A class of pairs of weights related to the boundedness of the Fractional Integral Operator between L^p and Lipschitz spaces

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Abstract. In [P] we characterize the pairs of weights for which the fractional integral operator I_γ of order γ from a weighted Lebesgue space into a suitable weighted BMO and Lipschitz integral space is bounded.

In this paper we consider other weighted Lipschitz integral spaces that contain those defined in [P], and we obtain results on pairs of weights related to the boundedness of I_γ acting from weighted Lebesgue spaces into these spaces. Also, we study the properties of those classes of weights and compare them with the classes given in [P]. Then, under additional assumptions on the weights, we obtain necessary and sufficient conditions for the boundedness of I_γ between BMO and Lipschitz integral spaces. For the boundedness between Lipschitz integral spaces we obtain sufficient conditions.

Keywords: two-weighted inequalities, fractional integral, weighted Lebesgue spaces, weighted Lipschitz spaces, weighted BMO spaces.

Classification: Primary 42B25

1. Introduction and preliminary notation

In harmonic analysis, a question of considerable interest that arises in connection with the theory of partial differential equations, is to determine the classes of weights related to the boundedness of certain operators between weighted spaces. This type of problem was studied by several authors, see [CF], [HL], [MW1], [MW2], [S], [SWe] and others. For example, in [MW1], B. Muckenhoupt and R. Wheeden proved that the fractional integral of order γ, $0 < \gamma < n$, defined by

$$I_\gamma f(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{\gamma-n}\,dy \tag{1.1}$$

satisfies the inequality

$$\|v^{-1}\chi_B\|_\infty \frac{1}{|B|}\int_B \left|I_\gamma f(x) - m_B\left(I_\gamma f\right)\right|\,dx \le C\|f/v\|_{n/\gamma}, \tag{1.2}$$

The author was supported by Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.

if and only if $v^{(n/\gamma)'} \in A_1$, where $m_B f = (1/|B|) \int_B f$. This inequality may be viewed as the boundedness of I_γ from $L_v^{n/\gamma}$ into a weighted version of the space of functions with bounded mean oscillation.

In the unweighted case, it is well known that I_γ is a bounded linear operator from BMO, the space of the function with bounded mean oscillation, into the classical Lipschitz spaces $\Lambda(\gamma/n)$. See for example [Pe].

In 1997, Harboure, Salinas and Viviani in [HSV], gave necessary and sufficient conditions on the weights for the boundedness of the fractional integral operator I_γ from weighted strong and weak L^p spaces within the range $p \geq n/\gamma$ into weighted versions of BMO and Lipschitz integral spaces. Under an additional assumption on the weight, they also obtain necessary and sufficient conditions for the boundedness between weighted Lipschitz spaces.

An extension to the case of two weights can be found in [P], where the author characterizes the pairs of weights for which I_γ is bounded from weighted Lebesgue spaces L_v^p into a weighted version of BMO and Lipschitz integral spaces of parameter δ, with a weight w, called $\mathcal{L}_w(\delta)$ spaces, defined as the locally integrable functions f such that for every ball $B \subset \mathbb{R}^n$ the inequality

$$\frac{\|(1/w)\chi_B\|_\infty}{|B|^{1+\delta/n}} \int_B |f(x) - m_B f| \, dx \leq C \tag{1.3}$$

holds. For $\delta = 0$, this space coincides with that one of the weighted bounded mean oscillation spaces introduced in [MW2]. The case $w = 1$ gives the known Lipschitz integral spaces for $0 < \delta < 1$, and the Morrey spaces given in [Pe], for $-n < \delta < 0$. The work includes a study of the properties of the classes of weights that arise in connection with the boundedness of I_γ.

Our aim in this work is to give a two weighted characterization for the boundedness of the fractional integral operator I_γ, $0 < \gamma < n$, generalizing the one-weighted results obtained in [HSV]. More precisely, we characterize the pairs of weights for which I_γ is bounded from weighted Lebesgue spaces L_v^p into a weighted version of Lipschitz integral spaces that contain those defined in [P]. Then, we give necessary and sufficient conditions for the boundedness of I_γ between weighted BMO and Lipschitz integral spaces. For the boundedness between Lipschitz spaces we obtain sufficient conditions. We also deal with the classes of pairs of weights that arise from these conditions and we determine their properties.

We shall give the basic notation used through this paper. As usual, we say that w is a weight if it is a nonnegative locally integrable function defined on $\mathbb{R}^n$. We also say that w satisfies the doubling condition if there exists a constant C such that the inequality

$$0 < w(2B) \leq C w(B) < \infty$$

holds for every ball $B \subset \mathbb{R}^n$. For a measurable set $E \subset \mathbb{R}^n$, we denote $w(E) = \int_E w(x)\, dx$. The open ball centered at x_B with radius R will be denoted by

$B(x_B, R)$ and θB will mean $B(x_B, \theta R)$. By L^p we mean the usual strong Lebesgue space on $\mathbb{R}^n$, and we denote by $\|\cdot\|_p$, the corresponding norm, that is

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.$$

Finally, we denote by L^p_w the class of functions f such that $f/w \in L^p$.

Section 2 of this paper contains the basic properties of the spaces that we are going to consider and the relations to those defined in [P]. In Section 3 we introduce the classes of pairs of weights related to the boundedness of I_γ between weighted Lebesgue spaces and the spaces given in Section 2. The properties of such classes of weights are given in Section 5. The proofs of the main results of this paper can be found in Section 4.

2. On the weighted Lipschitz integral spaces $\mathbb{L}_w(\delta)$

In this section we shall introduce the Lipschitz integral spaces that we are going to consider in our work.

2.1 Definition. *Let w be a weight and $\delta \in \mathbb{R}$. We say that a locally integrable function f belongs to $\mathbb{L}_w(\delta)$ if there exists a constant C such that the inequality*

$$\frac{1}{w(B)\,|B|^{\delta/n}} \int_B |f(x) - m_B f| \, dx \le C \tag{2.2}$$

holds for every ball $B \subset \mathbb{R}^n$. The least constant C with this property will be denoted by $|||f|||_{\mathbb{L}_w(\delta)}$.

It can be seen that, for each δ, γ and p, the space $\mathcal{L}_w(\delta)$ defined in [P] as the set of locally integrable functions f such that for every ball $B \subset \mathbb{R}^n$ the inequality

$$\frac{\|(1/w)\chi_B\|_\infty}{|B|^{1+\delta/n}} \int_B |f(x) - m_B f| \, dx \le C \tag{2.3}$$

holds, is contained in $\mathbb{L}_w(\delta)$. Moreover, if $\delta = 0$, the space $\mathbb{L}_w(\delta)$ coincides (as $\mathcal{L}_w(\delta)$) with one of the weighted bounded mean oscillation spaces, introduced by Muckenhoupt and Wheeden in [MW2], and for $-n < \delta < 1$, this definition agrees with one of the versions given in [HSV]. For the case $w = 1$, $\mathbb{L}_w(\delta)$ is the known Lipschitz integral space for $0 < \delta < 1$, and the Morrey space for $-n < \delta < 0$.

Now we shall establish the relation between $\mathbb{L}_w(\delta)$, the space $\mathcal{L}_w(\delta)$ defined in [P] and certain pointwise version of Lipschitz spaces.

2.4 Proposition. *Let $\delta \in \mathbb{R}$ and let w be a weight, then*

(2.5) *The space $\mathcal{L}_w(\delta)$ is contained in the space $\mathbb{L}_w(\delta)$. Moreover, if $w \in A_1$, then both spaces coincide.*

(2.6) *Let $\delta > 0$. If w satisfies the doubling condition, then the space $\mathbb{L}_w(\delta)$ coincides with the pointwise version $\Delta^w(\delta)$ consisting of all the functions f such that there exists a constant C satisfying*

$$(2.7)\quad |f(x) - f(y)| \le C\Big(\int\limits_{B(x,2|x-y|)} \frac{w(z)}{|z-x|^{n-\delta}}\, dz + \int\limits_{B(y,2|x-y|)} \frac{w(z)}{|z-y|^{n-\delta}}\, dz \Big)$$

for almost every x and y in $\mathbb{R}^n$.

Proof: Let us show first (2.5). From the inequality

$$\inf_B w \le \frac{w(B)}{|B|}$$

it is clear that $\mathcal{L}_w(\delta) \subset \mathbb{L}_w(\delta)$. The other inclusion is also immediate by our assumption that $w \in A_1$.

In order to prove (2.6), we first check (2.7) for $f \in \mathbb{L}_w(\delta)$. Given x and y in $\mathbb{R}^n$ Lebesgue points of f, $x \ne y$, take $B = B(x, |x-y|)$ and $B' = B(y, |x-y|)$. Then

$$|f(x) - f(y)| \le |f(x) - m_B f| + |f(y) - m_{B'} f| + |m_{B'} f - m_B f|.$$

We estimate only the first term of the right side. The estimates for the other terms are similar. Letting $B_i = 2^{-i}B$, $i \ge 0$, we get from the assumption

$$\begin{aligned}
|f(x) - m_B f| &\le \lim_{k\to\infty} \Big(\big|f(x) - m_{B_k} f\big| + \sum_{i=0}^{k-1} \big|m_{B_{i+1}} f - m_{B_i} f\big| \Big) \\
&\le C \sum_{i=0}^{\infty} |B_i|^{-1} \int\limits_{B_i} \big|f(z) - m_{B_i} f\big|\, dz \\
&\le C\, |||f|||_{\mathbb{L}_w(\delta)} \sum_{i=0}^{\infty} |B_i|^{\delta/n-1}\, w(B_i) \\
&\le C\, |||f|||_{\mathbb{L}_w(\delta)} \sum_{i=0}^{\infty} \int\limits_{B_i - B_{i+1}} \frac{w(z)}{|z-x|^{n-\delta}}\, dz \\
&\le C\, |||f|||_{\mathbb{L}_w(\delta)} \int\limits_{B(x,2\rho)} \frac{w(z)}{|z-x|^{n-\delta}}\, dz
\end{aligned}$$

for almost all $x \in \mathbb{R}^n$. Then (2.7) follows.

Conversely, integrating (2.7) over a ball B with respect to both variables, x and y, and changing the order of integration, we obtain that f belongs to $\mathbb{L}_w(\delta)$. Thus (2.6) is proved. □

3. Statement of the main results

First, we introduce the classes of pairs of weights that we are going to consider.

3.1 Definition. *Let $0 < \gamma < n$, $\delta \in \mathbb{R}$ and $1 < p \leq \infty$. We say that a pair of weights (w, v) belongs to $\mathbb{H}(p, \gamma, \delta)$, if there exists a constant C such that*

$$\frac{|B|^{1+(1-\delta)/n}}{w(B)} \left(\int_{\mathbb{R}^n} \frac{v^{p'}(y)}{\left(|B|^{1/n} + |x_B - y|\right)^{p'(n-\gamma+1)}} dy \right)^{1/p'} \leq C \tag{3.2}$$

holds for every ball $B \subset \mathbb{R}^n$, where x_B is the center of B. In the case $p = 1$, (3.2) should be understood as

$$\frac{|B|^{1+(1-\delta)/n}}{w(B)} \left\| \frac{v}{\left(|B|^{1/n} + |x_B - \cdot|\right)^{(n-\gamma+1)}} \right\|_{\infty} \leq C.$$

3.3 Remark. Keeping in mind that

$$\|(1/w)\chi_B\|_{\infty} = \frac{1}{\inf\limits_{x \in B} w} \geq \frac{|B|}{w(B)}$$

it is easy to check that the classes $\mathcal{H}(p, \gamma, \delta)$ defined in [P] are contained in the classes $\mathbb{H}(p, \gamma, \delta)$ given in the above definition. However, the reciprocal inclusion is not valid. We postpone the proof of this assertion to Section 5, where we shall study the properties of the classes $\mathbb{H}(p, \gamma, \delta)$.

Also, if $w = v$ and $\delta = \gamma - n/p$, it can be seen that the classes $\mathbb{H}(p, \gamma, \delta)$ coincide with the classes $H(p, \gamma)$ defined in [HSV].

Now we state the results on the boundedness of the operator I_γ involving the spaces $\mathbb{L}_w(\delta)$ and the corresponding classes $\mathbb{H}(p, \gamma, \delta)$.

3.4 Theorem. *Let $0 < \gamma < n$, $1 \leq p \leq \infty$, $\delta \in \mathbb{R}$ and let (w, v) be a pair of weights. The following statements are equivalent:*

(3.5) *The operator I_γ is a bounded linear operator from L^p_v into $\mathbb{L}_w(\delta)$.*

(3.6) *The pair (w, v) belongs to $\mathbb{H}(p, \gamma, \delta)$.*

In the following theorems we state results of boundedness of I_γ acting from suitable BMO and Lipschitz integral spaces into Lipschitz integral spaces. More precisely, under additional assumption on the weights, we obtain necessary and sufficient conditions for the boundedness of I_γ between BMO and Lipschitz integral spaces. On the other hand, we obtain sufficient conditions for the boundedness of I_γ between Lipschitz integral spaces. For the one weight case, similar results have been established in [HSV]. Our results are contained in the following theorems and the proofs are in Section 4.

3.7 Theorem. *Let $0 < \gamma < 1$ and (w, v) be a pair of weights. Then*

(3.8) *the condition $\mathbb{H}(\infty, \gamma, \gamma)$ is necessary for the boundedness of the operator I_γ from $\mathbb{L}_v(0)$ into $\mathbb{L}_w(\gamma)$;*

(3.9) *if w and v satisfy the doubling condition, and (w, v) belongs to $\mathbb{H}(\infty, \gamma, \gamma)$, then I_γ is a bounded linear operator from $\mathbb{L}_v(0)$ into $\mathbb{L}_w(\gamma)$.*

3.10 Theorem. *Let $\gamma > 0$ and $\delta \geq 0$ be such that $0 < \gamma + \delta < 1$, and let (w, v) be a pair of weights that satisfy the doubling condition and such that (w, v) belongs to $\mathbb{H}(\infty, \gamma + \delta, \gamma + \delta)$. Then, the operator I_γ is bounded from $\mathbb{L}_v(\delta)$ into $\mathbb{L}_w(\gamma + \delta)$.*

We note that Theorem 3.7 generalizes the classical unweighted results on the boundedness of I_γ between BMO and Lipschitz spaces $\Lambda(\alpha)$. For the one weight case, E. Harboure, O. Salinas and B. Viviani prove that the spaces $\mathbb{L}_w(\delta)$, $0 < \delta < 1$ coincide with the pointwise versions given in Proposition 2.4 because the weight in the classes they obtain satisfies the doubling condition.

4. Proof of the main results

Now we will restrict our attention to the boundedness of I_γ from BMO and Lipschitz integral spaces into Lipschitz integral spaces, the proof of Theorem 3.4 follows similar lines as in [P, Theorem 3.5] and we omit it. First, we shall consider the following expression for the operator I_γ (since the usual definition, i.e. (1.1), is not good to deal with $\mathbb{L}_w(\delta)$ spaces because of convergence problems, as can be seen in related classical results)

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \left(\frac{1}{|x_0 - y|^{n-\gamma}} - \frac{1}{|x - y|^{n-\gamma}} \right) f(y)\, dy, \tag{4.1}$$

where $x_0 \in \mathbb{R}^n$ is chosen adequately. It can be proved that, if both integrals (1.1) and (4.1) converge, then differ by a constant.

The next lemma was proved in [HSV] and we omit its proof here.

4.2 Lemma. *Let $\alpha \in \mathbb{R}^+$ and $\delta \geq 0$ be such that $0 < \alpha + \delta < 1$. Let v be a weight satisfying the doubling condition. Then there exists a constant C such that the inequality*

$$\int_{\mathbb{R}^n - B} \frac{|f(y) - m_B f|}{|x_B - y|^{n+1-\alpha}}\, dy \leq C |||f|||_{\mathbb{L}_v(\delta)} \int_{\mathbb{R}^n - B} \frac{v(y)}{|x_B - y|^{n+1-\alpha-\delta}}\, dy$$

holds for every $f \in \mathbb{L}_v(\delta)$ and every $B = B(x_B, R) \subset \mathbb{R}^n$.

Now, we prove the finiteness of (4.1) for every $f \in \mathbb{L}_v(\delta)$.

4.3 Lemma. *Given $\gamma > 0$ and $\delta \geq 0$ such that $0 < \gamma + \delta < 1$, let (w, v) be a pair of weights belonging to $\mathbb{H}(\infty, \gamma+\delta, \gamma+\delta)$, with v satisfying the doubling condition. If $x_0 \in \mathbb{R}^n$ is a point such that, for all $R \in \mathbb{R}^+$*

$$\int\limits_{B(x_0,R)} \frac{v(y)}{|x_0 - y|^{n-(\gamma+\delta)}}\, dy < \infty \qquad \text{and} \qquad \int\limits_{B(x_0,R)} \frac{w(y)}{|x_0 - y|^{n-(\gamma+\delta)}}\, dy < \infty$$

hold, and $f \in \mathbb{L}_v(\delta)$, then (4.1) is finite for almost every $x \in \mathbb{R}^n$.

Proof: Since, for every $v \in L_{loc}(\mathbb{R}^n)$, $R > 0$ and $m \in \mathbb{N}$

$$\begin{aligned}\int\limits_{B(0,m)} \Bigg(\int\limits_{B(x,R)} \frac{v(y)}{|x-y|^{n-(\gamma+\delta)}}\, dy\, dx \Bigg) &\leq \int\limits_{B(0,R+m)} v(y) \Bigg(\int\limits_{B(0,m)} \frac{dx}{|x-y|^{n-(\gamma+\delta)}} \Bigg) dy \\ &\leq C(m,\alpha) \int\limits_{B(0,R+m)} v(y)\, dy < \infty,\end{aligned}$$

we can choose x_0 and $x \in \mathbb{R}^n$, with $x \neq x_0$ as in the hypotheses of the lemma. Then we take $B = B(x_0, |x - x_0|)$. Since the expression in parentheses of (4.1) has zero integral over $\mathbb{R}^n$ as a function of y, we have

$$\begin{aligned}&\int\limits_{\mathbb{R}^n} \left(\frac{1}{|x_0 - y|^{n-\gamma}} - \frac{1}{|x-y|^{n-\gamma}} \right) f(y)\, dy \\ &\qquad = \int\limits_{\mathbb{R}^n} \left(\frac{1}{|x_0 - y|^{n-\gamma}} - \frac{1}{|x-y|^{n-\gamma}} \right) (f(y) - m_B f)\, dy \\ &\qquad = I_1(x) + I_2(x),\end{aligned} \tag{4.4}$$

where I_1 is the integral over the ball B and I_2 is the integral over the complement of B.

Let us first estimate I_1. Setting $\tilde{B} = B(x, 2|x-x_0|)$, we have

$$\begin{aligned}|I_1(x)| &\leq \int\limits_B \frac{|f(y) - m_B f|}{|x_0 - y|^{n-\gamma}}\, dy + \int\limits_{\tilde{B}} \frac{|f(y) - m_B f|}{|x-y|^{n-\gamma}}\, dy \\ &\leq \int\limits_B \frac{|f(y) - m_B f|}{|x_0 - y|^{n-\gamma}}\, dy + \int\limits_{\tilde{B}} \frac{|f(y) - m_{\tilde{B}} f|}{|x-y|^{n-\gamma}}\, dy \\ &\quad + |||f|||_{\mathbb{L}_v(\delta)}\, v(\tilde{B})\, |B|^{\frac{\gamma+\delta}{n} - 1}.\end{aligned}$$

Both integrals can be estimated in the same way, so we do only the first one. Thus, denoting $B_k = 2^{-k}B$, $k \in \mathbb{N}$, we get

$$\begin{aligned}
\int_B \frac{|f(y) - m_B f|}{|x_0 - y|^{n-\gamma}}\, dy &\le C\,|B|^{\frac{\gamma}{n}} \sum_{k=0}^{\infty} 2^{-k\gamma}\,|B_k|^{-1} \int_{B_k - B_{k+1}} |f(y) - m_B f|\, dy \\
&\le C\,|B|^{\frac{\gamma}{n}} \sum_{k=0}^{\infty} 2^{-k\gamma} \sum_{j=0}^{k} |B_j|^{-1} \int_{B_j} \left| f(y) - m_{B_j} f \right| dy \\
&\le C\|\!|f|\!\|_{\mathbb{L}_v(\delta)}\,|B|^{\frac{\gamma}{n}} \sum_{k=0}^{\infty} 2^{-k\gamma} \sum_{j=0}^{k} |B_j|^{\delta/n-1}\, v(B_j) \\
&\le C\|\!|f|\!\|_{\mathbb{L}_v(\delta)}\,|B|^{\frac{\gamma+\delta}{n}-1} \sum_{j=0}^{\infty} 2^{j(n-\delta)} v(B_j) \sum_{k=j}^{\infty} 2^{-k\gamma} \\
&\le C\|\!|f|\!\|_{\mathbb{L}_v(\delta)}\,|B|^{\frac{\gamma+\delta}{n}-1} \sum_{j=0}^{\infty} 2^{j(n-\gamma-\delta)} v(B_j) \\
&\le C\|\!|f|\!\|_{\mathbb{L}_v(\delta)}\,|B|^{\frac{\gamma+\delta}{n}-1} \sum_{j=0}^{\infty} 2^{j(n-\gamma-\delta)} v(B_j - B_{j+1}) \\
&\le C\|\!|f|\!\|_{\mathbb{L}_v(\delta)} \int_B \frac{v(y)}{|x_0 - y|^{n-\gamma-\delta}}\, dy.
\end{aligned}$$

Therefore

$$|I_1(x)| \le C\|\!|f|\!\|_{\mathbb{L}_v(\delta)} \left(\int_B \frac{v(y)}{|x_0 - y|^{n-(\gamma+\delta)}}\, dy + \int_{\tilde{B}} \frac{v(y)}{|x - y|^{n-(\gamma+\delta)}}\, dy \right). \tag{4.5}$$

Next, let us estimate I_2. Applying Lemma 4.2 with γ and δ and the fact that $(w, v) \in \mathbb{H}(\infty, \gamma + \delta, \gamma + \delta)$ we get

$$\begin{aligned}
|I_2(x)| &\le \int_{\mathbb{R}^n - B} \left| \frac{1}{|x_0 - y|^{n-\gamma}} - \frac{1}{|x - y|^{n-\gamma}} \right| |f(y) - m_B f|\, dy \\
&\le C\,|B|^{1/n} \int_{\mathbb{R}^n - B} \frac{|f(y) - m_B f|}{|x_0 - y|^{n-\gamma+1}}\, dy \\
&\le C\|\!|f|\!\|_{\mathbb{L}_v(\delta)}\,|B|^{1/n} \int_{\mathbb{R}^n - B} \frac{v(y)}{|x_0 - y|^{n-\gamma-\delta+1}}\, dy \qquad (4.6) \\
&\le C\|\!|f|\!\|_{\mathbb{L}_v(\delta)}\,|B|^{(\gamma+\delta)/n-1}\, w(B) \\
&\le C\|\!|f|\!\|_{\mathbb{L}_v(\delta)} \int_B \frac{w(y)}{|x_0 - y|^{n-(\gamma+\delta)}}\, dy.
\end{aligned}$$

Then, it follows from the assumptions that I_2 is finite almost everywhere. Finally, combining (4.5) and (4.6) we get the lemma. □

Now, we are going to prove the theorems that involve the boundedness of I_γ between Lipschitz spaces, that is, Theorems 3.7 and 3.10.

Proof of Theorem 3.7: Let us first see (3.9). In order to prove the boundedness of I_γ we note that, by Proposition 2.4 it is enough to get a pointwise estimate as in (2.7) for I_γ instead of f. Given x_1 and x_2 in $\mathbb{R}^n$ with $x_1 \neq x_2$ let $B = B\left(x_1, |x_1 - x_2|\right)$. Since the kernel of I_γ has zero integral over $\mathbb{R}^n$, we have

$$\begin{aligned}\left|I_\gamma f\left(x_1\right) - I_\gamma f\left(x_2\right)\right| &\leq \int_{\mathbb{R}^n} \left|\frac{1}{\left|x_1 - y\right|^{n-\gamma}} - \frac{1}{\left|x_2 - y\right|^{n-\gamma}}\right| \left|f\left(y\right) - m_B f\right| \, dy \\ &= I_1 + I_2,\end{aligned}$$

where I_1 is the integral over B and I_2 is the integral over $\mathbb{R}^n \backslash B$. Thus, with arguments similar to the one used for (4.5) and (4.6), we get

$$\begin{aligned}&\left|I_\gamma f\left(x_1\right) - I_\gamma f\left(x_2\right)\right| \\ &\quad \leq C|||f|||_{\mathbb{L}_v(0)} \Bigg(\int_{B(x_1, 2|x_1 - x_2|)} \frac{w(z)}{\left|z - x_1\right|^{n-\delta}} \, dz + \int_{B(x_2, 2|x_1 - x_2|)} \frac{w(z)}{\left|z - x_2\right|^{n-\delta}} \, dz \Bigg).\end{aligned}$$

Then, by integrating over a ball with respect to x_1 and x_2 we obtain the desired result.

In order to prove (3.8) we observe that, by the assumptions,

$$\frac{1}{w(B) \left|B\right|^{\gamma/n}} \int_B \left|I_\gamma f(x) - m_B I_\gamma f\right| \, dx \leq C|||f|||_{\mathbb{L}_v(0)}$$

holds for every $B \subset \mathbb{R}^n$ and $f \in \mathbb{L}_v(0)$, with C independent of f. Following similar arguments as in the proof of Theorem 3.5 of [P], it can be seen that there exists a constant C such that the inequality

$$\frac{\left|B\right|^{1+\frac{1-\gamma}{n}}}{w\left(B\right)} \int_{\mathbb{R}^n} \frac{f\left(y\right)}{\left(\left|x_B - y\right| + \left|B\right|^{1/n}\right)^{n-\gamma+1}} \, dy \leq C|||f|||_{\mathbb{L}_v(0)} \tag{4.7}$$

holds for every $f \in \mathbb{L}_v(0)$. Let us show that $v \in \mathbb{L}_v(0)$. In fact

$$|||v|||_{\mathbb{L}_v(0)} = \sup_B \frac{1}{v(B)} \int_B \left|v(x) - m_B v\right| \, dx \leq 2.$$

Then taking $f = v$ in (4.7) we have

$$\int_{\mathbb{R}^n} \frac{v(y)}{\left(|B|^{1/n} + |x_B - y|\right)^{n-\gamma+1}}\, dy \le C \frac{w(B)}{|B|^{1+\frac{1-\gamma}{n}}}$$

so we obtain that $(w, v) \in \mathbb{H}(\infty, \gamma, \gamma)$. $\square$

Proof of Theorem 3.10: To obtain the boundedness of I_γ we proceed as in the proof of (3.9). Then we have

$$\begin{aligned}
&\left|I_\gamma f\,(x_1) - I_\gamma f\,(x_2)\right| \\
&\qquad \le C \||f\||_{\mathbb{L}_v(\delta)} \Bigg(\int_{B(x_1, 2|x_1 - x_2|)} \frac{w(z)}{|z - x_1|^{n-\delta}}\, dz \int_{B(x_2, 2|x_1 - x_2|)} \frac{w(z)}{|z - x_2|^{n-\delta}}\, dz \Bigg).
\end{aligned}$$

The desired inequality is obtained by integrating over a ball $B(x_B, R)$ with respect to x_1 and x_2. $\square$

5. Properties of the classes $\mathbb{H}(p, \gamma, \delta)$

We begin with technical lemmas that establish some properties of the classes $\mathbb{H}(p, \gamma, \delta)$.

5.1 Lemma. *Let $0 < \gamma < n$, $1 \le p \le \infty$ and $\delta \in \mathbb{R}$. The condition $\mathbb{H}(p, \gamma, \delta)$ is equivalent to the existence of a constant C such that the inequalities*

$$\frac{|B|^{(\gamma-\delta)/n}}{w\,(B)} \left(\int_B v^{p'}\,(y)\; dy \right)^{\frac{1}{p'}} \le C \tag{5.2}$$

and

$$\frac{|B|^{1+(1-\delta)/n}}{w(B)} \left(\int_{\mathbb{R}^n - B} \frac{v^{p'}(y)}{|x_B - y|^{(n-\gamma+1)p'}}\, dy \right)^{\frac{1}{p'}} \le C \tag{5.3}$$

hold simultaneously for every ball $B \subset \mathbb{R}^n$, where x_B is the center of B.

In [P] it is proved that, when $\delta < 1$, the condition $\mathcal{H}(p, \gamma, \delta)$ can be reduced to a condition over a ball B. This is not possible for the condition $\mathbb{H}(p, \gamma, \delta)$. In fact, we get

5.4 Lemma. *Let p and γ be as in Lemma 5.1. There exist nontrivial pairs of weights (w,v) that satisfy (5.2) but not (5.3) for δ in the range*

$$\delta \le \min(1, \gamma - n/p),$$

excluding the case $\delta = 1$ when $\gamma - n/p = 1$.

PROOF: Let us first consider $\delta = 1 < \gamma - n/p$. The pair (w,v) given by

$$w = 1 \quad \text{and} \quad v(x) = |x|^{n/p-\gamma+1}$$

satisfies (5.2) for every ball $B \subset \mathbb{R}^n$ because, if $|x_B| \le R$ we have

$$\frac{|B|^{(\gamma-1)/n}}{w(B)} \Big(\int_B v^{p'}(y)\, dy \Big)^{\frac{1}{p'}} \le CR^{\gamma-1-n} R^{n/p-\gamma+1+n/p'} = C,$$

and for $|x_B| \ge R$ we get

$$\begin{aligned}\frac{|B|^{(\gamma-1)/n}}{w(B)} \Big(\int_B v^{p'}(y)\, dy \Big)^{\frac{1}{p'}} &\le CR^{\gamma-1-n} |x_B|^{n/p-\gamma+1} R^{n/p'} \\ &\le CR^{\gamma-1-n+n/p-\gamma+1+n/p'} \\ &= C.\end{aligned}$$

On the other hand, if $B = B(0,R)$, we obtain

$$\begin{aligned}\frac{|B|}{w(B)} \Big(\int_{\mathbb{R}^n - B} \frac{v^{p'}(y)}{|y|^{(n-\gamma+1)p'}}\, dy \Big)^{\frac{1}{p'}} &\ge \Big(\int_{\{|y|>R\}} \frac{|y|^{(n/p-\gamma+1)p'}}{|y|^{(n-\gamma+1)p'}}\, dy \Big)^{\frac{1}{p'}} \\ &= \Big(\int_{\{|y|>R\}} \frac{1}{|y|^n}\, dy \Big)^{\frac{1}{p'}},\end{aligned}$$

where the last integral is infinite and, thus, (w,v) does not satisfy (5.3).

Similar estimates can be obtained for the case $\delta < 1 \le \gamma - n/p$ by considering the pair (w,v) defined by

$$w(x) = |x|^{\gamma-\delta-n/p} \quad \text{and} \quad v \equiv 1.$$

The same is true for the case $\delta \le \gamma - n/p < 1$ and (w,v) defined by

$$w(x) = |x|^{\beta} \quad \text{and} \quad v(x) = |x|^{\alpha}$$

with

$$\alpha > n/p - \gamma + 1 \quad \text{and} \quad \beta = \alpha + \gamma - \delta - n/p. \qquad \square$$

We have proved that $\mathbb{H}(p,\gamma,\delta)$ cannot be reduced to (5.2). However, if $v^{p'}$ satisfies the doubling property then $\mathbb{H}(p,\gamma,\delta)$ can be reduced to (5.3). This was already proved in [HSV] for the case $w = v$ and $\delta = \gamma - n/p$, where the condition imposed to $v^{p'}$ arises naturally.

5.5 Lemma. *Let (w,v) be a pair of weights that satisfy (5.3) such that $v^{p'}$ satisfies the doubling property for $1 < p \leq \infty$. Then (w,v) satisfy (5.2).*

Proof: Since (5.3) holds and $v^{p'}$ satisfies a doubling property, then, given a ball $B\,(x_B, R)$, we have

$$\begin{aligned}\frac{w\,(B)}{|B|} &\geq C|B|^{(1-\delta)/n}\Bigg(\int\limits_{\mathbb{R}^n - B} \frac{v^{p'}(y)}{|x_B - y|^{(n-\gamma+1)p'}}\,dy\Bigg)^{\frac{1}{p'}} \\ &\geq \frac{|B|^{(1-\delta)/n}}{|B|^{1+(1-\gamma)/n}}\left(v^{p'}\,(2B - B)\right)^{1/p'} \\ &\geq \frac{\left(v^{p'}\,(2B)\right)^{1/p'}}{|B|^{1+(\delta-\gamma)/n}}\,,\end{aligned}$$

and thus (5.2) holds and, in view of Lemma 5.1, we have $(w,v) \in \mathbb{H}(p,\gamma,\delta)$. ☐

It is important to note that, as distinguished from the case $\delta = \gamma - n/p$ and $w = v$, the doubling property of $v^{p'}$ does not arise naturally from the condition $\mathbb{H}(p,\gamma,\delta)$. In fact, in Theorem 5.13 of [P] it is proved that the pairs $(1,v)$ with v any function in $L^{p'}$ belong to $\mathcal{H}(p,\gamma,\gamma-n)$ for $1 < p \leq \infty$. Then, by Remark 3.3, the same holds for $\mathbb{H}(p,\gamma,\gamma-n)$ and it is clear that there exist functions in $L^{p'}$ that do not satisfy the doubling condition.

Now we shall determine the range of p and δ for which the pairs of weights that satisfy $\mathbb{H}(p,\gamma,\delta)$ are trivial, i.e. $v = 0$ *a.e.*

5.6 Theorem. *Given $\gamma \in (0,n)$, we have*

(5.7) *if $\delta > 1$ or $\delta > \gamma - n/p$, the condition $\mathbb{H}(p,\gamma,\delta)$ is satisfied if and only if $v = 0$ a.e. $x \in \mathbb{R}^n$;*

(5.8) *the same conclusion holds if $\delta = \gamma - n/p = 1$.*

Proof: Let us first show (5.7). In both cases, $\delta > 1$ and $\delta > \gamma - n/p$, the proof follows similar lines as in (5.7) of Theorem 5.6 given in [P], by observing that the condition $\mathbb{H}(p,\gamma,\delta)$ is

$$\Bigg(\int\limits_{\mathbb{R}^n} \frac{v^{p'}\,(y)}{\left(|B|^{1/n} + |x_B - y|\right)^{p'(n-\gamma+1)}}\,dy\Bigg)^{\frac{1}{p'}} \leq C\,\frac{w\,(B)}{|B|}\,|B|^{(\delta-1)/n}\,,$$

and from this condition it can be deduced that

$$\left(\frac{v^{p'}(B)}{|B|}\right)^{\frac{1}{p'}} \leq C\,\frac{w\,(B)}{|B|}\,|B|^{(\delta-\gamma)/n+1/p}\,.$$

To prove (5.8) we proceed as in (5.8) of Theorem 5.6 of [P], by observing that the condition $\mathbb{H}(p,\gamma,1)$ is given by

$$\left(\int_{\mathbb{R}^n} \frac{v^{p'}(y)}{\left(|x_B-y|+|B|^{1/n}\right)^{(n-\gamma+1)p'}}\,dy\right)^{1/p'} \leq C\frac{w(B)}{|B|},$$

that is, the same inequality used in the proof of that theorem. □

5.9 Remark. In Remark 3.3 we proved that $\mathcal{H}(p,\gamma,\delta)\subset \mathbb{H}(p,\gamma,\delta)$. Let us see that the reciprocal inclusion is not valid. In fact, let us consider

$$\begin{gathered}(2(\gamma-n/p)-1)^+ \leq \alpha \leq \gamma - n/p,\\ \gamma-n/p-\alpha<\delta<\min\{\gamma-n/p, n/p-\gamma+1\},\\ n/\gamma<p<n/(\gamma-1)^+\end{gathered}$$

and the pair (w,v) defined by

$$w(x)=\begin{cases}|x|^{\alpha} & \text{if } |x|\leq 1\\ |x|^{\alpha+\delta} & \text{if } |x|>1\end{cases} \qquad \text{and} \qquad v(x)=|x|^{\delta}.$$

It is easy to check that (w,v) does not belong to $\mathcal{H}(p,\gamma,\delta)$. However, we shall see that (w,v) belongs to $\mathbb{H}(p,\gamma,\delta)$. We use Lemma 5.5 to estimate only (5.3). Letting $B_i = 2^iB$, we have

$$\begin{aligned}\text{(5.10)}\quad & \frac{|B|^{1+\frac{1-\delta}{n}}}{w(B)}\left(\int_{\mathbb{R}^n-B}\frac{v^{p'}(y)}{|x_B-y|^{(n-\gamma+1)p'}}\,dy\right)^{\frac{1}{p'}}\\ &\leq C\frac{R^{\gamma-\delta}}{w(B)}\sum_{i=1}^{\infty}\frac{1}{2^{i(n-\gamma+1)}}\left(\int_{B_i}v^{p'}\right)^{\frac{1}{p'}}.\end{aligned}$$

Let us first consider $|x_B|\leq R$. Then, from (5.10) we obtain

$$\begin{aligned}&\frac{|B|^{1+\frac{1-\delta}{n}}}{w(B)}\left(\int_{\mathbb{R}^n-B}\frac{v^{p'}(y)}{|x_B-y|^{(n-\gamma+1)p'}}\,dy\right)^{\frac{1}{p'}}\\ \text{(5.11)}\quad &\leq C\frac{R^{\gamma+n/p'}}{w(B)}\sum_{i=1}^{\infty}\frac{1}{2^{i(n/p-\delta-\gamma+1)}}\\ &\leq C\frac{R^{\gamma+n/p'}}{w(B)}.\end{aligned}$$

Thus, since $w(B)\geq C\max\{R^{\alpha+n},R^{\alpha+\delta+n}\}$ we obtain that (5.3) holds for this case.

Let us now suppose that $|x_B| > R$. Then there exists N_1 such that $2^{N_1}R \leq |x_B| < 2^{N_1+1}R$. The right hand side of (5.10) can be divided into S_1 and S_2 where

$$(5.12)\qquad \begin{aligned} S_1 &= C\frac{R^{\gamma-\delta}}{w\,(B)}\sum_{i=1}^{N_1}\frac{1}{2^{i(n-\gamma+1)}}\left(\int\limits_{B_i} v^{p'}\right)^{\frac{1}{p'}} \\ S_2 &= C\frac{R^{\gamma-\delta}}{w\,(B)}\sum_{i=N_1+1}^{\infty}\frac{1}{2^{i(n-\gamma+1)}}\left(\int\limits_{B_i} v^{p'}\right)^{\frac{1}{p'}}. \end{aligned}$$

Let us first estimate S_1. Since $i \leq N_1$ and $n/p - \gamma + 1 > 0$ we have

$$S_1 \leq C\frac{R^{\gamma-\delta+n/p'}}{w\,(B)}\,|x_B|^{\delta}\,.$$

Using that $w(B) \geq C\,\max\left\{|x_B|^{\alpha}R^n, |x_B|^{\alpha+\delta}R^n\right\}$ we obtain

$$S_1 \leq C.$$

To estimate S_2, first we observe that

$$S_2 \leq C\frac{R^{\gamma+n/p'}}{w\,(B)}$$

and then we proceed as in the estimate of S_1 to obtain that $S_2 \leq C$. This concludes the proof.

Now we give the ranges for which there exist nontrivial pairs of weights that satisfy $\mathbb{H}(p,\gamma,\delta)$.

5.13 Theorem. *Given $\gamma \in (0,n)$, there exist pairs of weights with v not identically equal to zero, that verify the condition $\mathbb{H}(p,\gamma,\delta)$ in the range of p and δ given by*

$$\delta \leq \min\{1, \gamma - n/p\}$$

excluding the case $\delta = 1$ when $\gamma - n/p = 1$.

Proof: From Remark 3.3 the pairs of weights given in the proof of Theorem 5.13 of [P] satisfy the condition $\mathbb{H}(p,\gamma,\delta)$ for $\gamma - n \leq \delta \leq \min\{1, \gamma - n/p\}$ excluding the case $\delta = 1$ when $\gamma - n/p = 1$. However note that both classes $\mathcal{H}(p,\gamma,\delta)$ and $\mathbb{H}(p,\gamma,\delta)$ do not coincide even for p and δ in this range.

Now we give examples of pairs of weights for the case $\delta < \gamma - n$. First, we consider $1 < p \leq \infty$. We divide the range $\delta < \gamma - n$ in two regions

(i) $\gamma - n - k < \delta \leq \min\{\gamma - n/p - k, \gamma - n - k + 1\}$, $k \in \mathbb{N}$,
(ii) $\gamma - n/p - k - 1 < \delta \leq \gamma - n - k$, $k \in \mathbb{N}_0$.

For (i) we consider the pairs (w, v) given by

$$w(x) = |x|^k \qquad \text{and} \qquad v(x) = |x|^{n/p-\gamma+\delta+k}$$

with

$$\gamma - n - k < \delta \le \min\{\gamma - n/p - k, \gamma - n - k + 1\}, \;\; k \in \mathbb{N}.$$

Since $v^{p'}$ satisfies the doubling condition, we use Lemma 5.5 to estimate only (5.3).

First we let $|x_B| \le R$ and $B_i = 2^i B$. Then

$$\begin{aligned}
&\frac{|B|^{1+\frac{1-\delta}{n}}}{w(B)} \left(\int\limits_{\mathbb{R}^n - B} \frac{v^{p'}(y)}{|x_B - y|^{(n-\gamma+1)p'}}\, dy \right)^{\frac{1}{p'}} \\
&\qquad \le \frac{|B|^{1+\frac{1-\delta}{n}}}{w(B)} \sum_{i=1}^{\infty} \left(\int\limits_{B_i - B_{i-1}} \frac{v^{p'}(y)}{|x_B - y|^{(n-\gamma+1)p'}}\, dty \right) \\
&\qquad \le C \frac{R^{n+1-\delta}}{R^{n+k}} \sum_{i=1}^{\infty} \frac{\left(2^i R\right)^{n/p-\gamma+\delta+k+n/p'}}{\left(2^i R\right)^{n-\gamma+1}} \\
&\qquad = C \sum_{i=1}^{\infty} \frac{1}{2^{i(1-\delta-k)}}
\end{aligned}$$

and since $\delta + k < \gamma - n + 1 < 1$, the last sum is finite.

Now let $|x_B| > R$. Then there exists N_1 such that $\frac{|x_B|}{R} \cong 2^{N_1}$. On the other hand we have

$$\begin{aligned}
(5.14) \quad &\frac{|B|^{1+\frac{1-\delta}{n}}}{w(B)} \left(\int\limits_{\mathbb{R}^n - B} \frac{v^{p'}(y)}{|x_B - y|^{(n-\gamma+1)p'}}\, dy \right)^{\frac{1}{p'}} \\
&\qquad\qquad \le C \frac{R^{\gamma-\delta-n}}{|x_B|^k} \sum_{i=1}^{\infty} \frac{1}{2^{(n-\gamma+1)i}} \left(\int\limits_{B_i} v^{p'} \right)^{1/p'}.
\end{aligned}$$

The last term in (5.14) can be divided into S_1 and S_2 where S_1 is the sum up to the N_1-th term and S_2 is the sum of the remaining terms. We first estimate S_1

$$\begin{aligned}
S_1 &\le C \frac{R^{\gamma-\delta-n}}{|x_B|^k} \sum_{i=1}^{N_1} \frac{|x_B|^{n/p-\gamma+\delta+k} \left(2^i R\right)^{n/p'}}{\left(2^i\right)^{n-\gamma+1}} \\
&\le C R^{\gamma-\delta-n} \sum_{i=1}^{N_1} \frac{\left(2^i R\right)^{n/p-\gamma+\delta+n/p'}}{\left(2^i\right)^{n-\gamma+1}} \\
&= C \sum_{i=1}^{N_1} \frac{1}{2^{i(1-\delta)}}
\end{aligned}$$

and the last sum is finite because $\delta < 1$.

For S_2 we have

$$S_2 \le C \frac{R^{\gamma-\delta-n}}{|x_B|^k} \sum_{i=N_1+1}^{\infty} \frac{\left(2^i R\right)^{n-\gamma+\delta+k}}{\left(2^i\right)^{n-\gamma+1}} \tag{5.15}$$
$$= C \frac{R^k}{|x_B|^k} \sum_{i=N_1+1}^{\infty} \frac{1}{\left(2^i\right)^{1-\delta-k}} .$$

Since $\delta + k < 1$, the last term of (5.15) is less than or equal to $C\left(\frac{R}{|x_B|}\right)^k$, which is bounded by a constant.

Let us now consider (ii). For

$$\gamma - n/p - k - 1 < \delta \le \gamma - n - k, \ \ k \in \mathbb{N}_0$$

we consider the pair (w, v) defined by

$$w(x) = |x|^{\alpha} \qquad \text{and} \qquad v(x) = |x|^{\beta}$$

with

$$\alpha = \gamma - n/p - k - 2\delta \qquad \text{and} \qquad \beta = -k - \delta.$$

Since $v^{p'}$ satisfies the doubling condition, by Lemma 5.5 we only need to estimate (5.3). Let us take $B = B(x_B, R)$ with $|x_B| \le R$. Then, if $B_i = 2^i B$ we have

$$\frac{|B|^{1+\frac{1-\delta}{n}}}{w(B)} \left(\int_{\mathbb{R}^n - B} \frac{v^{p'}(y)}{|x_B - y|^{(n-\gamma+1)p'}} \, dy \right)^{\frac{1}{p'}}$$
$$\le \frac{|B|^{1+\frac{1-\delta}{n}}}{w(B)} \sum_{i=1}^{\infty} \frac{1}{\left(2^i R\right)^{n-\gamma+1}} \left(\int_{B_i} v^{p'} \right)^{\frac{1}{p'}} \tag{5.16}$$
$$= C R^{1-\delta-\alpha+\beta-n/p+\gamma-1} \sum_{i=1}^{\infty} \frac{1}{2^{i(n/p-\gamma+1-\beta)}} .$$

Noting that

$$1 - \delta - \alpha + \beta - n/p + \gamma - 1 = 0 \qquad \text{and} \qquad n/p - \gamma + 1 > 0,$$

it is immediate that the last sum in (5.16) is bounded by a constant independent of B.

Let us now consider $|x_B| > R$. As in the case (i), we obtain

$$\frac{|B|^{1+\frac{1-\delta}{n}}}{w(B)}\left(\int_{\mathbb{R}^n-B}\frac{v^{p'}(y)}{|x_B-y|^{(n-\gamma+1)p'}}\,dy\right)^{\frac{1}{p'}} \tag{5.17}$$

$$\le \frac{R^{n+1-\delta}}{|x_B|^{\alpha}R^n}\sum_{i=1}^{\infty}\frac{1}{(2^iR)^{n-\gamma+1}}\left(\int_{B_i}v^{p'}\right)^{\frac{1}{p'}}$$

and then we divide the last term of the above inequality into S_1 and S_2, in similar way as in that case.

To estimate S_1, since $i \le N_1$, we have

$$\begin{aligned} S_1 &\le C\frac{R^{n+1-\delta}}{|x_B|^{\alpha}R^n}\sum_{i=1}^{N_1}\frac{1}{(2^iR)^{n-\gamma+1}}\left(\int_{B_i}v^{p'}\right)^{1/p'} \\ &\le C\frac{R^{n+1-\delta}}{|x_B|^{\alpha}R^n}\sum_{i=1}^{N_1}\frac{|x_B|^{\beta}(2^iR)^{n/p'}}{(2^iR)^{n-\gamma+1}} \qquad (5.18)\\ &= C\frac{R^{1-\delta}}{|x_B|^{\gamma-n/p-\delta}}\sum_{i=1}^{N_1}\frac{1}{(2^iR)^{n/p-\gamma+1}}. \end{aligned}$$

Since $\delta < \gamma - n/p$ and $|x_B| > 2^iR$ we have that the last sum in the above inequality is bounded by

$$\sum_{i=1}^{N_1}\frac{1}{2^{i(1-\delta)}}$$

which is finite since $\delta < 1$.

For S_2 we have

$$\begin{aligned} S_2 &\le \frac{R^{n+1-\delta}}{|x_B|^{\alpha}R^n}\sum_{i=N_1+1}^{\infty}\frac{1}{(2^iR)^{n-\gamma+1}}\left(\int_{B_i}v^{p'}\right)^{1/p'} \\ &\le C\frac{R^{1-\delta}}{|x_B|^{\alpha}}\sum_{i=N_1+1}^{\infty}\frac{(2^iR)^{\beta+n/p'}}{(2^iR)^{n-\gamma+1}} \\ &= C\frac{R^{1-\delta+\beta-n/p+\gamma-1}}{|x_B|^{\alpha}}\sum_{i=N_1+1}^{\infty}\frac{1}{(2^i)^{n/p-\gamma+1-\beta}}. \end{aligned}$$

Now, since $1-\delta+\beta-n/p+\gamma-1=\alpha$ and $n/p-\gamma+1-\beta>0$ we obtain

$$S_2 \le C\left(\frac{R}{|x_B|}\right)^{\alpha}$$

which is bounded because $\alpha > 0$ and $|x_B| > R$. This concludes the proof of (ii).

For the case $p = 1$ and $\delta < \gamma - n$ we set

$$w(x) = |x|^{-\delta} \quad \text{and} \quad v(x) = |x|^{n-\gamma}.$$

We shall see that $(w, v) \in \mathbb{H}(1, \gamma, \delta)$. From Lemma 5.1, we have to estimate the first terms of the two inequalities (5.2) and (5.3). Let us first see (5.2). Given $B = B(x_B, R)$, with $|x_B| \leq R$, we obtain

$$\frac{|B|^{(\gamma-\delta)/n}}{w(B)} \, ||\chi_B v||_\infty \leq C \frac{R^{\gamma-\delta+n-\gamma}}{R^{n-\delta}} = C$$

and if $|x_B| > R$ then

$$\begin{aligned}\frac{|B|^{(\gamma-\delta)/n}}{w(B)} \, ||\chi_B v||_\infty &\leq C \frac{R^{\gamma-\delta} |x_B|^{n-\gamma}}{|x_B|^{-\delta} R^n} \\ &= C R^{\gamma-\delta-n} |x_B|^{n-\gamma+\delta},\end{aligned}$$

which is bounded because $\gamma - \delta - n > 0$ and $|x_B| > R$.

We shall now estimate (5.3). First we consider $|x_B| \leq R$. Then

$$\begin{aligned}&\frac{|B|^{1+\frac{1-\delta}{n}}}{w(B)} \left\| \frac{\chi_{\mathbb{R}^n - B} v}{\left(|B|^{1/n} + |x_B - \cdot|\right)^{(n-\gamma+1)}} \right\|_\infty \\ &\quad \leq \frac{|B|^{1+\frac{1-\delta}{n}}}{w(B)} \sum_{i=1}^{\infty} \frac{1}{\left(2^i R\right)^{n-\gamma+1}} \left\|\chi_{B_i} v\right\|_\infty \\ &\quad \leq C \frac{R^{n+1-\delta}}{R^{n-\delta}} \sum_{i=1}^{\infty} \frac{\left(2^i R\right)^{n-\gamma}}{\left(2^i R\right)^{n-\gamma+1}} \\ &\quad = C \sum_{i=1}^{\infty} \frac{1}{2^i} = C.\end{aligned}$$

On the other hand, if $|x_B| > R$ we proceed as in the case $p > 1$ to obtain that the first term of the above inequality is bounded by S_1 and S_2 where

$$\begin{aligned}S_1 &= C \frac{R^{1-\delta+n}}{|x_B|^{-\delta} R^n} \sum_{i=1}^{N_1} \frac{\left\|\chi_{B_k} v\right\|_\infty}{\left(2^i R\right)^{n-\gamma+1}}, \\ S_2 &= C \frac{R^{1-\delta}}{|x_B|^{-\delta}} \sum_{i=N_1+1}^{\infty} \frac{\left\|\chi_{B_k} v\right\|_\infty}{\left(2^i R\right)^{n-\gamma+1}}.\end{aligned}$$

To estimate S_1, since $|x_B| > 2^i R$ for $i \le N_1$, we have

$$S_1 \le C \frac{R^{1-\delta}}{|x_B|^{-\delta}} \sum_{i=1}^{N_1} \frac{|x_B|^{n-\gamma}}{(2^i R)^{n-\gamma+1}}$$
$$\le C R^{\gamma-\delta-n} |x_B|^{\delta+n-\gamma},$$

which is bounded by a constant.

For S_2 we have $i > N_1$ and thus $|x_B| \le 2^i R$. Then we obtain

$$S_2 \le C \frac{R^{1-\delta}}{|x_B|^{-\delta}} \sum_{i=N_1+1}^{\infty} \frac{(2^i R)^{n-\gamma}}{(2^i R)^{n-\gamma+1}}$$
$$= C \left(\frac{R}{|x_B|}\right)^{-\delta} \sum_{i=2}^{\infty} \frac{1}{2^i},$$

and since $\delta < \gamma - n < 0$ and $|x_B| > R$, the last term is bounded by a constant. This proves that $(w, v) \in \mathbb{H}(1, \gamma, \delta)$ and concludes the proof of the theorem. $\square$

In Theorem 5.25 of [P], we prove that $\delta = \gamma - n/p$ is a necessary condition for the case $w = v$ in condition $\mathcal{H}(p, \gamma, \delta)$. The same is true for the classes $\mathbb{H}(p, \gamma, \delta)$. The above assertion is proved in the following theorem.

5.19 Theorem. *Let $0 < \gamma < n$ and $1 \le p \le \infty$. If $(w, v) \in \mathbb{H}(p, \gamma, \delta)$ and $w = v$ then $\delta = \gamma - n/p$.*

Proof: The proof follows by arguments similar to those from Theorem 5.25 of [P], replacing $\|(1/w)\chi_B\|_\infty$ by $|B|/w(B)$, and we omit it. $\square$

In the next theorem we prove that, as in the case of the classes $\mathcal{H}(p, \gamma, \delta)$ given in [P], the classes $\mathbb{H}(p, \gamma, \delta)$ are not open in the parameter p.

5.20 Theorem. *Given $0 < \gamma < n$, and $1 \le p < \infty$, there exist pairs of weights (w, v) belonging to $\mathbb{H}(p, \gamma, \delta)$ such that (w, v) does not belong to $\mathbb{H}((p'r)', \gamma, \delta)$ for any $r > 0$, with $r \ne 1$.*

Proof: We only need to prove the statement of the theorem for the case $p = 1$ and $\delta < \gamma - n$ since the other cases are the same as in Theorem (5.27) of [P]. Then, let $p = 1$ and $\delta < \gamma - n$, and consider the pair

$$w(x) = |x|^{-\delta} \quad \text{and} \quad v(x) = |x|^{n-\gamma}$$

given in Theorem 5.13. We proved there that (w, v) belongs to $\mathbb{H}(1, \gamma, \delta)$. Let us see that (w, v) does not belong to $\mathbb{H}(1 + \epsilon, \gamma, \delta)$ for any $\epsilon > 0$. From Lemma 5.1 it is enough to show that (w, v) does not satisfy condition (5.2) with $p = 1 + \epsilon$. In fact, if $B = B(0, R)$, we get

$$\frac{|B|^{(\gamma-\delta)/n}}{w(B)} \|v\chi_B\|_{(1+\epsilon)'} \ge R^{n/(1+\epsilon)'}$$

and the last expression tends to ∞ when R tends to ∞. We are done. $\square$

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(*Received* July 8, 1999, *revised* October 16, 2000)