On a certain converse statement of the Filippov-Ważewski relaxation theorem

Aurelain Cernea

Abstract. A certain converse statement of the Filippov-Ważewski theorem is proved. This result extends to the case of time dependent differential inclusions a previous result of Joó and Tallos in [5] obtained for autonomous differential inclusions.

Keywords: differential inclusion, relaxation property, tangent cone

Classification: 34A60

1. Introduction

A fundamental result in the qualitative theory of differential inclusions and their applications (especially in control theory) is the celebrated Filippov-Ważewski relaxation theorem ([1], [4], [6], etc.). This theorem states that the solution set of a Lipschitzian differential inclusion is dense in the set of relaxed solutions (i.e. the set of solutions of the differential inclusion whose right-hand side is the convex hull of the original multifunction).

Recently, Joó and Tallos ([5]) proved a certain converse of this result. More exactly, given an autonomous differential inclusion with convex valued right-hand side, a smaller set-valued map which essentially yields the same reachable sets is found. The key tool in this approach is a property of the contingent derivative of the reachable set.

As usual in the theory of differential inclusions, important difficulties occur when one passes from autonomous problems to non-autonomous ones. Moreover, the main applications of the relaxation theorem concern non-autonomous differential inclusions.

The aim of this paper is to prove a similar converse statement of the relaxation theorem in the time dependent case. Even if the proof of our result follows the same ideas as in [5], the basic tool in our approach is another property ([4]) of the reachable set of time dependent differential inclusions.

At the same time, we point out that the proof of the main result in [5] may be done using quasitangent (intermediate) derivatives instead of contingent derivatives of the reachable set.

The paper is organized as follows: in Section 2 we present preliminary results to be used in the next section and in Section 3 we prove our main result.

2. Preliminaries

In this section we recall some basic notations and concepts concerning differential inclusions.

Let S be a metric space, $X \subset R^n$ and $x \in cl(X)$ (the closure of X).

Definition 2.1. Let $(K_s)_{s\in S}$ be a family of subsets of a metric space Y. The upper limit and the lower limit of K_s at $s_0 \in S$ are closed sets defined by

$$\operatorname*{Limsup}_{s\to s_0} K_s := \{x \in Y; \quad \liminf_{s\to s_0} d(x, K_s) = 0\},$$
$$\operatorname*{Liminf}_{s\to s_0} K_s := \{x \in Y; \quad \lim_{s\to s_0} d(x, K_s) = 0\}.$$

A subset K is said to be the limit or the set limit of the sequence K_s if:

$$K = \operatorname*{Limsup}_{s\to s_0} K_s = \operatorname*{Liminf}_{s\to s_0} K_s =: \operatorname*{Lim}_{s\to s_0} K_s.$$

From the multitude of the tangent cones in the literature (e.g. [2], etc.) we recall only the contingent and the quasitangent (intermediate) cone, defined, respectively by:

$$K_xX = \{v \in R^n; \quad \exists s_m \to 0+,\ \exists v_m \to v:\ x + s_m v_m \in X\},$$

$$Q_xX = \{v \in R^n; \quad \forall\, s_m \to 0+,\ \exists v_m \to v:\ x + s_m v_m \in X\}.$$

These cones are related as follows: $Q_xX \subset K_xX$.

Corresponding to each type of tangent cone, say τ_xX, one may introduce a *set-valued directional derivative* of a multifunction $G(\cdot) : X \subset R^n \to \mathcal{P}(R^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in \mathrm{Graph}(G)$ as follows

$$\tau_yG(x; v) = \{w \in R^n; : (v, w) \in \tau_{(x,y)}\ \mathrm{Graph}(G)\},\ v \in \tau_xX.$$

Consider $T > 0$, $I = [0, T]$, a set-valued map $F(\cdot, \cdot) : I \times R^n \to R^n$ and let $t_0 \in I, x_0 \in R^n, \Omega \subset R^n$ be a nonempty open set. We denote by $S_F(t_0, x_0)$ the set of all absolutely continuous functions that are solutions of the differential inclusion:

$$x' \in F(t, x), \quad x(t_0) = x_0. \tag{2.1}$$

For all $0 \le t_0 \le t_1 \le T$ and $\xi \in R^n$ set

$$R_F(t_1, t_0, \xi) = \{x(t_1); \quad x(\cdot) \in S_F(t_0, \xi)\} \tag{2.2}$$

the reachable set of the inclusion (2.1) from (t_0, ξ) at time t_0.

Let B be the unit ball in R^n.

In what follows we assume the following:

Hypothesis 2.2. (i) $\forall (t,x) \in I \times R^n$ $F(t,x)$ *is closed.*
(ii) $\forall x \in R^n$ $F(\cdot, x)$ *is a measurable set-valued map.*
(iii) $\forall (t,x) \in I \times \Omega$ $F(t,x)$ *is nonempty.*
(iv) $F(t,\cdot)$ *is Lipschitz on* Ω, *i.e.* $\exists L > 0$ *such that* $\forall t \in I, \forall x, y \in \Omega$ $F(t,x) \subset F(t,y) + L\|x-y\|B$.

The set-valued map R_F enjoys the semigroup properties:

$$\forall\, 0 \le t_1 \le t_2 \le t_2 \le T, \quad \forall \xi \in R^n \quad R_F(t_3, t_2, R_F(t_2, t_1, \xi)) = R_F(t_3, t_1, \xi)$$
$$\forall\, 0 \le t \le T, \quad \forall \xi \in R^n \quad R_F(t,t,\xi) = \xi.$$

By $\overline{co}F(\cdot,\cdot)$ we denote the set-valued map whose values are the closed convex hulls of the values of $F(\cdot,\cdot)$ at every point. Let us note that if $F(\cdot,\cdot)$ is upper semicontinuous (resp. continuous, locally Lipschitz) then so is $\overline{co}F(\cdot,\cdot)$ (e.g. [1]).

When $F(\cdot,\cdot)$ does not depend on the first variable (2.1) reduces to

$$x' \in F(x), \quad x(0) = x_0. \tag{2.3}$$

We denote by $S_F(x_0)$ the solution set of the differential inclusion (2.3). In this case, the following result is proved in [5]:

Theorem 2.3 ([5]). *Assume that* $F(\cdot)$ *is locally Lipschitz on* Ω *with convex compact values and consider a compact valued locally Lipschitz set-valued map* $G(\cdot)$ *with the same Lipschitz constant such that* $G(x) \subset F(x)$ *for every* $x \in \Omega$. *Then* $S_G(x)$ *is dense in* $S_F(x)$ *(in the metric of uniform convergence) for all* $x \in \Omega$ *if and only if* $\overline{co}G(x) = F(x)$ *for each* $x \in \Omega$.

The proof is essentially based on the following property of the contingent derivatives of the set valued map $t \to R_F(t, 0, x)$:

Theorem 2.4 ([5]). *If* $F(\cdot)$ *is continuous on* Ω, *then*

$$F(x) \subset K_x R_F(\cdot, 0, x)(0; 1) \subset \overline{co}F(x) \quad \forall x \in \Omega. \tag{2.4}$$

Remark 2.5. According to Theorem 2.5 in [3], if $F(\cdot)$ is continuous on Ω, then

$$F(x) \subset Q_x R_F(\cdot, 0, x)(0; 1) \subset coF(x) \quad \forall x \in \Omega. \tag{2.5}$$

Obviously, the first inclusion in (2.5) is stronger than the corresponding one in (2.4).

Let us note that the proof of Theorem 2.3 may be performed through the same arguments employed in [5], but using the quasitangent derivative of the reachable set and the property in (2.5) instead of the contingent derivative and Theorem 2.4.

3. The main result

In order to establish a corresponding relaxation result for nonautonomous differential inclusion (2.1) one needs the following property, which states that when $F(\cdot,\cdot)$ is sufficiently regular, the set-valued map $\overline{co}F(\cdot,\cdot)$ is the infinitesimal generator of the semigroup $R_F(\cdot,\cdot)$ in the sense that the difference quotions $(R_F(t+h,t,x)-x)/h$ converge to $\overline{co}F(t,x)$.

Theorem 3.1 ([4])**.** *Assume that Hypothesis 2.2 holds true and let $t_0 \in [0,T)$ and $x_0 \in \Omega$.*

If $F(\cdot,\cdot)$ is lower semicontinuous at (t_0,x_0), then

$$\overline{co}F(t_0,x_0) \subset \operatorname*{Liminf}_{h\to 0+} \frac{R_F(t_0+h,t_0,x_0)-x_0}{h}. \tag{3.1}$$

If $F(\cdot,\cdot)$ is upper semicontinuous at (t_0,x_0) and $F(t_0,x_0)$ is bounded, then

$$\operatorname*{Limsup}_{h\to 0+} \frac{R_F(t_0+h,t_0,x_0)-x_0}{h} \subset \overline{co}F(t_0,x_0). \tag{3.2}$$

Consequently, if $F(\cdot,\cdot)$ is continuous at (t_0,x_0) and $F(t_0,x_0)$ is bounded, then

$$\operatorname*{Lim}_{h\to 0+} \frac{R_F(t_0+h,t_0,x_0)-x_0}{h} = \overline{co}F(t_0,x_0).$$

We show first that if a continuous convex set-valued map is given, that any smaller upper semicontinuous map that essentially retains the same reachable sets, necessarily contains all extremal points of the convex set-valued map.

Theorem 3.2. *Assume that $F(\cdot,\cdot): I\times\Omega \to \mathcal{P}(R^n)$ is continuous with convex compact values and consider an upper semicontinuous compact valued set-valued map $G(\cdot,\cdot): I\times\Omega \to \mathcal{P}(R^n)$ such that $G(t,x)\subset F(t,x)\quad \forall\,(t,x)\in I\times\Omega$. Suppose that $R_G(\tau,t,x)$ is dense in $R_F(\tau,t,x)\quad \forall\, t,\tau\in I$, $\forall\, x\in\Omega$.*

Then $\overline{co}G(t,x) = F(t,x)$ for every $(t,x)\in I\times\Omega$.

Proof: It is an easy consequence of the definition of the upper limit that if $R_G(\tau,t,x)$ is dense in $R_F(\tau,t,x)$ then

$$\operatorname*{Limsup}_{h\to 0+} \frac{R_F(t+h,t,x)-x}{h} = \operatorname*{Limsup}_{h\to 0+} \frac{R_G(t+h,t,x)-x}{h}. \tag{3.3}$$

Indeed, one has $v \in \operatorname{Limsup}_{h\to 0+} \frac{R_F(t+h,t,x)-x}{h}$ if and only if $v \in K_x R_F((t,t,x);(1,0,0))$ if and only if $((1,0,0),v) \in K_{((t,t,x),x)}\operatorname{Graph} R_F(\cdot,\cdot,\cdot)$ and it is well known that for any $X\subset R^n$ and any $x\in X$, $K_x X = K_x \overline{X}$.

By applying Theorem 3.1 and (3.3) we obtain for $(t,x)\in[0,t)\times\Omega$

$$F(t,x) = \operatorname*{Lim}_{h\to 0+} \frac{R_F(t+h,t,x)-x}{h} = \operatorname*{Limsup}_{h\to 0+} \frac{R_F(t+h,t,x)-x}{h}$$

$$= \operatorname*{Limsup}_{h\to 0+} \frac{R_G(t+h,t,x)-x}{h} \subset \overline{co}G(t,x)$$

and this proves the theorem. □

In what follows the density of the solution set will be understood with respect to the norm in the Banach space $C(I,R^n)$ of continuous functions.

Theorem 3.3. *Assume that Hypothesis 2.2 is satisfied and $F(\cdot,\cdot) : I\times\Omega \to \mathcal{P}(R^n)$ is continuous with convex compact values. Consider an upper semicontinuous compact valued set-valued map $G(\cdot,\cdot) : I\times\Omega \to \mathcal{P}(R^n)$ such that $G(t,\cdot)$ is locally Lipschitz with the same Lipschitz constant as $F(t,\cdot)$ and $G(t,x)\subset F(t,x)$ $\forall\,(t,x)\in I\times\Omega$.*

Then $S_G(t,x)$ is dense in $S_F(t,x)$ $\forall\, t\in I$, $x\in\Omega$ if and only if $\overline{co}G(t,x) = F(t,x)$ for every $(t,x)\in I\times\Omega$.

Proof: The sufficiency is the classical relaxation theorem. Let us note that this statement is valid without the assumptions that $F(\cdot,\cdot)$ is continuous and $G(\cdot,\cdot)$ is upper semicontinuous.

For the necessity, since $S_G(t,x)$ and $S_F(t,x)$ have the same closure with respect to the norm of $C(I,R^n)$ then so do $R_G(\tau,t)(x)$ and $R_F(\tau,t)(x)$ in the norm of R^n $\forall\, t,\tau\in I, \forall\, x\in\Omega$, hence we apply Theorem 3.2 and the proof is complete. □

Remark 3.4. Obviously, a key tool in the proof of Theorem 3.2 is Theorem 3.1.

Theorem 3.1 (Theorem 2.2.11 in [4]) remains valid under the more general assumption that $F(t,\cdot)$ is $L(t)$-Lipschitz, with $L(\cdot)\in L^1(I,R)$ (more exactly, the proof of Theorem 3.1 can be adapted using the absolute continuity of the Lebesgue integral).

So, Hypothesis 2.2(iv) can be improved by assuming that there exists $L(\cdot)\in L^1(I,R)$ such that $\forall\, t\in I$, $F(t,\cdot)$ is $L(t)$-Lipschitz on Ω.

Acknowledgments. The author would like to thank to an anonymous referee for his helpful comments which improved the presentation of the paper.

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Faculty of Mathematics, University of Bucharest, Academiei 14, 70109 Bucharest, Romania

(*Received* January 18, 2000)