For a dense set of equivalent norms, a non-reflexive Banach space contains a triangle with no Chebyshev center

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Abstract. Let X be a non-reflexive real Banach space. Then for each norm $|\cdot|$ from a dense set of equivalent norms on X (in the metric of uniform convergence on the unit ball of X), there exists a three-point set that has no Chebyshev center in $(X, |\cdot|)$. This result strengthens theorems by Davis and Johnson, van Dulst and Singer, and Konyagin.

Keywords: renormings, non-reflexive Banach spaces, Chebyshev centers

Classification: Primary 46B03; Secondary 41A65

The aim of the present paper is to show that, if X is a non-reflexive real Banach space, every equivalent norm on X can be arbitrarily approximated by another equivalent norm $|\cdot|$ such that some three-point set in X has no Chebyshev center in $(X, |\cdot|)$.

Recall that, roughly speaking, a Chebyshev center of a bounded set is the center of a minimal ball containing the set (see Definition below). Observe that Chebyshev centers of a set coincide with those of the convex hull of the set; this is the reason why we can write shortly "triangle" instead of "three-point set" in the title.

Our result strengthens the following three theorems (see Corollary).

[DJ] (*Davis and Johnson*). Every non-reflexive Banach space X admits an equivalent norm $|\cdot|$ such that $(X, |\cdot|)$ is not isometric to any dual Banach space. (See [Ve] for a short geometric proof of this result.)

[vDS] (*van Dulst and Singer*). Every non-reflexive Banach space X admits an equivalent norm $|\cdot|$ such that $(X, |\cdot|)$ is not norm-one complemented in its bidual.

[Ko] (*Konyagin*). Every non-reflexive Banach space X admits an equivalent norm $|\cdot|$ such that some three-point subset of X has no Chebyshev center in $(X, |\cdot|)$.

Our construction is very geometrical. While the geometric idea is quite simple, it could seem less simple when written with all technical details. We encourage, therefore, the reader to sketch simple pictures while reading the proof.

We consider only spaces over the reals. For a normed linear space X, let B_X denote its closed unit ball and X^* its dual Banach space. We start with the definition of Chebyshev centers and some preparatory facts.

Definition. *Let A be a bounded subset of a normed space $(X, \|\cdot\|)$. A point $x_0 \in X$ is called a* Chebyshev center of A *if $r(x_0) = \inf r(X)$ where $r(x) = \sup_{a\in A} \|x-a\|$.*

Proposition. *Let X be a Banach space. If X is norm-one complemented in its bidual* (*in particular, if X is a dual space*) *then every bounded set in X admits a Chebyshev center.* (Cf., e.g., [Ho].)

Remark. *Let A be a bounded subset of a normed space X. Suppose that, for some $\varrho_0 > 0$,*

$$\bigcap_{a\in A}(a+\varrho_0 B_X) = \emptyset\,, \qquad \textit{while} \qquad \bigcap_{a\in A}(a+\varrho B_X) \neq \emptyset \quad \forall \varrho > \varrho_0.$$

Then A has no Chebyshev center. Indeed, if r is the function from Definition, the two conditions say that $r^{-1}(\varrho_0) = \emptyset$, and $r^{-1}(\varrho) \neq \emptyset$ whenever $\varrho > \varrho_0$. Since r is convex, this implies that $\inf r(X) = \varrho_0$ and the infimum is not attained.

Observation. *Let A, B be two nonempty subsets of a normed space X. For every $f \in X^*$,*

$$\begin{aligned}\sup f\left(\overline{\mathrm{conv}}(A\cup B)\right) = \sup f\big(\mathrm{conv}(A\cup B)\big) &= \sup f(A\cup B)\\ &= \max\big\{\sup f(A), \sup f(B)\,\big\}.\end{aligned}$$

Lemma. *Let A, B be two bounded nonempty sets in a normed space X. Let $f \in X^*$ be such that $\sup f(A) < \sup f(B)$. If f attains its supremum over $C = \overline{\mathrm{conv}}(A\cup B)$ at some point $x \in C$, then x belongs to $\overline{\mathrm{conv}}(B)$.*

Proof: Denote $\alpha = \sup f(A)$ and $\beta = \sup f(B)$. By Observation, $f(x) = \sup f(C) = \beta$. Since $C = \overline{\mathrm{conv}}\big[\overline{\mathrm{conv}}(A) \cup \overline{\mathrm{conv}}(B)\big]$, we can write

$$x = \lambda_n a_n + (1-\lambda_n)b_n + e_n$$

where $0 \le \lambda_n \le 1$, $a_n \in \overline{\mathrm{conv}}(A)$, $b_n \in \overline{\mathrm{conv}}(B)$ (for each n) and $\|e_n\| \to 0$ as $n \to \infty$. Then, for each n,

$$\beta = f(x) = \lambda_n f(a_n) + (1-\lambda_n)f(b_n) + f(e_n) \le \lambda_n\alpha + (1-\lambda_n)\beta + |f(e_n)|,$$

which implies $(\beta-\alpha)\lambda_n \le |f(e_n)| \to 0$. Consequently, $\lambda_n \to 0$ and $b_n \to x$. Thus $x \in \overline{\mathrm{conv}}(B)$. □

We are ready to state the main result of the present paper.

Theorem. *Let $(X,\|\cdot\|)$ be a non-reflexive Banach space. Then, for each $\varepsilon > 0$, X admits an equivalent norm $|\cdot|_\varepsilon$ such that $(1-\varepsilon)\|\cdot\| \le |\cdot|_\varepsilon \le \|\cdot\|$ and some three-point subset of X has no Chebyshev center in $(X,|\cdot|_\varepsilon)$.*

PROOF: Let $\varepsilon \in (0,1/2)$ be arbitrary. Fix a functional $\varphi \in X^*$ with $\|\varphi\| = 1$, and denote $Y = \varphi^{-1}(0)$. Since (obviously) Y is not reflexive, the James theorem ([Ja]) provides a functional $f \in Y^*$ such that $\|f\| = 1$ and f does not attain its norm on $B_{(Y,\|\cdot\|)}$. Let $F \in X^*$ be any norm-one extension of f. (Of course, F may attain its $\|\cdot\|$-norm, but if so, the corresponding point cannot lie in Y.)

For simplicity, let us denote

$$C = B_{(X,\|\cdot\|)}, \qquad B = B_{(Y,\|\cdot\|)} = C \cap Y, \qquad L = f^{-1}(0) \cap B.$$

Claim 1. There exists $z_0 \in X$ such that

$$\varphi(z_0) > 1, \qquad F(z_0) < 1 \quad \text{and} \quad \|z_0\| \le 1+\varepsilon.$$

To prove this, fix a point $u \in X$ such that $F(u) = 0$, $\|u\| \le \varepsilon$ and $0 < \varphi(u) < \varepsilon$; this is possible since φ is not identically zero on $F^{-1}(0)$. Choose $z \in X$ such that $\|z\| < 1$ and $\varphi(z) > 1 - \varphi(u)/2$. Then the point $z_0 = u + z$ satisfies our needs. Indeed, $\varphi(z_0) = \varphi(u) + \varphi(z) > \varphi(u) + 1 - \varphi(u)/2 > 1$, $F(z_0) = F(z) < 1$ (as $\|F\| = 1$) and $\|z_0\| \le \|z\| + \|u\| < 1+\varepsilon$. Claim 1 is thus proved.

Fix arbitrarily $\eta \in (0,\varepsilon)$. Let $y_0 \in Y$ be such that $f(y_0) = 1+\eta$ and $\|y_0\| \le 1+2\eta$. Define

$$\widetilde{C} = \overline{\mathrm{conv}}\big[\, C \cup (z_0 + \varepsilon L) \cup (-z_0 + \varepsilon L) \cup (y_0 + \eta^2 B) \cup (-y_0 + \eta^2 B)\,\big].$$

Claim 2. If η was taken small enough, the following properties are satisfied:

(1) $C \subset \widetilde{C} \subset (1+2\varepsilon)C$;
(2) $(z_0 + \widetilde{C}) \cap (-z_0 + \widetilde{C}) = \varepsilon L$;
(3) f does not attain its supremum on $\widetilde{B} = \widetilde{C} \cap Y$;
(4) $\|\cdot\|$-diam $\Big[\widetilde{B} \cap f^{-1}\big([t,+\infty)\big)\Big] < \varepsilon$ for some $t < \sup f(\widetilde{B})$.

Assume for a while that we have already proved Claim 2. Then we can complete the argument as follows. Let $|\cdot|$ be the norm on X whose unit ball is the set $\widetilde{C}$. (Hence $\sup f(\widetilde{B}) = |f|$.) Note that, in (4), $t < |f|$ can be taken arbitrarily close to $|f|$. Let $t \in (0,|f|)$ be so close to $|f|$ that, for $s_0 := \frac{|f|}{t}$, we have

$$(4') \quad \|\cdot\|\text{-diam}\Big[s_0\widetilde{B} \cap f^{-1}\big([|f|,+\infty)\big)\Big] = s_0 \cdot \|\cdot\|\text{-diam}\Big[\widetilde{B} \cap f^{-1}\big([t,+\infty)\big)\Big] < \varepsilon.$$

By (3), $(w+\widetilde{B}) \cap f^{-1}(0) = \emptyset$ whenever $w \in f^{-1}(|f|)$. Fix any $w_0 \in s_0\widetilde{B} \cap f^{-1}(|f|)$. Since (4′) implies

$$\|\cdot\|\text{-diam}\big[s_0\widetilde{B} \cap f^{-1}(|f|)\big] < \varepsilon,$$

we have $s_0\widetilde{B}\cap f^{-1}(|f|)\subset w_0+\varepsilon B$. Consequently,

$$(5)\qquad \begin{aligned}(w_0+s_0\widetilde{B})\cap f^{-1}(0)&=w_0+\big[s_0\widetilde{B}\cap f^{-1}(-|f|)\big]=w_0-\big[s_0\widetilde{B}\cap f^{-1}(|f|)\big]\\ &\subset \Big(w_0-[w_0+\varepsilon B]\Big)\cap f^{-1}(0)=\varepsilon B\cap f^{-1}(0)=\varepsilon L.\end{aligned}$$

Consider the set $A=\{\,z_0,-z_0,w_0\,\}$. Then, for each $s>1$,

$$\begin{aligned}(z_0+s\widetilde{C})\cap(-z_0+s\widetilde{C})\cap(w_0+s\widetilde{C})&\supset(z_0+\widetilde{C})\cap(-z_0+\widetilde{C})\cap(w_0+s\widetilde{C})\\ &\overset{\text{by (2)}}{=}\varepsilon L\cap(w_0+s\widetilde{C})=\varepsilon L\cap(w_0+s\widetilde{B}),\end{aligned}$$

and the last set is nonempty, since, by (5), for $s\in(1,s_0]$ it contains

$$(w_0+s\widetilde{B})\cap f^{-1}(0)=w_0+\big[s\widetilde{B}\cap f^{-1}(-|f|)\big]\neq\emptyset$$

(and it is even larger for $s>s_0$). On the other hand, for $s=1$,

$$(z_0+\widetilde{C})\cap(-z_0+\widetilde{C})\cap(w_0+\widetilde{C})\overset{(2)}{=}\varepsilon L\cap(w_0+\widetilde{C})\subset f^{-1}(0)\cap(w_0+\widetilde{B})=\emptyset.$$

By Remark, A has no Chebyshev center in $(X,|\cdot|)$. Moreover, (1) implies $\frac{1}{1+2\varepsilon}\|\cdot\|\le|\cdot|\le\|\cdot\|$; this will give the assertion of Theorem since ε was an arbitrary number from $(0,1/2)$.

It remains to prove Claim 2, it is, that (1), (2), (3), (4) are satisfied if $\eta\in(0,\varepsilon)$ was taken sufficiently small. To see (1), observe that

$$\begin{aligned}\pm z_0+\varepsilon L&\subset(1+\varepsilon)C+\varepsilon C=(1+2\varepsilon)C\\ \pm y_0+\eta^2B\subset(1+2\eta)C+\eta^2C&=(1+\eta)^2C\subset(1+2\varepsilon)C\end{aligned}$$

for all sufficiently small $\eta>0$.

Let us show (2). Observe that φ attains its supremum on $\widetilde{C}$ (at the points of $z_0+\varepsilon L$). Applying Lemma to the functional φ and to the two sets

$$C\cup(-z_0+\varepsilon L)\cup(y_0+\eta^2B)\cup(-y_0+\eta^2B)\quad\text{ and }\quad z_0+\varepsilon L,$$

we obtain that φ attains its $|\cdot|$-norm (i.e., its supremum over $\widetilde{C}$) exactly at the points of the set $z_0+\varepsilon L$. Moreover, $|\varphi|=\varphi(z_0)$. Since

$$(z_0+\widetilde{C})\cap(-z_0+\widetilde{C})\subset\varphi^{-1}\big([0,+\infty)\big)\cap\varphi^{-1}\big((-\infty,0]\big)=\varphi^{-1}(0),$$

we have

$$\begin{aligned}(z_0+\widetilde{C})\cap(-z_0+\widetilde{C})&=\big[(z_0+\widetilde{C})\cap\varphi^{-1}(0)\big]\cap\big[(-z_0+\widetilde{C})\cap\varphi^{-1}(0)\big]\\ &=\Big[z_0+\Big(\widetilde{C}\cap\varphi^{-1}(-|\varphi|)\Big)\Big]\cap\Big[-z_0+\Big(\widetilde{C}\cap\varphi^{-1}(|\varphi|)\Big)\Big]\\ &=\big[z_0+(-z_0+\varepsilon L)\big]\cap\big[-z_0+(z_0+\varepsilon L)\big]=\varepsilon L.\end{aligned}$$

Suppose that (3) does not hold, in other words, that f attains its supremum (i.e., its $|\cdot|$-norm) on $\widetilde{B}$. By Observation, $|F| = \sup F(\widetilde{C}) = \sup F(y_0 + \eta^2 B)$. Since $y_0 + \eta^2 B \subset Y$, we obtain

$$|F| = \sup(\widetilde{C} \cap Y) = \sup f(\widetilde{B}) = |f|.$$

Hence F attains its $|\cdot|$-norm on $\widetilde{C}$. It follows easily from Lemma that F attains its supremum on $y_0 + \eta^2 B$. But this means that f attains its supremum on B, which contradicts our choice of f.

To prove (4), take $t = 1 + \eta = f(y_0)$. Denote

$$D = \overline{\mathrm{conv}}\big[\, C \cup (z_0 + \varepsilon L) \cup (-z_0 + \varepsilon L) \cup (-y_0 + \eta^2 B)\,\big].$$

It is easy to see that $\sup f(D) = \|f\| = 1$ and $\widetilde{C} = \overline{\mathrm{conv}}\big[D \cup (y_0 + \eta^2 B)\big]$. Let us consider the slice $S = \widetilde{C} \cap F^{-1}\big([1+\eta, +\infty)\big)$ (which contains the set $\widetilde{B} \cap f^{-1}\big([1+\eta, +\infty)\big)$ and the point y_0). Every point $x \in S$ can be written in the form

$$x = \lambda_n d_n + (1 - \lambda_n)(y_0 + \eta^2 b_n) + e_n,$$

where $\lambda_n \in [0,1]$, $d_n \in D$, $b_n \in B$ (for each n) and $\|e_n\| \to 0$ as $n \to \infty$. We have

$$\begin{aligned} 1 + \eta \le F(x) &\le \lambda_n F(d_n) + (1 - \lambda_n)\big(F(y_0) + \eta^2 \|F\|\big) + F(e_n) \\ &\le \lambda_n + (1 - \lambda_n)(1 + \eta + \eta^2) + |F(e_n)|\,, \end{aligned}$$

which implies $\lambda_n \le \eta(1+\eta)^{-1} + \eta^{-1}(1+\eta)^{-1}\,|F(e_n)| \le \eta + (1/\eta)\,|F(e_n)|$. Using the facts that $d_n, y_0 \in \widetilde{C}$ and $\|\cdot\| \le (1 + 2\varepsilon)|\cdot|$, we obtain

$$\begin{aligned} \|x - y_0\| &\le \lambda_n \|d_n - y_0\| + (1 - \lambda_n)\eta^2 \|b_n\| + \|e_n\| \\ &\le 2(1 + 2\varepsilon)\lambda_n + \eta^2 + \|e_n\| \\ &\le 2(1 + 2\varepsilon)\big(\eta + (1/\eta)|F(e_n)|\big) + \eta^2 + \|e_n\|. \end{aligned}$$

Passing to limit for $n \to \infty$ gives $\|x - y_0\| \le 2\eta(1 + 2\varepsilon) + \eta^2$. Since x was an arbitrary element of S, we conclude that

$$\|\cdot\|\text{-diam}(S) \le 4\eta(1 + 2\varepsilon) + 2\eta^2 < \varepsilon$$

for η small enough. This proves (4), and completes the proof of Theorem. $\square$

In view of Proposition above, we can state the following corollary which relates our Theorem to the previously known results mentioned in the beginning of this paper.

Corollary. *Let $\mathcal{N}$ be the set of all equivalent norms on a non-reflexive Banach space X, equipped with the metric $d(p,q) = \sup\{|p(x) - q(x)| : x \in B_X\}$, $p, q \in \mathcal{N}$. There exists a dense subset $\mathcal{N}_0$ of $\mathcal{N}$ such that each norm $p \in \mathcal{N}_0$ satisfies:*

(X,p) contains a triangle with no Chebyshev center,
hence (X,p) is not norm-one complemented in its bidual,
hence (X,p) is not isometric to any dual space.

Problem. *In the notation of Corollary, is the set $\mathcal{N}_0$ residual in $\mathcal{N}$?*

Acknowledgment. The author thanks the referee for several remarks that improved the readability of the present paper.

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(*Received* March 17, 2000, *revised* October 6, 2000)