,Q}^{(\alpha/2)}, \quad \alpha,\beta\in(0,1).$$

$\|u\|_{m,D}$ denotes the norm of u in the space $L^m(D)$, $m\in[1,\infty]$.

Here and below we write $\mathcal{B}(Q)$ instead of $\mathcal{B}(Q;\mathbb{R}^N)$ for brevity.

$\mathcal{H}^{\alpha,\alpha/2}(\overline{Q})$ is the space of all continuous in $\overline{Q}$ functions with finite norm

$$\|v\|_{\mathcal{H}^{\alpha,\alpha/2}(\bar{Q})} = \|v\|_{\infty,Q} + \langle v\rangle_{x,Q}^{(\alpha)} + \langle v\rangle_{t,Q}^{(\alpha/2)}.$$

(So $\mathcal{H}^{\alpha,\alpha/2}(\overline{Q}) = C^{\alpha,\alpha/2}(\overline{Q})$).

$\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})$ is the space of functions u continuous on $\overline{Q}$ with derivatives u_t, u_x, u_{xx} and finite norm:

$$\begin{aligned}\|u\|_{\mathcal{H}^{2+\alpha,1+\alpha/2}(\bar{Q})} &= \|u\|_{\infty,Q} + \|u_x\|_{\infty,Q} + \|u_{xx}\|_{\infty,Q} + \|u_t\|_{\infty,Q}\\ &+ [u_t]_Q^{(\alpha)} + [u_{xx}]_Q^{(\alpha)} + \langle u_x\rangle_{t,Q}^{((1+\alpha)/2)}.\end{aligned}$$

We also consider the space $\mathcal{H}^{1+\alpha,(1+\alpha)/2}(\overline{\Gamma})$ of functions v that are continuous and have continuous derivatives v_x on $\overline{\Gamma}$. Here we define this space as the trace space for $\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})$ [5]. Let $\partial\Omega$ be a $C^{2+\alpha}$ surface. We denote by $V_1,\dots,V_m\subset\mathbb{R}^n$ a system of neighborhoods with the following properties: (1) $\bigcup_{j=1}^m V_j \supset \partial\Omega$, (2) there exists a system of $C^{2+\alpha}$ diffeomorphisms P_j on V_j, $j=1,\dots,m$, such that $P_j(V_j\cap\Omega) = B_1^+$, $P_j(V_j\cap\partial\Omega) = \sigma$. Here $B_1 = \{x\in\mathbb{R}^n \mid |x|<1\}$, $B_1^+ = B_1\cap\{x_n>0\}$, $\sigma = B_1\cap\{x_n=0\}$.

For a function $v\in\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})$, $x\in V_j\cap\partial\Omega$, $t\in[0,T_1]$, we define the function $v^{(j)}(y,t) = v(P_j^{-1}(y),t)$ on $\overline{Q^+} = \overline{B_1^+}\times[0,T_1]$.

Let $y' = (y_1,\dots,y_{n-1})$, $\Sigma = \bar{\sigma}\times[0,T_1]$, we put

$$\|v\|_{\mathcal{H}^{1+\alpha,(1+\alpha)/2}(\bar{\Gamma})} = \sup_{j\leq m}\|v^{(j)}(y',0,t)\|_{\mathcal{H}^{1+\alpha,(1+\alpha)/2}(\Sigma)}, \tag{1.9}$$

where

$$(1.10)\quad \|w(y',t)\|_{\mathcal{H}^{1+\alpha,(1+\alpha)/2}(\Sigma)} = \|w\|_{\infty,\Sigma} + \|w_{y'}\|_{\infty,\Sigma} + [w_{y'}]_{\Sigma}^{(\alpha)} + \langle w\rangle_{t,\Sigma}^{(1+\alpha)/2}.$$

It is obvious that definition (1.9),(1.10) depends on the fixed atlas $\{V_j, P_j\}_{j=1}^m$ but all the norms are equivalent.

Now we fix an $\alpha_0 > 0$ and some $T \in (0, T_1]$ and introduce the space

$$X_T = \left\{v : Q \to \mathbb{R}^N, v \in \mathcal{H}^{2+\alpha_0,1+\alpha_0/2}(\overline{Q^T})\middle|\ \ v|_{t=0} = 0\right\}, \quad Q^T = \Omega \times (0,T).$$

For a fixed $v \in X_T$ we put

$$\begin{aligned}
&\Delta A_{kl}^{\alpha\beta}(x,t,v,v_x) = A_{kl}^{\alpha\beta}(x,t,v,v_x) - A_{kl}^{\alpha\beta}(x,0,0,0),\\
&\Delta\varkappa_{kl}^{\beta}(x,t,v,v_x) = \varkappa_{kl}^{\beta}(x,t,v,v_x) - \varkappa_{kk}^{\beta}(x,0,0,0),\\
&\Delta\psi_{kl}^{\beta}(x,t,v) = \psi_{kl}^{\beta}(x,t,v) - \psi_{kl}^{\beta}(x,0,0)
\end{aligned}$$

and consider the linear problem

(1.11)

$$\begin{aligned}
&w_t^k - A_{kl}^{\alpha\beta}(x,0,0,0)w_{x_\beta x_\alpha}^l + b^k(x,t,v,v_x) + \Delta A_{kl}^{\alpha\beta}(x,t,v,v_x)v_{x_\beta x_\alpha}^l = 0,\ (x,t)\in Q^T,\\
&\Big(\varkappa_{kl}^{\beta}(x,0,0,0) + \psi_{kl}^{\beta}(x,0,0)\Big)\, w_{x_\beta}^l + G^k(x,t,v) + \big(\Delta\varkappa_{kl}^{\beta}(x,t,v,v_x)\\
&\ + \Delta\psi_{kl}^{\beta}(x,t,v)\big)v_{x_\beta}^l\big|_{\Gamma^T} = 0, \quad \Gamma^T = \partial\Omega\times(0,T),\ k = 1,\dots,N,\\
&w\big|_{t=0} = 0.
\end{aligned}$$

We write $\Delta A v_{xx} = \{\Delta A_{kl}^{\alpha\beta}(x,t,v,v_x)v_{x_\beta x_\alpha}^l\}^{k\le N}$, $\Delta\psi\cdot v_x = \{\Delta\psi_{kl}^{\beta}(x,t,v)v_{x_\beta}^l\}^{k\le N}$, $\Delta\varkappa v_x = \{\Delta\varkappa_{kl}^{\beta}(x,t,v,v_x)v_{x_\beta}^l\}^{k\le N}$, $G = \{G^k(x,t,v)\}^{k\le N}$ for brevity.

We assume that the complementing conditions hold for problem (1.11).

If the data are smooth enough then according to the linear theory there exists a unique solution $w \in X_T$ of (1.11) [5, Chapter VII, Theorem 10.1] and the following estimate is valid:

$$(1.12)\quad \begin{aligned}
\|w\|_{X^T} \le c_0\big\{&\|b\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\bar{Q}^T)} + \|\Delta A\cdot v_{xx}\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\bar{Q}^T)}\\
&+ \|\Delta\varkappa\cdot v_x\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Gamma^T)} + \|\Delta\psi\cdot v_x\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Gamma^T)}\\
&+ \|G\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Gamma^T)}\big\}.
\end{aligned}$$

The constant c_0 in (1.12) depends on the parameters ν, ν_0 from conditions (1.5), (1.6), $\|A(x,0,0,0)\|_{C^{\alpha_0}(\bar{\Omega})}$, $\|\varkappa(x,0,0,0)\|_{C^{1+\alpha_0}(\partial\Omega)}$, $\|\psi(x,0,0)\|_{C^{1+\alpha_0}(\bar{\Omega})}$, $C^{2+\alpha_0}$ characteristics of $\partial\Omega$ and T_1, but it does not depend on the fixed T and any characteristic of the function v.

Thus, problem (1.11) defines the map $F{:}X_T \to X_T$,

(1.13) $$w = F(v), \quad \forall\, v \in X_T.$$

We shall prove that if $T < T_1$ is small enough then there exists a fixed point u of F. The function $u \in X_T$ is a solution of the problem (1.1)–(1.3).

Now we impose precise conditions on the data.

Let $M > 0$ be an arbitrary fixed number.

$\mathbb{A}_1$. On $\mathcal{M}_M = \{(x,t,u,p) \in \mathcal{M} |\ |u| + |p| \le M\}$

- the functions $A = \{A_{kl}^{\alpha\beta}\}_{k,l\le N}^{\alpha,\beta\le n}$ are continuous and have continuous derivatives A_u, A_p;
- the functions A, A_u, A_p are Hölder continuous in x,t,u,p with the exponents α_0, $\alpha_0/2$, α_0, α_0, respectively.

$\mathbb{A}_2$. On $\mathcal{M}_M$

- the functions $b = \{b^k(x,t,u,p)\}^{k\le N}$ are continuous with derivatives b_u and b_p,
- the functions b are Hölder continuous in x,t with the exponents α_0, $\alpha_0/2$, respectively,
- b_u, b_p are Hölder continuous in x,t,u,p with the exponents α_0, $\alpha_0/2$, α_0, α_0, respectively.

$\mathbb{A}_3$. On $\mathcal{N}_M = \{(x,t,u,p) \in \mathcal{N} |\ |u| + |p| \le M\}$,

- the functions $\Phi = \{\Phi^k(x,t,u,p)\}^{k\le N}$ are continuous with derivatives Φ_p,
- the functions $\varkappa = \{\varkappa_{kl}^{\beta}(x,t,u,p)\}_{k,l\le N}^{\beta\le n}$, $\varkappa_{kl}^{\beta}(x,t,u,p) = \int_0^1 \frac{\partial \Phi^k(x,t,u,sp)}{\partial p_\beta^l}\,ds$ have continuous derivatives $\varkappa_x$, $\varkappa_u$, $\varkappa_p$,
- the functions $\varkappa$, $\varkappa_u$, $\varkappa_p$ are Hölder continuous in t with the exponent $(1+\alpha_0)/2$,
- the derivatives $\varkappa_x$ are Hölder continuous in x,t with the exponents α_0 and $\alpha_0/2$ correspondently,
- the derivatives $\varkappa_{xu}$, $\varkappa_{xp}$, $\varkappa_{uu}$, $\varkappa_{up}$, $\varkappa_{pp}$ exist and are Hölder continuous in x,t,u,p with the exponents α_0, $\alpha_0/2$, α_0, α_0.

$\mathbb{A}_4$. On $\mathcal{N}_M^0 = \{(x,t,u) \in \mathcal{N}^0 ||u| \le M\}$,

- the function $G(x,t,u) = \Phi(x,t,u,0) + g(x,t,u)$, $G = \{G^k\}^{k\le N}$, is continuous with derivatives G_x, G_u, G_{xu}, G_{uu},
- the functions G, G_u are Hölder continuous in t with the exponent $(1+\alpha_0)/2$,
- the function G_x is Hölder continuous in x and t with the exponent α_0, $\alpha_0/2$,
- G_{xu}, G_{uu} are Hölder continuous in x,t,u with the exponents α_0, $\alpha_0/2$, α_0.

$\mathbb{A}_5$. On $\mathcal{N}_M^0$,

- $\psi = \{\psi_{kl}^\beta(x,t,u)\}_{k,l\le N}^{\beta\le n}$ are continuous functions with derivatives ψ_x, ψ_u,
- the functions ψ are Hölder continuous in t with the exponent $(1+\alpha_0)/2$,
- ψ_x, ψ_u are Hölder continuous in x,t,u with the exponents α_0, $\alpha_0/2$, α_0, respectively.

Remark 1. All Hölder constants h in conditions $\mathbb{A}_1$–$\mathbb{A}_5$ depend on M, i.e., $h = h(M)$.

We put

$$\begin{aligned} H_0 = \|b(x,0,0,0)\|_{\infty,\Omega} + \left[b(x,t,0,0)\right]_Q^{(\alpha_0)} + \|G_x(x,t,0)\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\Gamma)} \\ + \|G_v(x,t,0)\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\Gamma)} + \langle G(x,t,o)\rangle_{t,\Gamma}^{(1+\alpha_0)/2}, \end{aligned} \tag{1.14}$$

and note that H_0 depends on T_1 but it does not depend on T and M.

Now we formulate the main local result.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 2$, with $C^{2+\alpha_0}$-smooth boundary $\partial\Omega$, $\alpha_0 \in (0,1)$ is a fixed number. Suppose that conditions* (1.5), (1.6), *and* (1.8) *hold and linear problem* (1.11) *satisfies the complementing conditions. Then there exist numbers M_0 and $T_0 = T_0(M_0) \in (0, T_1]$ such that if assumptions $\mathbb{A}_1$–$\mathbb{A}_5$ hold with $M = M_0$ then problem* (1.1)–(1.3) *is uniquely solvable in X_T for any fixed $T < T_0$. Numbers M_0 and $T_0(M_0)$ depend on the given problem data.*

We split the proof of Theorem 1 into the following lemmas.

Lemma 1. *There exist numbers M_0 and $\hat{T} = \hat{T}(M_0) \le T_1$, such that for any $T \le \hat{T}$ the map F transforms $\mathcal{B}_{M_0}$ in $\mathcal{B}_{M_0}$, where $\mathcal{B}_{M_0} = \{v \in X_T | \, \|v\|_{X_T} \le M_0\}$. The numbers M_0 and $\hat{T}$ depend on H_0 (see* (1.14)) *and on the same values as the constant c_0 in* (1.12).

Lemma 2. *Let M_0 and $\hat{T}$ be fixed as in Lemma* 1. *There exists a positive number $\theta = \theta(M_0)$ such that for every $v_1, v_2 \in \mathcal{B}_{M_0} \subset X_T$, $T \le \hat{T}$, we have*

$$\left\|F(v_1) - F(v_2)\right\|_{X_T} \le \theta \left(T^{\alpha_0/2} + T^{(1-\alpha_0)/2}\right) \|v_1 - v_2\|_{X_T}. \tag{1.15}$$

Indeed, if these lemmas are proved then we fix T' from the condition

$$\theta \left[(T')^{\alpha_0/2} + (T')^{(1-\alpha_0/2}\right] = 1.$$

For $T < T_0 = \min\{\hat{T}, T'\}$, the mapping F is a contraction in $\mathcal{B}_{M_0} \subset X_T$, which implies the existence of a unique $u \in \mathcal{B}_{M_0}$ such that $u = F(u)$. Certainly, u is the solution to (1.1), (1.2), (1.7) or (1.1)–(1.3) and Theorem 1 is proved.

Proof of Lemma 1: Fix $T \le T_1$, $M > 0$ and $v \in \mathcal{B}_M \subset X_T$ arbitrary. For all summands J_k in the braces of (1.12), we shall derive the following inequalities:

$$J_k \le h_k(M)\left(T^{\alpha_0/2} + T^{(1-\alpha_0)/2}\right) + H_k, \quad k = 1, \dots, 5. \tag{1.16}$$

In what follows, we denote by $h_k(M)$, $h(M)$ different but nondecreasing in M functions. They may depend on the data and T_1 but not on the fixed T. All parameters H_k and H are independent of M and T.

1. *Estimation of* $J_1 = \|b\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\bar{Q}^T)}$. We split J_1 in

$$J_1 = \|b\|_{\infty,Q^T} + \langle b\rangle^{(\alpha_0)}_{x,Q^T} + \langle b\rangle^{(\alpha_0/2)}_{t,Q^T} = j_1 + j_2 + j_3.$$

First of all, note that $v\big|_{t=0} = 0$ and

$$\begin{aligned} &\|v\|_{\infty,Q^T} \le \|v_t\|_{\infty,Q^T} T, \qquad \|v_x\|_{\infty,Q^T} \le \langle v_x\rangle^{(1+\alpha_0)/2}_{t,Q^T} T^{(1+\alpha_0)/2}, \\ &\|v_{xx}\|_{\infty,Q^T} \le \langle v_{xx}\rangle^{(\alpha_0/2)}_{t,Q^T} T^{\alpha_0/2}. \end{aligned} \tag{1.17}$$

It is evident that

$$\begin{aligned} j_1 &\le \|b(x,t,v,v_x) - b(x,0,0,0)\|_{\infty,Q^T} + \|b(x,0,0,0)\|_{\infty,Q^T} \\ &\le h(M)\left(T^{\alpha_0/2} + \|v\|_{\infty,Q^T} + \|v_x\|_{\infty,Q^T}\right) + H \\ &\le h(M)\left(T^{\alpha_0/2} + T + T^{(1-\alpha_0)/2}\right) + H. \end{aligned}$$

To estimate $j_2 = \langle b\rangle^{(\alpha_0)}_{x,Q^T}$ we write the inequalities

$$\begin{aligned} &|b(x,t,v(x,t),v_x(x,t)) - b(x',t,v(x',t),v_x(x',t)| \\ &\le |b(x,t,0,0) - b(x',t,0,0)| + \left| \int_0^1 \frac{db(x,t,sv(x,t),sv_x(x,t))}{ds}\, ds \right. \\ &\left. - \int_0^1 \frac{db(x',t,sv(x',t),sv_x(x',t))}{ds}\, ds \right| \le H|\Delta x|^{\alpha_0} \\ &+ \int_0^1 |b_v(x,t,sv(x,t),sv_x(x,t)) - b_v(x',t,sv(x',t),sv_x(x',t))|\, |v(x,t)|\, ds \\ &+ \int_0^1 |b_v(x',t,sv(x',t),sv_x(x',t))|\, ds |v(x,t) - v(x',t)| \end{aligned}$$

$$+\int_0^1 |b_p(x,t,sv(x,t),sv_x(x,t)) - b_p(x',t,sv(x',t),sv_x(x',t))|\,ds\cdot|v_x(x,t)|$$

$$+\int_0^1 |b_p(x',t,sv(x',t),sv_x(x',t))|\,ds|v_x(x,t)-v_x(x',t)| \underset{(*)}{\le} H|\Delta x|^{\alpha_0}$$

$$+h(M)|\Delta x|^{\alpha_0}\left(T+T^{\alpha_0}+T^{(1+\alpha_0)/2}\right),\quad |\Delta x| = |x-x'|.$$

To justify inequality $(*)$, we have used (1.17) and the following inequalities:

$$\begin{aligned} |v(x,t)-v(x',t)| &\le \langle v_x\rangle^{(1+\alpha_0)/2}_{t,Q^T} T^{(1+\alpha_0)/2}|\Delta x|,\\ |v_x(x,t)-v_x(x',t)| &\le \langle v_{xx}\rangle^{\alpha_0/2}_{t,Q^T} T^{\alpha_0/2}|\Delta x|. \end{aligned} \tag{1.18}$$

We arrive at the inequality

$$j_2 \le H + h(M)\left(T^{\alpha_0}+T^{(1+\alpha_0)/2}\right).$$

j_3 is estimated in a similar way:

$$|b(x,t,v(x,t),v_x(x,t)) - b(x,t',v(x,t'),v_x(x,t'))| \le |b(x,t,0,0)-b(x,t',0,0)|$$

$$+\left|\int_0^1\left[\frac{db(x,t,sv(x,t),sv_x(x,t))}{ds} - \frac{db(x,t',sv(x,t'),sv_x(x,t'))}{ds}\right]ds\right|$$

$$\le H|\Delta t|^{\alpha_0/2}$$

$$+\left|\int_0^1 \left[b_v(x,t,sv(x,t),sv_x(x,t))v(x,t) - b_v(x,t',sv(x,t'),sv_x(x,t'))v(x,t')\right]ds\right|$$

$$+\left|\int_0^1 \left[b_p(x,t,sv(x,t),sv_x(x,t))v_x(x,t) - b_p(x,t',sv(x,t'),sv_x(x,t'))v_x(x,t')\right]ds\right|$$

$$\le H|\Delta t|^{\alpha_0/2} + h(M)\left(\|v\|_{\infty,Q^T}+\|v_x\|_{\infty,Q^T}\right)\left\{|\Delta t|^{\alpha_0/2}+|v(x,t)-v(x,t')|^{\alpha_0}\right.$$

$$\left.+|v_x(x,t)-v_x(x,t')|^{\alpha_0}\right\} + h(M)\{|v(x,t)-v(x,t')|+|v_x(x,t)-v_x(x,t')|\}$$

$$\le H|\Delta t|^{\alpha_0/2} + h(M)\left(T+T^{(1+\alpha_0)/2}\right)\left\{|\Delta t|^{\alpha_0/2}+|\Delta t|^{\alpha_0}+|\Delta t|^{\alpha_0\cdot(1+\alpha_0)/2}\right\}$$

$$+M\{|\Delta t|+|\Delta t|^{(1+\alpha_0)/2}\}.$$

It follows that $j_3 \le H + h(M)T^{\alpha_0/2}$ and we get (1.16) for $k=1$, where H_1 is defined by $\|b(x,0,0,0)\|_{\infty,\Omega}$ and $[b(x,t,0,0)]^{(\alpha_0)}_{Q^T}$.

2. *Estimation of* $J_2 = \|\Delta A \cdot v_{xx}\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\bar{Q}^T)}$. We have

$$J_2 = \|\Delta A \cdot v_{xx}\|_{\infty,Q^T} + \langle \Delta A \cdot v_{xx}\rangle^{(\alpha_0)}_{x,Q^T} + \langle \Delta A \cdot v_{xx}\rangle^{(\alpha_0/2)}_{t,Q^T} = i_1 + i_2 + i_3.$$

It is easy to see that $i_1 \le h(M)\|v_{xx}\|_{\infty,Q} \le h(M)T^{\alpha_0/2}$.

Further,

$$i_2 \le \langle \Delta A\rangle^{(\alpha_0)}_{x,Q^T}\|v_{xx}\|_{\infty,Q^T} + \|\Delta A\|_{\infty,Q^T}\langle v_{xx}\rangle^{(\alpha_0)}_{x,Q},$$

where

$$\big\langle A^{\alpha\beta}_{kl}(x,t,v,v_x) - A^{\alpha\beta}_{kl}(x,0,0,0)\big\rangle^{(\alpha_0)}_{x,Q^T} \le \big\langle A^{\alpha\beta}_{kl}(x,t,v,v_x)\big\rangle^{(\alpha_0)}_{x,Q^T} + H \le h(M) + H,$$

$\|\Delta A\|_{\infty,Q} \le h(M)T^{\alpha_0/2}$.

Whence,

$$i_2 \le h(M)\|v_{xx}\|_{\infty,Q^T} + h(M)T^{\alpha_0/2} \underset{(1.17)}{\le} h(M)T^{\alpha_0/2}.$$

At last,

$$i_3 \le \langle \Delta A\rangle^{(\alpha_0/2)}_{t,Q^T}\|v_{xx}\|_{\infty,Q^T} + \|\Delta A\|_{\infty,Q^T}\langle v_{xx}\rangle^{(\alpha_0/2)}_{t,Q^T} \le h(M)T^{\alpha_0/2}.$$

Consequently, $J_2 \le h(M)T^{\alpha_0/2}$ and (1.18) is proved for $k = 2$.

3. *Estimation of* $J_3 = \|\Delta\varkappa v_x\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Gamma^T)}$ *and* $J_4 = \|\Delta\psi v_x\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Gamma^T)}$.

For a fixed atlas $\{V_j, P_j\}_{j=1}^m$ we choose a neighborhood V_j and a mapping P_j and then express $\Delta\varkappa v_x$ in the local coordinate system $(y_1,\dots,y_n)$. We shall write $y = y(x)$ and $x = x(y)$ for $y = P_j(x)$ and $x = P_j^{-1}(y)$, respectively, $\hat{v}(y,t) = v(x(y),t)$, $y \in \overline{B_1^+}$, $t \in [0,T]$. In the new coordinates

$$\Delta\varkappa^{\beta}_{kl}(x,t,v(x,t),v_x(x,t))v^l_{x_\beta}(x,t) = \bigg[\varkappa^{\beta}_{kl}\bigg(x(y),t,\hat{v}(y,t),\hat{v}_y(y,t)\,\frac{\partial y}{\partial x}\bigg) - \varkappa^{\beta}_{kl}(x(y),0,0,0)\bigg]\hat{v}^l_{y_\gamma}(y,t)\frac{\partial y_\gamma}{\partial x_\beta}, \quad \hat{v}\big|_{t=0} = 0 \text{ in } B_1^+.$$

Putting

$$\hat{\varkappa}^{\gamma}_{kl}(y,t,\hat{v}(y,t),\hat{v}_y(y,t)) = \varkappa^{\beta}_{kl}\bigg(x(y),t,\hat{v}(y,t),\hat{v}_y(y,t)\,\frac{\partial y}{\partial x}\bigg)\,\frac{\partial y_\gamma}{\partial x_\beta},$$

we have $\Delta\varkappa v_x\big|_{x=x(y)} = \Delta\hat{\varkappa}\hat{v}_y$.

According to definition (1.9), (1.10), we have to estimate the expression

$$\hat{J}_3 = \|\Delta\hat{\varkappa}\hat{v}_y\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Sigma^T)} = \|\Delta\hat{\varkappa}\hat{v}_y\|_{\infty,\Sigma^T} + \|(\Delta\hat{\varkappa}\hat{v}_y)_{y'}\|_{\infty,\Sigma^T}$$
$$+ \left[(\Delta\hat{\varkappa}\hat{v}_y)_{y'}\right]^{(\alpha_0)}_{\Sigma^T} + \langle\Delta\hat{\varkappa}\hat{v}_y\rangle^{(1+\alpha_0)/2}_{t,\Sigma^T}.$$

Here $\Sigma^T = \sigma \times (0,T)$, $\sigma = \{y' \in \mathbb{R}^{n-1} |\ |y'| < 1\}$.

We have the following estimates for $\hat{v}$:

$$\begin{aligned}
&\|\hat{v}\|_{\infty,\Sigma^T} \le MT, \quad [\hat{v}]^{(\alpha_0)}_{\Sigma^T} \le h(M)\left(T^{1-\alpha_0/2} + T^{(1+\alpha_0)/2}\right),\\
&\|\hat{v}_y\|_{\infty,\Sigma^T} \le h(M)T^{(1+\alpha_0)/2}, \quad [\hat{v}_y]^{(\alpha_0)}_{\Sigma^T} \le h(M)T^{\alpha_0/2},\\
&\langle\hat{v}_y\rangle^{(1+\alpha_0)/2}_{t,\Sigma^T} \le h(M), \quad \|(\hat{v}_y)_{y'}\|_{\infty,\Sigma^T} \le h(M)\left(T^{(1+\alpha_0)/2} + T^{\alpha_0/2}\right),\\
&\left[\left(\hat{v}_y\right)_{y'}\right]^{(\alpha_0)}_{\Sigma^T} \le h(M),
\end{aligned} \tag{1.19}$$

where $h(M)$ depends on the same parameters as in (1.17), (1.18) and $C^{2+\alpha_0}$ characteristics of the maps $y = y(x)$ and $x = x(y)$.

Now for $\hat{J}_3$ we deduce the estimate

$$\hat{J}_3 \le h(M)T^{(1+\alpha_0)/2} + h(M)T^{\alpha_0/2}\left(\|\Delta\hat{\varkappa}\|_{\infty,\Sigma^T} + \|(\Delta\varkappa)_{y'}\|_{\infty,\Sigma^T}\right)$$
$$+ \|\Delta\hat{\varkappa}\|_{\infty,\Sigma^T}h(M)T^{\alpha_0/2} + \left[(\Delta\hat{\varkappa})_{y'}\,\hat{v}_y\right]^{(\alpha_0)}_{\Sigma^T} + \left[\Delta\hat{\varkappa}(\hat{v}_y)_{y'}\right]^{(\alpha_0)}_{\Sigma^T}$$
$$+ \langle\Delta\varkappa\rangle^{(1+\alpha_0)/2}_{t,\Sigma^T}h(M)T^{(1+\alpha_0)/2} + h(M)\|\Delta\hat{\varkappa}\|_{\infty,\Sigma^T}.$$

Here

$$\|\Delta\hat{\varkappa}\|_{\infty,\Sigma^T} \le h(M)\left(T^{\alpha_0/2} + \|\hat{v}\|_{\infty,\Sigma} + \|\hat{v}_y\|_{\infty,\Sigma^T}\right) \underset{(1.19)}{\le} h(M)T^{\alpha_0/2};$$

$$\|(\Delta\hat{\varkappa})_{y'}\|_{\infty,\Sigma^T} \le h(M)T^{\alpha_0/2} + h(M)[\|\hat{v}_{y'}\|_{\infty,\Sigma^T} + \|(\hat{v}_y)_{y'}\|_{\infty,\Sigma^T}]$$
$$\underset{(1.19)}{\le} h(M)T^{\alpha_0/2};$$

$$[\Delta\hat{\varkappa}]^{(\alpha_0)}_{\Sigma^T} + [(\Delta\hat{\varkappa})_{y'}]^{(\alpha_0)}_{\Sigma^T} + \langle\Delta\hat{\varkappa}\rangle^{(1+\alpha_0)/2}_{t,\Sigma^T} \underset{(1.19)}{\le} h(M) + H.$$

From the above it follows that $\hat{J}_3 \le h(M)T^{\alpha_0/2}$. This implies (1.16) for $k = 3$. The expression J_4 is estimated in the same way as J_3.

4. *Estimation of* $J_5 = \|G\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Gamma^T)}$.

In the local coordinates system we estimate $\hat{J}_5 = \|\hat{G}\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Sigma^T)}$, where $\hat{G}(y', t, \hat{v}(y', 0, t)) = G(x(y', 0), t, v(x(y', 0), t))$, $y' \in \sigma$, $t \in [0, T]$.

From the compatibility condition (1.8) it follows that $\hat{G}\big|_{t=0} = \hat{G}_{y'}\big|_{t=0} = 0$.
It is easy to see that

$$\begin{aligned}\hat{J}_5 &\le \|\hat{G}\|_{\infty,\Sigma^T} + \|\hat{G}_{y'}\|_{\infty,\Sigma^T} + \|\hat{G}_{\hat{v}}\|_{\infty,\Sigma^T}\|\hat{v}_{y'}\|_{\infty,\Sigma^T} \\ &+ [\hat{G}_{y'}]^{(\alpha_0)}_{\Sigma^T} + [\hat{G}_{\hat{v}}]^{(\alpha_0)}_{\Sigma^T}\|v_y\|_{\infty,\Sigma^T} + \|\hat{G}_{\hat{v}}\|_{\infty,\Sigma^T}[\hat{v}_y]^{(\alpha_0}_{\Sigma^T} \\ &+ \langle\hat{G}\rangle^{(1+\alpha_0)/2}_{t,\Sigma^T} \underset{(1.19)}{\le} h(M)T^{\alpha_0/2} + [\hat{G}_{y'}]^{(\alpha_0)}_{\Sigma^T} + \langle\hat{G}\rangle^{(1+\alpha_0)/2}_{t,\Sigma^T}.\end{aligned} \tag{1.20}$$

To estimate $[\hat{G}_{y'}]^{(\alpha_0)}_{\Sigma^T}$ we consider the expression

$$\begin{aligned}\langle\hat{G}_{y'}\rangle^{(\alpha_0)}_{y',\Sigma^T} &= \sup_{\left\{\substack{y',y''\in\sigma \\ t\in[0,T], \\ y'\neq y''}\right\}} \frac{|\hat{G}_{y'}(y',t,\hat{v}(y',0,t)) - \hat{G}_{y'}(y'',t,\hat{v}(y'',0,t))|}{|\Delta y'|^{\alpha_0}} \\ &\le \sup_{\{\dots\}} \frac{\left|\int_0^1 \left[\frac{d}{ds}\hat{G}_{y'}(y',t,s\hat{v}(y',0,t)) - \frac{d}{ds}\hat{G}_{y'}(y'',t,s\hat{v}(y'',0,t))\right] ds\right|}{|\Delta y'|^{\alpha_0}} \\ &+ \langle\hat{G}_{y'}(y',t,0)\rangle^{(\alpha_0)}_{y',\Sigma^T} \\ &\le h\left(\|\hat{v}\|_{\infty,\Sigma^T} + \langle\hat{v}\rangle^{(\alpha_0)}_{y',\Sigma^T}\right) + H \le h(M)\left(T + T^{(1+\alpha_0)/2}\right) + H.\end{aligned}$$

In the same way we derive that

$$\langle\hat{G}_{y'}\rangle^{(\alpha_0/2)}_{t,\Sigma^T} \le h(M)\left(T + T^{1-\alpha_0/2}\right) + H.$$

This implies

$$[\hat{G}_{y'}]^{(\alpha_0)}_{\Sigma^T} \le h(M)\left(T^{1-\alpha_0/2} + T^{(1+\alpha_0)/2}\right) + H. \tag{1.21}$$

To estimate $[\hat{G}_{\hat{v}}]^{(\alpha_0)}_{\Sigma^T}$ we argue in the same way.
At last, we shall derive

$$\langle\hat{G}\rangle^{(1+\alpha_0)/2}_{t,\Sigma^T} \le h(M)\left(T + T^{(1-\alpha_0)/2}\right) + H. \tag{1.22}$$

Indeed,

$$
\begin{aligned}
&|\hat{G}(y',t',\hat{v}(y',0,t')) - \hat{G}(y',t'',\hat{v}(y',0,t''))| \\
&\le \left| \int_0^1 \left[\frac{d\hat{G}(y',t',s\hat{v}(y',0,t'))}{ds} - \frac{d\hat{G}(y',t'',s\hat{v}(y',0,t''))}{ds} \right] ds \right| \\
&+ |\hat{G}(y',t',0) - \hat{G}(y',t'',0)| \le \int_0^1 \big|\hat{G}_{\hat{v}}(y',t',s\hat{v}(y',0,t')) \\
&- \hat{G}_{\hat{v}}(y',t'',s\hat{v}(y',0,t''))\big|\, ds \|\hat{v}\|_{\infty,\Sigma^T} \\
&+ \int_0^1 \big|\hat{G}_{\hat{v}}(y',t'',s\hat{v}(y',0,t''))\big|\, ds \|\hat{v}_t\|_{\infty,\Sigma^T} |\Delta t| \\
&+ H|\Delta t|^{(1+\alpha_0)/2} \le h(M)T \left\{ |\Delta t|^{(1+\alpha_0)/2} + |\Delta t| \right\} + h(M)|\Delta t| + H|\Delta t|^{(1+\alpha_0)/2} \\
&\le \left(h(M)T^{(1-\alpha_0)/2} + H \right) |\Delta t|^{(1+\alpha_0)/2}.
\end{aligned}
$$

This proves (1.22). Now from (1.20)–(1.22) it follows that

$$
\hat{J}_5 \le h(M) \left(T^{\alpha_0/2} + T^{(1-\alpha_0)/2} \right) + H,
$$

verifying (1.16) for J_5.

Thus, by (1.12), (1.16), we obtain that

$$
\|w\|_{X^T} \le c_0 h(M) \left(T^{\alpha_0/2} + T^{(1-\alpha_0)/2} \right) + c_0 H_0, \tag{1.23}
$$

where c_0 is the constant from (1.12) and H_0 is defined in (1.14).

Now we put $M_0 = 2c_0H_0$ and fix $\hat{T} \le T_1$ from the condition

$$
c_0 h(M_0) \left(\hat{T}^{\alpha_0/2} + \hat{T}^{(1-\alpha_0)/2} \right) \le c_0 H_0.
$$

Then from (1.23) it follows that for any $T \le \hat{T}$ and $v \in \mathcal{B}_{M_0} \subset X_T$, $\|w\|_{X_T} = \|F(v)\|_{X_T} \le M_0$. Lemma 1 is proved. □

Proof of Lemma 2: Let M and $\hat{T}(M)$ be fixed as pointed in the statement of Lemma 1. For any $T \le \hat{T}$ we fix v' and $v'' \in \mathcal{B}_M \subset X_T$. Put $w' = F(v')$, $w'' = F(v'')$, $\hat{w} = w' - w''$, $\hat{v} = v' - v''$. According to (1.14) we have the following system:

$$
\begin{aligned}
&\hat{w}^k_t - A^{\alpha\beta}_{kl}(x,0,0,0)\hat{w}^l_{x_\beta x_\alpha} + \left[b^k(z,v',v'_x) - b^k(z,v'',v''_x) \right] \\
&+ \left[\Delta A^{\alpha\beta}_{kl}(z,v',v'_x)(v'^l)_{x_\beta x_\alpha} - \Delta A^{\alpha\beta}_{kl}(z,v'',v''_x)(v''^l)_{x_\beta x_\alpha} \right] = 0, \\
&z \in Q^T, \quad k \le N,
\end{aligned}
$$

$$
\begin{aligned}
&\Big(\varkappa^\beta_{kl}(x,0,0,0)+\psi^\beta_{kl}(x,0,0)\Big)\,\hat w^l_{x_\beta}+\left[G^k(z,v')-G^k(z,v'')\right]\\
&+\Big[\Big(\Delta\varkappa^\beta_{kl}(z,v',v'_x)+\Delta\psi^\beta_{kl}(z,v')\Big)\,(v'^l)_{x_\beta}-\Big(\Delta\varkappa^\beta_{kl}(z,v'',v''_x)\\
&+\Delta\psi^\beta_{kl}(z,v'')\Big)\,(v''^l)_{x_\beta}\Big]=0\quad\text{on}\quad\Gamma_T,\\
&\hat w\big|_{t=0}=0.
\end{aligned}\tag{1.24}
$$

Problem (1.24) can be written in the short form:

$$
\begin{aligned}
&\hat w^k_t-A^{\alpha\beta}_{kl}(x,0,0,0)\hat w^l_{x_\beta x_\alpha}+D^k(z)+E^k(z)=0,\quad z\in Q_T,\ k\le N,\\
&\Big(\varkappa^\beta_{kl}(x,0,0,0)+\psi^\beta_{kl}(x,0,0)\Big)\,\hat w^l_{x_\beta}+Z^k(z)+Y^k(z)\big|_{\Gamma_T}=0,
\end{aligned}\tag{1.24^0}
$$

where D, E, Z, Y denote the correspondent expressions in the square brackets of (1.24). For example, $D=\{D^k\}^{k\le N}$,
$D^k(z)=b^k(z,v'(z),v'_x(z))-b^k(z,v''(z),v''_x(z))$ and so on.

For the linear problem (1.24^0), the following estimate holds:

$$
\begin{aligned}
\|\hat w\|_{X_T}\le c_0\Big\{&\|D\|_{\mathcal H^{\alpha_0,\alpha_0/2}(\bar Q^T)}+\|E\|_{\mathcal H^{\alpha_0,\alpha_0/2}(\bar Q^T)}\\
&+\|Z\|_{\mathcal H^{1+\alpha_0,(1+\alpha_0)/2}(\bar\Gamma^T)}+\|Y\|_{\mathcal H^{1+\alpha_0,(1+\alpha_0)/2}(\bar\Gamma^T)}\Big\}.
\end{aligned}\tag{1.25}
$$

We shall prove that every term on the right-hand side of (1.25) is estimated by $h(M)(T^{\alpha_0/2}+T^{(1-\alpha_0)/2})\|\hat v\|_{X_T}$ with some nondecreasing function $h(M)>0$.

1) *Estimation of* $\|D\|_{\mathcal H^{\alpha_0,\alpha_0/2}(\overline{Q^T})}$.

First of all we note that $\hat v|_{t=0}=0$ and

$$
\begin{aligned}
&\|\hat v\|_{\infty,Q^T}\le\|\hat v\|_{X^T}T,\quad [\hat v]^{(\alpha_0)}_{Q^T}\le c(\Omega)\|\hat v\|_{X^T}\Big(T^{(1+\alpha_0)/2}+T^{1-\alpha_0/2}\Big),\\
&\|\hat v_x\|_{\infty,Q^T}\le\|\hat v\|_{X^T}T^{(1+\alpha_0)/2},\quad [\hat v_x]^{(\alpha_0)}_{Q^T}\le c(\Omega,T_1)\|\hat v\|_{X^T}T^{\alpha_0/2},\\
&\|\hat v_{xx}\|_{\infty,Q^T}\le\|\hat v\|_{X^T}T^{\alpha_0/2}.
\end{aligned}\tag{1.26}
$$

We write D^k in the form:

$$
D^k(z)=\int_0^1\frac{\partial b^k(z,\tilde v,\tilde v_x)}{\partial v^m}\,ds\ \hat v^m(z)+\int_0^1\frac{\partial b^k(z,\tilde v,\tilde v_x)}{\partial v^m_{x_\gamma}}\,ds\ \hat v^m_{x_\gamma}(z),
$$

where $\tilde{v} = v'' + s\hat{v}$, $\tilde{v}_x = v''_x + s\hat{v}_x$. By condition $\mathbb{A}_2$, we obtain the inequality

$$\|D\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\bar{Q}^T)} \leq h(M)\Big(\|\hat{v}\|_{\infty,Q^T} + \|\hat{v}_x\|_{\infty,Q^T}\Big) + \Big[\int_0^1 b_v(\dots)\,ds\Big]^{(\alpha_0)}_{\overline{Q}^T} \|\hat{v}\|_{\infty,Q^T}$$
$$+ \Big\|\int_0^1 b_v(\dots)\,ds\Big\|_{\infty,Q^T} [\hat{v}]^{(\alpha_0)}_{Q^T} + \Big[\int_0^1 b_p(\dots)\,ds\Big]^{(\alpha_0)}_{\bar{Q}^T} \|\hat{v}_x\|_{\infty,Q^T}$$
$$+ \Big\|\int_0^1 b_p(\dots)\,ds\Big\|_{\infty,Q^T} [\hat{v}_x]^{(\alpha_0)}_{Q^T} \leq h_1(M)\|\hat{v}\|_{X^T} T^{\alpha_0/2}.$$

Here and below, we denote by $h(M)$ different positive nondecreasing on M functions. They do not depend on T but may be depend on T_1.

2) *Estimation of* $\|E\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\bar{Q}^T)}$.

$E^k(z)$ can be written in the form
$E^k(z) = \Delta A^{\alpha\beta}_{kl}(z, v', v'_x)\hat{v}^l_{x_\beta x_\alpha} + [A^{\alpha\beta}_{kl}(z, v', v'_x) - A^{\alpha\beta}_{kl}(z, v'', v''_x)](v''^l)_{x_\beta x_\alpha}$, then by condition $\mathbb{A}_1$ and inequalities (1.26), we obtain the estimate

$$\|E\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\bar{Q}^T)} \leq \|\Delta A(z, v', v'_x)\|_{\infty,Q^T}\|\hat{v}_{xx}\|_{\infty,Q^T}$$
$$+ M\Big(\Big\|\int_0^1 A_v(\dots)\,ds\Big\|_{\infty,Q^T}\|\hat{v}\|_{\infty,Q^T} + \Big\|\int_0^1 A_p(\dots)\,ds\Big\|_{\infty,Q^T}\|\hat{v}_x\|_{\infty,Q^T}$$
$$+ \|\Delta A\|_{\infty,Q^T}[\hat{v}_{xx}]^{(\alpha_0)}_{Q^T} + [\Delta A]^{(\alpha_0)}_{Q^T}\|\hat{v}_{xx}\|_{\infty,Q^T} +$$
$$+ \Big[\int_0^1 A_v(\dots)\,ds\Big]^{(\alpha_0)}_{Q^T}\|\hat{v}\|_{\infty,Q^T} M + \Big\|\int_0^1 A_v(\dots)\,ds\Big]_{\infty,Q^T}[\hat{v}\hat{v}_{xx}]^{(\alpha_0)}_{Q^T}$$
$$+ \Big[\int_0^1 A_p(\dots)\,ds\Big]^{(\alpha_0)}_{Q^T}\|\hat{v}_x\|_{\infty,Q^T} M + \Big[\int_0^1 A_p(\dots)\,ds\Big]^{(\alpha_0)}_{Q^T}[\hat{v}_x\hat{v}_{xx}]^{\alpha_0}_{Q_T}$$
$$\leq h(M)\|\hat{v}\|_{X^T} T^{\alpha_0/2}.$$

3) *Estimation of* $\|Y\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Gamma^T)}$.

To estimate this expression we straighten a part of $\partial\Omega$ and obtain the corresponding local estimate of the norm as it was done in the proof of Lemma 1. Using the same notation we introduce an atlas $\{V_j, P_j\}_{j\leq m}$ for $\overline{\Omega}$, where $P_j : V_j \cap \Omega \to B_1^+$, $P_j(V_j \cap \partial\Omega) = \sigma$, $\Sigma_T = \sigma \times (0, T)$.

For some fixed $j \le m$, we write $y = y(x)$ for $y = P_j(x)$ and $x = x(y)$ for $x = P_j^{-1}(y)$.

Note that $Y^k(z) = [\Delta\varkappa_{kl}^{\beta}(z, v', v'_x){v'_{x_\beta}}^l - \Delta\varkappa_{kl}^{\beta}(z, v'', v''_x){v''_{x_\beta}}^l] +$ $[\Delta\psi_{kl}^{\beta}(z, v'){v'_{x_\beta}}^l - \Delta\psi_{kl}^{\beta}(z, v''){v''_{x_\beta}}^l] \equiv Y_I^k(z) + Y_{II}^k(z)$. It is sufficient to prove that

$$\|Y_I\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Gamma_T)} \le h(M)\|\hat{v}\|_{X_T}\left(T^{\alpha_0/2} + T^{(1-\alpha_0)/2}\right). \tag{1.27}$$

An analogous estimate for Y_{II} is more easily derived in the same way.

In the local coordinates $y \in B_1^+, y = (y', y_n), y' \in \sigma$, we have

$$\begin{aligned}
&\widehat{Y}_I^k(y', t) \equiv \Delta\varkappa_{kl}^{\beta}\left(x(y), t, v'(x(y), t), v'_y(x(y), t)\cdot\frac{\partial y}{\partial x}\right)\hat{v}_{y_\gamma}^l \cdot \frac{\partial y_\gamma}{\partial x_\beta}\\
&+\left[\varkappa_{kl}^{\beta}\left(x(y), t, v'(x(y), t), v'_y(x(y), t)\cdot\frac{\partial y}{\partial x}\right) - \varkappa_{kl}^{\beta}\left(x(y), t, v''(x(y), t),\right.\right.\\
&\left.\left.v''_y(x(y), t)\cdot\frac{\partial y}{\partial x}\right)\right]{v''_{y_\gamma}}^l\frac{\partial y_\gamma}{\partial x_\beta}\bigg|_{\substack{y'\in\sigma\\ y_n=0}} \equiv \Delta\hat{\varkappa}_{kl}^{\gamma}(y', t, v', v'_y)\hat{v}_{y_\gamma}^l\\
&+\left(\hat{\varkappa}_{kl}^{\gamma}(y', t, v', v'_y) - \hat{\varkappa}_{kl}^{\gamma}(y', t, v'', v''_y)\right){v''_{y_\gamma}}^l,
\end{aligned} \tag{1.28}$$

where in the last equality we set $v = v(x(y', 0), t)$.

We shall estimate $\|\hat{Y}_I\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Sigma_T)}$ according to definition (1.9), (1.10).

In the local coordinates

$$\begin{aligned}
&\|\hat{v}\|_{\infty,\Sigma_T} \le \|\hat{v}\|_{X_T}\cdot T; \quad \|\hat{v}_y\|_{\infty,\Sigma_T} \le K_1\|\hat{v}\|_{X_T}\cdot T^{\alpha_0/2};\\
&[\hat{v}]_{\Sigma_T}^{(\alpha_0)} + [\hat{v}_y]_{\Sigma_T}^{(\alpha_0)} \le K_2\|\hat{v}\|_{X_T}\cdot T^{\alpha_0/2};\\
&\|(\hat{v}_y)_{y'}\|_{\infty,\Sigma_T} \le K_3\|\hat{v}\|_{X_T}\cdot T^{\alpha_0/2}; \qquad [(\hat{v}_y)_{y'}\|_{\Sigma_T}^{(\alpha_0)} \le K_4,
\end{aligned} \tag{1.29}$$

where constants K_i depend on T_1 and $C^{2+\alpha_0}$ characteristics of $x(y)$ and $y(x)$.

It is easy to see that $l_1 = \|\hat{Y}_I\|_{\infty,\Sigma_T} \le h(M)\|\hat{v}\|_{X_T}T^{\alpha_0/2}$.

For the next step we omit indexes of functions and write

$$\begin{aligned}
\left(\hat{Y}_I\right)_{y'} &= \Delta\hat{\varkappa}_{y'}\hat{v}_y + \Delta\hat{\varkappa}(\hat{v}_y)_{y'} + \int_0^1 \frac{\partial^2\varkappa(\dots)}{\partial v\partial y'}\,ds\ \hat{v}v''_y\\
&+\int_0^1 \frac{\partial^2\hat{\varkappa}(\dots)}{\partial v\partial v}\tilde{v}_{y'}ds\ \hat{v}v''_y + \int_0^1 \frac{\partial^2\hat{\varkappa}(\dots)}{\partial v\partial p}\tilde{v}_{yy'}ds\ \hat{v}v''_y\\
&+\int_0^1 \frac{\partial\hat{\varkappa}(\dots)}{\partial v}\,ds\ (\hat{v}_{y'}v''_y + \hat{v}(v''_y)_{y'}) + \int_0^1 \frac{\partial^2\hat{\varkappa}(\dots)}{\partial p\partial v}\tilde{v}_{y'}\,ds\ \hat{v}_{y'}v''_y
\end{aligned} \tag{1.30}$$

$$+\int_0^1 \frac{\partial^2 \hat{\varkappa}(\dots)}{\partial p \partial p}\,(\tilde{v}_y)_{y'}\,ds\ \hat{v}_{y'} v''_y + \int_0^1 \frac{\partial^2 \varkappa(\dots)}{\partial p \partial y'}\,ds\ \hat{v}_y v''_y$$

$$+\int_0^1 \frac{\partial \hat{\varkappa}(\dots)}{\partial p}\,ds\ ((\hat{v}_y)_{y'} v''_y + \hat{v}_y (v''_y)_{y'}).$$

Now we apply definition (1.28), conditions $\mathbb{A}_3$, estimates (1.29) and equality (1.30) to deduce that

$$l_2 = \|(\hat{Y}_I)_{y'}\|_{\infty,\Sigma_T} \le h(M)\|\hat{v}\|_{X_T} T^{\alpha_0/2}.$$

Further,

$$l_3 = \left[\left(\hat{Y}_I\right)_{y'}\right]^{(\alpha_0)}_{\Sigma_T} \le \Big\{[\Delta\varkappa_{y'}]^{(\alpha_0}_{\Sigma_T}\|\hat{v}_y\|_{\infty,\Sigma_T} + \|\Delta\hat{\varkappa}_{y'}\|_{\infty,\Sigma_T}[\hat{v}_y]^{(\alpha_0)}_{\Sigma_T}$$

$$+[\Delta\hat{\varkappa}]^{(\alpha_0)}_{\Sigma_T}\|(\hat{v}_y)_{y'}\|_{\infty,\Sigma_T} + \|\Delta\hat{\varkappa}\|_{\infty,\Sigma_T}[(\hat{v}_y)_{y'}]^{(\alpha_0)}_{\Sigma_T} + \dots$$

$$\dots + \left\|\int_0^1 \frac{\partial \hat{\varkappa}(\dots)}{\partial p}\,ds \cdot (\hat{v}''_y)_{y'}\right\|_{\infty,\Sigma_T} \cdot [\hat{v}_y]^{(\alpha_0)}_{\Sigma_T}\Big\},$$

where there are twenty two terms in the braces. We have not enough place to write and calculate all of them. Note only that all the terms are estimated by $h(M)\|\hat{v}\|_{X_T} T^{\alpha_0/2}$ by using $\mathbb{A}_3$ and inequalities (1.29).

At last,

$$l_4 = \left\langle \hat{Y}_I \right\rangle^{\left(\frac{1+\alpha_0}{2}\right)}_{t,\Sigma_T} \le \left\langle \Delta\hat{\varkappa}(y',t,v',v'_y)\hat{v}_y \right\rangle^{\left(\frac{1+\alpha_0}{2}\right)}_{t,\Sigma_T}$$

$$+\left\langle \int_0^1 \frac{\partial \hat{\varkappa}(y',t,\tilde{v},\tilde{v}_y)}{\partial v}\,ds\ v''_y \right\rangle^{\left(\frac{1+\alpha_0}{2}\right)}_{t,\Sigma_T} \|\hat{v}\|_{\infty,\Sigma_T}$$

$$+\left\|\int_0^1 \frac{\partial \hat{\varkappa}(\dots)}{\partial v}\,ds\ v''_y\right\|_{\infty,\Sigma_T} \langle \hat{v}\rangle^{\left(\frac{1+\alpha_0}{2}\right)}_{t,\Sigma_T}$$

$$+\left\langle \int_0^1 \frac{\partial \hat{\varkappa}(\dots)}{\partial p}\,ds\ v''_y \right\rangle^{\left(\frac{1+\alpha_0}{2}\right)}_{t,\Sigma_T} \|\hat{v}_y\|_{\infty,\Sigma_T}$$

$$+\left\|\int_0^1 \frac{\partial \hat{\varkappa}(\dots)}{\partial p}\,ds\ v''_y\right\|_{\infty,\Sigma_T} \langle \hat{v}_y\rangle^{\left(\frac{1+\alpha_0}{2}\right)}_{t,\Sigma_T} = j_1 + j_2 + j_3 + j_4 + j_5.$$

For example, we estimate j_1:

$$j_1 \le \langle \Delta\hat{\varkappa}\rangle_{t,\Sigma_T}^{\left(\frac{1+\alpha_0}{2}\right)} \|\hat{v}_y\|_{\infty,\Sigma_T} + \|\Delta\hat{\varkappa}\|_{\infty,\Sigma_T} \langle \hat{v}_y\rangle_{t,\Sigma_T}^{\left(\frac{1+\alpha_0}{2}\right)}.$$

Here $\langle \Delta\hat{\varkappa}\rangle_{t,\Sigma_T}^{\left(\frac{1+\alpha_0}{2}\right)} = \langle \Delta\hat{\varkappa}(y',t,v'(y',t),v'_y(y',t))\rangle_{t,\Sigma_T}^{\left(\frac{1+\alpha_0}{2}\right)} \le h(M)$, $\|\Delta\hat{\varkappa}\|_{\infty,\Sigma_T} \le h(M)T^{(1+\alpha_0)/2}$. Now j_1 is estimated by $h(M)\|\hat{v}\|_{X_T}T^{\alpha_0/2}$ with the help of (1.29). All other j_k's are estimated in the same way (about the arguments $(\dots)$ see (1.28)). Summarizing, we have estimated l_1–l_4 and $\|\hat{Y}_I\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Sigma_T)}$ by $h(M)\|\hat{v}\|_{X_T}T^{\alpha_0/2}$ with some $h(M) > 0$, hence (1.27) follows.

4) *Estimation of* $\|Z\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Gamma_T)}$.

We write Z_k in the form $Z^k(z) = \int_0^1 \frac{\partial G^k(z,\tilde{v}(z))}{\partial v^m}\, ds\ \hat{v}^m(z)$, $\tilde{v} = v'' + s\hat{v}$ and put $\widehat{Z}(y',t) = Z(x(y',0),t)$, $y' \in \sigma$, $t \in (0,T)$.

To deduce the estimate

$$\|\hat{Z}\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Sigma_T)} \le h(M)\|\hat{v}\|_{X_T}\left(T^{\alpha_0/2} + T^{(1-\alpha_0)/2}\right) \tag{1.31}$$

we make use of conditions $\mathbb{A}_4$, inequalities (1.29) and argue in the same way as at the previous step. From (1.31) and definition (1.9), (1.10), the estimate of $\|Z\|_{\mathcal{H}^{1+\alpha_0,(1+\alpha_0)/2}(\Gamma_T)}$ follows.

Now we go back to estimate (1.25) and obtain that for some $h(M) > 0$

$$\|\hat{w}\|_{X_T} \le c_0 h(M)\|\hat{v}\|_{X_T}\left(T^{\alpha_0/2} + T^{(1-\alpha_0)/2}\right). \tag{1.32}$$

Here c_0 depends on the same data as in the statement of Theorem 1. By inequality (1.32) with $\theta = c_0 h(M)$, (1.17) follows. Lemma 2 is proved. □

2. Weak global in time solvability

Using M. Struwe's idea [6], we shall construct a global solution of the Cauchy-Neumann problem to the class of parabolic systems studied in [9].

Suppose that Ω is a bounded domain in $\mathbb{R}^2$ and $T > 0$ is fixed arbitrarily, $Q = \Omega \times (0,T)$. For some functions $f\colon \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{2N} \to \mathbb{R}^1$ and $G\colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^1$, $N > 1$, we consider a solution $u\colon Q \to \mathbb{R}^N$, $u = (u^1,\dots,u^N)$, of the problem

$$\begin{aligned}
&u_t^k - \frac{d}{dx_\alpha} f_{p_\alpha^k}(x,u,u_x) + f_{u^k}(x,u,u_x) = 0, \quad z = (x,t) \in Q,\\
&f_{p_\alpha^k}(x,u,u_x)\cos(\mathbf{n},x_\alpha) + g^k(x,u)\big|_\Gamma = 0, \quad \Gamma = \partial\Omega \times (0,T),\ k \le N,\\
&u\big|_{t=0} = \varphi(x),
\end{aligned} \tag{2.1}$$

where $g = \nabla_u G$, $\varphi{:}\overline{\Omega} \to \mathbb{R}^N$ is a given function, $\mathbf{n} = \mathbf{n}(x)$ is the outward to Ω normal vector at a point $x \in \partial\Omega$.

It is easy to see that the corresponding to (2.1) stationary problem describes stationary points of the functional

$$\mathcal{E}[u] = \int\limits_{\Omega} f(x, u, u_x)\, dx + \int\limits_{\partial\Omega} G(x, u)\, ds. \tag{2.2}$$

Now we fix a number $\alpha_0 \in (0, 1)$ and formulate all assumptions on $\partial\Omega$, φ, f, G and g.

$\mathbb{D}_1$. $\partial\Omega \in \mathbb{C}^{3+\alpha_0}$, $\varphi \in W_2^1(\Omega)$.

$\mathbb{D}_2$. f is defined on the set $\mathcal{M} = \partial\Omega \times \mathbb{R}^N \times \mathbb{R}^{2N}$ with the derivatives mentioned below and satisfies the following conditions:

(1)

$$\begin{gathered} \nu_0|p|^2 \le f \le \mu_1 + \mu_0|p|^2, \\ |f_u| + |f_{ux}| + |f_{uu}| \le \mu_2(1 + |p|^2), \quad |f_p| + |f_{px}| + |f_{pu}| \le \mu_2(1 + |p|), \\ |f_{pp}| + |f_{ppx}| \le \mu_2, \quad \langle f_{pp}(x, u, p)\xi, \xi\rangle \ge \nu|\xi|^2, \ \forall \xi \in \mathbb{R}^{2N}, \end{gathered} \tag{2.3}$$

with positive constants ν_0, μ_0, μ_2, ν and $\mu_1 \ge 0$.

(2) Derivatives f_{px}, f_{ppx} are continuous on $\mathcal{M}$ and are Hölder continuous in x with the exponent α_0 on any compact set of $\mathcal{M}$.

(3) $\psi(x, u, p) = f_{up}(x, u, p)$ is continuously differentiable in x, u, p on the set $\mathcal{M}$.

(4) On any compact subset of $\mathcal{M}$, the function $\Lambda(x, u, p) = f_{pp}(x, u, p)$ is twice continuously differentiable in all arguments and Λ_{xu}, Λ_{xp}, Λ_{uu}, Λ_{up}, Λ_{pp} are Hölder continuous in all arguments with the exponent α_0.

$\mathbb{D}_3$. (1) $G(x, u)$ is a continuous function on the set $\mathcal{M}_0 = \overline{\Omega} \times \mathbb{R}^N$, it has continuous derivative G_x and satisfies

$$G \ge h_0|u|^2 - h_1, \quad |G| + |G_x| \le h_2(1 + |u|^2), \tag{2.4}$$

$h_0, h_1 = \text{const} \ge 0$, $h_2 = \text{const} > 0$.

(2) The function $g(x, u) = \nabla_u G(x, u)$ and its derivatives g, g_x, g_{xx}, g_u, g_{ux}, g_{uu} are continuous on $\mathcal{M}_0$ and

$$|g| + |g_x| + |g_{xx}| \le h_3(1 + |u|), \quad |g_u| + |g_{ux}| + |g_{uu}| \le h_3, \tag{2.5}$$

$h_3 = \text{const} > 0$.

(3) On any compact subset of $\mathcal{M}_0$, g_x is Hölder continuous in x with the exponent α_0 and g_{xu}, g_u are Hölder continuous in x, u with the exponent α_0.

It is evident that under assumptions $\mathbb{D}_1$–$\mathbb{D}_3$, the parabolic system (2.1) has *nondiagonal* main matrix and *quadratic nonlinearity* in the gradient. In general, the weak global solvability for such a type systems was not proved yet.

If we put $f(x,u,p) = \frac{1}{2}\, A^{\alpha\beta}_{kl}(x,u)p^l_\alpha p^k_\beta$, $G(x,u) = \frac{1}{2}\, h(x)|u|^2 + (u, r(x))$, where $A^{\alpha\beta}_{kl}$ are $C^{2+\alpha_0}$ smooth functions on $\overline{\Omega}\times\mathbb{R}^N$ and $A^{\alpha\beta}_{kl} = A^{\beta\alpha}_{lk}$,

$$A^{\alpha\beta}_{kl}(x,u)\xi^k_\alpha\xi^l_\beta \ge \nu|\xi|^2, \quad \forall\,\xi\in\mathbb{R}^{2N}, \quad \nu = \text{const} > 0,$$

$h, r \in C^2(\overline{\Omega})$, $h(x)\ge 0$, then conditions $\mathbb{D}_2$, $\mathbb{D}_3$ hold. In this case we have the quasilinear problem (2.1) in the form

(2.6)
$$\begin{aligned}
&u^k_t - (A^{\alpha\beta}_{kl}(x,u)u^l_{x_\beta})_{x_\alpha} + \frac{1}{2}\,(A^{\alpha\beta}_{ml}(x,u))'_{u^k} u^m_{x_\beta} u^l_{x_\alpha} = 0, \quad (x,t)\in Q,\\
&\left(\frac{\partial u}{\partial n_A}\right)^{(k)}\Big|_\Gamma \equiv A^{\alpha\beta}_{kl}(x,u)u^l_{x_\beta}\cos(\mathbf{n},x_\alpha) + h(x)u^k + r^k(x)\big|_\Gamma = 0, \quad k\le N,\\
&u\big|_{t=0} = \varphi.
\end{aligned}$$

We shall construct a weak global solution to (2.1) (and, in particular, to (2.6)) in five steps.

<u>Step 1.</u> First of all, we "smooth" the initial function φ.

Proposition 1. *Under conditions $\mathbb{D}_1$–$\mathbb{D}_3$ there exists a sequence $\{\varphi_m(x)\}_{m\in\mathcal{N}}$, $\varphi_m \in C^{2+\alpha_0}(\overline{\Omega})$, with $\varphi_m \to \varphi$ in $W^1_2(\Omega)$ and such that every function φ_m satisfies the compatibility condition:*

$$l^{[k]}[\varphi_m] = f_{p^k_\alpha}(x,\varphi_m(x),(\varphi_m(x))_x)\cos(\mathbf{n},x_\alpha) + g^k(x,\varphi_m(x))\big|_{x\in\partial\Omega} = 0, \quad k\le N. \tag{2.7}$$

As $\partial\Omega\in C^{3+\alpha_0}$, there exists a sequence $\{\psi_m\}$, $\psi_m\in C^{3+\alpha_0}(\overline{\Omega})$, $\psi_m\to\varphi$ in $W^1_2(\Omega)$. If some function ψ_m does not satisfy (2.7) then we can "correct" it in a boundary layer with the help of the distance function. The function belongs to $C^{2+\alpha_0}(\overline{\Omega})$ and the new sequence tends to φ in $W^1_2(\Omega)$. To save place we omit the proof of Proposition 1.

<u>Step 2.</u> Now we study problem (2.1) with the initial condition $u\big|_{t=0} = \varphi_m$, φ_m satisfies (2.7). To apply Theorem 1 we introduce the problem in the nondivergence form, $\hat{u} = u - \varphi_m$:

$$\begin{aligned}
&\hat{u}^k_t - A^{\alpha\beta}_{kl}(x,\hat{u},\hat{u}_x)\hat{u}^l_{x_\beta x_\alpha} + b^k(x,\hat{u},\hat{u}_x) = 0, \quad (x,t)\in Q,\\
&\varkappa^\beta_{kl}(x,\hat{u},\hat{u}_x)\hat{u}^l_{x_\beta} + \hat{g}^k(x,\hat{u})\big|_\Gamma = 0, \quad k = 1,\dots,N,\\
&\hat{u}\big|_{t=0} = 0,
\end{aligned} \tag{2.8}$$

where

$$A^{\alpha\beta}_{kl}(x,\hat u,\hat p)=f_{p^k_\alpha p^l_\beta}(x,\hat u+\varphi_m(x),\hat p+(\varphi_m(x))_x),$$
$$b^k(x,\hat u,\hat p)=f_{u^k}(x,\hat u+\varphi_m(x),\hat p+(\varphi_m(x))_x)-f_{p^k_\alpha u^m}(\dots)(\hat p^m_\alpha+$$
$$+(\varphi_m(x))_{x_\alpha})-f_{p^k_\alpha x_\alpha}(\dots)-f_{p^k_\alpha p^l_\beta}(\dots)\varphi^m_{x_\alpha x_\beta}$$

(by $(\dots)$ we denote the same arguments as function f_{u^k} has);

$$\varkappa^\beta_{kl}(x,\hat u,\hat p)=\int_0^1 f_{p^k_\alpha p^l_\beta}(x,\hat u+\varphi_m(x),(\varphi_m(x))_x+s\hat p)\,ds\cdot\cos(\mathbf{n},x_\alpha),$$
$$\hat g^k(x,\hat u)=g^k(x,\hat u+\varphi_m(x))+f_{p^k_\alpha}(x,\hat u+\varphi_m(x),(\varphi_m(x))_x)\cos(\mathbf{n},x_\alpha).$$

Conditions $\mathbb{D}_1$–$\mathbb{D}_3$ and (2.7) imply the validity of the assumptions of Theorem 1. Thus, for some $T_m>0$ there exists a unique smooth solution $\hat u_m$ to (2.8) in a cylinder $\hat Q_m=\overline{\Omega}\times[0,T_m)$, $\hat u_m\in\mathcal{H}^{2+\alpha_0,1+\alpha_0/2}(\hat Q_m)$. It implies the existence of a solution u_m to problem (2.1). We suppose that T_m defines the maximal interval of the smooth solution.

Step 3. We put $E[u(t)]=\|u_x(\cdot,t)\|^2_{2,\Omega}+\|u(\cdot,t)\|^2_{2,\partial\Omega}$,

$$E[u(t);\Omega_r(x^0)]=\|u_x(\cdot,t)\|^2_{2,\Omega_r(x^0)}+\|u(\cdot,t)\|^2_{2,\gamma_r(x^0)},$$
$$\gamma_r(x^0)=\partial\Omega\cap B_r(x^0).$$

The functions u_m, $m\in\mathcal{N}$, satisfy the following inequalities

$$\begin{aligned}&\|u^m_t\|^2_{2,\Omega\times(0,t)}+E[u_m(t)]\le c_1+c_2E[\varphi_m]\\&\le c_1+\hat c_2E[\varphi]\equiv\mathbf{e}_0,\qquad\forall\,t\in[0,T_m),\end{aligned}\tag{2.9}$$

$$\begin{aligned}&E[u_m(t'');\Omega_R(x^0)]\le c_3\big(R+(t''-t')\big)+c_4E[u_m(t');\Omega_{2R}(x^0)]\\&+\frac{c_5(t''-t')\mathbf{e}_0}{R^2},\quad\forall\,t'\le t''<T_m,\ \forall\,x^0\in\overline{\Omega},\ R<\min\{1,\operatorname{diam}\Omega/2\}.\end{aligned}\tag{2.10}$$

Inequalities (2.9), (2.10) follow from (13), (14) [9]. By Remark 13 [9], the constants $c_1,\dots,c_5$, do not depend on T_m.

Now we fix $R_0>0$ such that

$$E[\varphi;\Omega_{2R_0}(x^0)]<\frac{\varepsilon_0}{8c_4},\quad\forall\,x^0\in\overline{\Omega},\ \text{and}\ R_0<\min\left\{1,\ \frac{\varepsilon_0}{8c_3}\right\},$$

where ε_0 is as in Theorem 1 [9]; it depends on the data from conditions $\mathbb{D}_1$–$\mathbb{D}_3$.

Then there exists some number $m_0 \in \mathcal{N}$ such that

$$E[\varphi_m[\Omega_{2R_0}(x^0)] < \frac{\varepsilon_0}{4c_4}, \quad \forall\, m \geq m_0. \tag{2.11}$$

We put $\hat{T} = \theta R_0^2$, where $\theta < \frac{\varepsilon_0}{4(c_3 + c_5 \mathbf{e}_0)}$ and derive from (2.10) (with $t' = 0$, $t'' = t$, $R = R_0$) and (2.11) the inequality

$$\sup_{0 \leq t \leq \min\{\hat{T}, T_m\}} \sup_{x^0 \in \overline{\Omega}} E[u_m(t); \Omega_{R_0}(x)] < \varepsilon_0, \quad \forall\, m \geq m_0.$$

If $T_m < \hat{T}$ then all assumptions of Theorem 1 [9] are valid and it is possible to extend the solution u_m up to $t = T_m$. This contradicts the definition of T_m. Thus, $T_m > \hat{T} > 0$ and

$$\sup_{[0,\hat{T}]} \sup_{x \in \overline{\Omega}} E[u_m(t), \Omega_{R_0}(x)] < \varepsilon_0. \tag{2.12}$$

All functions $u_m(t)$, $m \geq m_0$, are smooth on $\overline{\Omega} \times [0, \hat{T}]$. According to Lemma 2 and Remark 7 [9], (2.12) guarantees that

$$\|(u_m)_{xx}\|^2_{2,\hat{Q}} \leq c + c_\varphi \left(1 + \hat{T} + \frac{\hat{T}}{R_0^2}\right), \quad \hat{Q} = \Omega \times (0, \hat{T}), \tag{2.13}$$

where the constants c and c_φ are defined by parameters from conditions (2.3)–(2.5) and C^{1+1} characteristics of $\partial\Omega$, c_φ also depends on $\|\varphi\|_{W_2^1(\Omega)}$.

By (2.9), (2.13), it follows that

$$\sup_{[0,\hat{T}]} \|u_m(\cdot, t)\|_{W_2^1(\Omega)} + \|u_m\|_{W_2^{2,1}(\hat{Q})} \leq c, \quad \forall\, m \geq m_0. \tag{2.14}$$

Whence, $u_m \to u$ weakly in $W_2^{2,1}(\hat{Q})$, $(u_m)_x \to u_x$ in $L^2(\hat{Q})$ for some sequence of $m \to +\infty$. The limit function u is a solution to (2.1), $u \in Y(\hat{Q}) = W_2^{2,1}(\hat{Q}) \cap L^\infty((0, \hat{T}), W_2^1(\Omega))$. From Theorem $2'$ [9] it follows that u is a unique solution in this class. Applying Theorem 2 [9], we find that $u \in \mathcal{H}^{2+\alpha_0, 1+\alpha_0/2}(\overline{\Omega} \times (0, \hat{T}])$ and $u_{xt} \in L^{2,2+2\alpha_0}(\Omega \times (\delta, \hat{T}))$, $\forall\, \delta > 0$.

Suppose that $T_1 > \hat{T}$ defines the maximal interval of the existence of smooth solution u. According to Theorem 3 and Remark 12 [9], u admits a smooth extension to the set $\overline{\Omega} \times (0, T_1] \setminus \Sigma_{T_1}$, where the singular set Σ_{T_1} consists of at most a finite number (M_1) points, $\Sigma_{T_1} = \{(x^1, T_1) \cup \ldots \cup (x^{M_1}, T_1)\}$. Analyzing the proof of Theorem 3 [9] and using (2.10) one can easy derive that $M_1 \leq \frac{4c_5 \mathbf{e}_0}{\varepsilon_0}$. Moreover, $u(\cdot, t) \underset{t \to T_1}{\rightarrow} u(\cdot, T_1)$ weakly in $W_2^1(\Omega)$ and in $W^1_{2,loc}(\Omega \backslash \{x^1 \cup \ldots \cup x^{M_1}\})$.

If $h_0 > 0$ in (2.4) or $G = 0$ on $\partial\Omega$ then the dominating constant M_1 does not depend on T_1 (see Remark 13).

Step 4. We denote $\varphi^{(1)}(x) = u(x, T_1) \in W_2^1(\Omega)$ and deduce that

$$\mathcal{E}[\varphi^{(1)}] \le \mathcal{E}[\varphi] - \frac{\nu_0\varepsilon_0}{4c_4} M_1, \tag{2.15}$$

in the same way as in [6]. Now we consider problem (2.1) for $t > T_1$ with initial function $\varphi^{(1)}(x)$. We argue precisely as we did at the previous step. As a result, we deduce the existence of a smooth solution $u^{(1)}(x,t)$ on some interval (T_1, T_2), $u^{(1)}(\cdot,t) \to u^{(1)}(\cdot, T_2)$ weakly in $W_2^1(\Omega)$, $t \to T_2$.

We construct a sequence of intervals $(T_m, T_{m+1}) \subset (0,T)$ and of solutions $u^{(m)}(\cdot,t)$, $m = 0, 1, 2, \ldots$, ($T_0 = 0$, $u^{(0)} = u$, $\varphi^{(0)} = \varphi$). Taking in consideration (2.15), we deduce that

$$\mathcal{E}[\varphi^{(m+1)}] \le \mathcal{E}[\varphi^{(0)}] - \left(\sum_{j=1}^{m+1} M_j\right)\frac{\nu_0\varepsilon_0}{4c_4}$$

and arrive at

$$M = \sum_{j=1}^{m+1} M_j \le \frac{\mathcal{E}[\varphi]\cdot 4c_4}{\nu_0\varepsilon_0} = m_0. \tag{2.16}$$

In the case when $h_0 > 0$ in (2.4) or $G = 0$ on $\partial\Omega$, m_0 in (2.16) does not depend on T.

Joining all the functions $u^{(m)}$, we obtain a solution u (2.1). The solution is smooth on $\hat{\Omega}\times(0,T]$, except of at most finitely many points. Further, $u_t \in L^2(Q)$, $\sup \mathcal{E}[u(t)] \le \mathcal{E}[\varphi]$, $u(\cdot,t) \to \varphi$ weakly in $W_2^1(\Omega)$ (indeed, one can prove that $u(\cdot,t) \to \varphi$ in the norm of $W_2^1(\Omega)$). The uniqueness of u with the mentioned properties follows from Theorem $2'$ [9] when applying the result to each interval $[T_j, T_{j+1})$, $\bigcup_{j=0}^{M} [T_j, T_{j+1}) = [0, T)$.

We have proved the following result.

Theorem 1. *Let conditions $\mathbb{D}_1$–$\mathbb{D}_3$ hold. Then for a fixed number $T > 0$ and any function $\varphi \in W_2^1(\Omega)$ there exists a global solution $u : \Omega\times(0,T) \to \mathbb{R}^N$ to the problem (2.1) such that u is $\mathcal{H}^{2+\alpha_0, 1+\alpha_0/2}$ smooth function in $\overline{\Omega}\times(0,T]\setminus\Sigma$. The singular set Σ consists of at most finitely many points $\{(x^j,t^j)\}_{j=1}^M$. The number M is estimated by the data from assumptions $\mathbb{D}_1$–$\mathbb{D}_3$ and T. If $h_0 \ne 0$ in (2.4) or $G = 0$ on $\partial\Omega$ then M is estimated by the data from $\mathbb{D}_1$–$\mathbb{D}_3$ only.*

Every point $(x^j, t^j) \in \Sigma$ is characterized by the condition

$$\varlimsup_{t\nearrow t^j} \|u_x(\cdot,t)\|^2_{2,\Omega_R(x^j)} \ge \varepsilon_0, \quad \forall\, R > 0,$$

(*number* $\varepsilon_0 > 0$ *is taken from Theorem* 1 [9]). *Furthermore,*

(1) $u \in L^\infty((0,T); W_2^1(\Omega))$, $u_t \in L^2(Q)$, $\sup_{[0,T]} \mathcal{E}[u(t)] \le \mathcal{E}[\varphi]$;
(2) u *is a unique solution with the above properties;*
(3) u *satisfies the integral identity*

$$\int\limits_Q u_t^k h^k + f_{p_\alpha^k}(x,u,u_x) h_{x_\alpha}^k + f_{u^k}(x,u,u_x)h^k)\,dQ$$

$$+ \int\limits_\Gamma g^k(x,u)h^k\,d\Gamma = 0, \quad \forall\, h \in L^2((0,T); W_2^1(\Omega)) \cap L^1((0,T); L^\infty(\Omega)),$$

$$u(\cdot,t) \underset{t\to 0}{\to} \varphi \quad in \quad W_2^1(\Omega).$$

On the behavior of the solution at infinity

Here we suppose that $h_0 \neq 0$ in (2.4) or $G = 0$ on $\partial\Omega$. In this case, the number m_0 in (2.16) does not depend on T. As $T > 0$ was fixed arbitrarily we may discuss the behavior of $u(\cdot,t)$ when $t \to +\infty$.

First, we assume that all singularities in $\overline{\Omega}$ are developed in a finite time interval. Then for some $T > 0$ and $R > 0$ we have the inequality

$$\sup_{t>T} \sup_{x\in\bar\Omega} \|u_x(\cdot,t)\|^2_{2,\Omega_R(x)} < \varepsilon_0.$$

Whence, (see [7, Chapter III]]) along a certain sequence of indices $j \to \infty$ the sequence $u(\cdot,t_j)$ weakly converges in $W_2^2(\Omega)$ to a function $u^\infty \in W_2^2(\Omega)$, $u_t(\cdot,t_j) \to 0$ in $L^2(\Omega)$. By the imbedding theorem, $u_x(\cdot,t_j) \to (u^\infty)_x$ in $L^s(\Omega)$, $s < \infty$. To justify these facts note that for any $t > T$ the following estimates are valid:

$$\int\limits_t^{t+1} \|u_{xx}(\cdot,\tau)\|^2_{2,\Omega} d\tau \le c + c_\varphi \left(1 + \frac{1}{R_0^2}\right); \quad \int\limits_t^{t+1} \|u_t(\cdot,\tau)\|^2_{2,\Omega} d\tau \underset{t\to\infty}{\longrightarrow} 0.$$

Furthermore, $u_{xx}(\cdot,t_j) \underset{t_j\to\infty}{\to} u_{xx}^\infty$ in the $L^2(\Omega)$ norm. To prove this assertion we treat the local setting of (2.1) (see (24) [9]). In such a case, the functions $u(\cdot,t_j), u^\infty$ transform to $v(\cdot,t_j)$, v^∞ in B_1^+. From the integral identity for $v(\cdot,t_j)$ and v^∞ we derive that $\|(v(\cdot,t_j) - v^\infty)_{yy}\|_{2,B_1^+} \to 0$. Returning to the functions $u(\cdot,t)$ and u^∞, we deduce that $\|u_{xx}(\cdot,t_j) - u_{xx}^\infty\|_{2,\Omega} \underset{t_j\to\infty}{\longrightarrow} 0$.

Known results on the smoothness of weak solutions of nonlinear elliptic systems guarantee that $u^\infty \in C^{2+\alpha_0}(\overline{\Omega})$, u^∞ is an extremal point of the functional $\mathcal{E}[u] = \int_\Omega f(x,u,u_x)\,dx + \int_{\partial\Omega} G(x,u)\,ds$.

In particular, if $\mathcal{E}[\varphi] < \varepsilon_0/\nu$ ($\varepsilon_0 > 0$ is defined in Theorem 1 [9] and ν is the constant from (2.3)), then from the monotonicity of $\mathcal{E}[u(t)]$ it follows that $\sup_{[0,\infty)} \|u_x(\cdot,t)\|^2_{2,\Omega} < \varepsilon_0$. In this case Theorem 1 [9] yields that solution $u(\cdot,t)$ to (2.1) is a smooth transformation of φ to an extremal point u^∞ when $t \in (0,\infty]$.

Suppose now that there exist singular points at the infinity. In this case u^∞ is a smooth in $\overline{\Omega} \setminus \{x^1 \cup \ldots \cup x^M\}$ solution to the problem

$$-\frac{d}{dx_\alpha} f_{p^k_\alpha}(x,u,u_x) + f_{u^k}(x,u,u_x) = 0, \quad x \in \Omega,$$

$$f_{p^k_\alpha}(x,u,u_x)\cos(\mathbf{n},x_\alpha) + g^k(x,u)\big|_{x\in\partial\Omega} = 0.$$

According to De Giorgi's lemma [8, Chapter II, Lemma 3.1], u^∞ satisfies the identity

$$\int\limits_\Omega (f_{p^k_\alpha}\eta^k_{x_\alpha} + f_{u^k}\eta^k)\,dx + \int\limits_{\partial\Omega} g^k\eta^k\,ds = 0, \quad \forall\,\eta \in W^1_2(\Omega) \cap L^\infty(\Omega).$$

Concluding, note that the boundedness of the solution constructed was not stated. The estimate $\sup_{[0,T]} \|u_x(\cdot,t)\|_{2,\Omega} \le$ const guarantees only that $\sup_{[0,T)} \|u(\cdot,t)\|_{\mathcal{L}^{2,n}(\Omega)} \le \text{const}$, $n = 2$.

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St.Petersburg State University, Department of Mathematics and Mechanics, Bibliotechnaya pl.2., 198904, St. Petersburg, Petrodvorets, Russia

E-mail: arina@aa1101.spb.edu

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