On α-normal and β-normal spaces

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Abstract. We define two natural normality type properties, α-normality and β-normality, and compare these notions to normality. A natural weakening of Jones Lemma immediately leads to generalizations of some important results on normal spaces. We observe that every β-normal, pseudocompact space is countably compact, and show that if X is a dense subspace of a product of metrizable spaces, then X is normal if and only if X is β-normal. All hereditarily separable spaces are α-normal. A space is normal if and only if it is κ-normal and β-normal.

Central results of the paper are contained in Sections 3 and 4. Several examples are given, including an example (identified by R.Z. Buzyakova) of an α-normal, κ-normal, and not β-normal space, which is, in fact, a pseudocompact topological group. We observe that under CH there exists a locally compact Hausdorff hereditarily α-normal non-normal space (Theorem 3.3). This example is related to the main result of Section 4, which is a version of the famous Katětov's theorem on metrizability of a compactum the third power of which is hereditarily normal (Corollary 4.3). We also present a Tychonoff space X such that no dense subspace of X is α-normal (Section 3).

Keywords: normal, α-normal, β-normal, κ-normal, weakly normal, extremally disconnected, $C_p(X)$, Lindelöf, compact, pseudocompact, countably compact, hereditarily separable, hereditarily α-normal, property wD, weakly perfect, first countable

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§0. Introduction

One of natural approaches to a systematic study of a notion of interest is to compare it with its weaker or stronger versions. For example, it helps to understand compactness better, when we study countable compactness, pseudocompactness, initial compactness, and so on. A deeper understanding of paracompactness can be achieved by a study of a great wealth of paracompactness type properties, such as metacompactness, strong paracompactness, countable paracompactness, and so on. Normality is no exception to this rule, and quite a few interesting normality type notions were introduced earlier (see, for example, [1], [8], [19], [21]).

Below we define two new normality type properties: α-normality and β-normality. Results in the first section of the paper are elementary; we just review from the new point of view some old classical examples of normal spaces and see that the classical proofs of non-normality in these examples show that the spaces under consideration are not α-normal.

Results in Section 2 are less obvious, though their proofs are not long. For example, this relates to Theorem 2.2, Theorem 2.3, and to Corollary 2.6. The proofs of Theorems 2.3 and Corollary 2.6 depend on some deep theorems on product spaces and C_p-spaces.

Central results of the paper are obtained in Sections 3 and 4. In particular, we present an argument, belonging to R.Z. Buzyakova, showing that there exists an α-normal space which is not β-normal. Theorems 3.9 and 3.10 also seem to be interesting. Under CH, it is demonstrated that there exists a locally compact Hausdorff hereditarily α-normal not normal space (Theorem 3.3). The main result of Section 4 is a version of the famous Katětov's theorem on metrizability of a compactum, the third power of which is hereditarily normal (Corollary 4.3).

Our notation and terminology are as in [10]. However, we prefer to use the expression "a closed discrete subset A of a space" instead of the shorter expression "a discrete subset A of a space", since the latter is quite often used when the subspace A is discrete in itself.

§1. Definitions and first results on α-normality

A space X will be called *α-normal* if for any two disjoint closed subsets A and B of X there exist disjoint open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B.

A space X will be called *β-normal* if for any two disjoint closed subsets A and B of X there exist open subsets U and V of X such that $A \cap U$ is dense in A, $B \cap V$ is dense in B, and $\overline{U} \cap \overline{V} = \emptyset$.

Clearly, any normal space is β-normal and any β-normal space is α-normal. Also, we have the following easy to prove statements:

Proposition 1.1. *If a β-normal space X satisfies the T_1 separation axiom, then the space X is regular.*

Proposition 1.2. *If an α-normal space X satisfies the T_1 separation axiom, then the space X is Hausdorff.*

From now on we assume all spaces under consideration to satisfy the T_1 separation axiom.

The following questions regarding the relationship between α-normality, β-normality, and the classical separation axioms come naturally to mind.

Question 1. Does there exist a Hausdorff α-normal non-regular space?

Question 2. Does there exist a regular Hausdorff α-normal not Tychonoff space?

Question 3. Does there exist a Tychonoff α-normal not normal space?

Question 4. Does there exist a Tychonoff β-normal not normal space?

Question 5. Does there exist an α-normal, Tychonoff, not β-normal space?

We answer some of these questions below. Somewhat unexpectedly, it turns out to be not too easy to get these answers. In this section, we present first results on α-normality. We start with two obvious general statements.

Proposition 1.3. *A closed subspace of an α-normal space (β-normal space) is α-normal (β-normal).*

Proposition 1.4. *If a space X is α-normal, then any two disjoint closed discrete subsets A and B of X can be separated by open disjoint subsets of X.*

By Proposition 1.4, we have the following refinement of F.B. Jones Lemma:

Proposition 1.5. *Let X be space with a discrete subspace S of cardinality λ and a dense subspace D of cardinality κ. Then*

1) *if X is hereditarily α-normal, then $2^\kappa \geq 2^\lambda$;*
2) *if X is α-normal and S is closed, then $2^\kappa \geq 2^\lambda$.*

Corollary 1.6. *If $2^\omega < 2^{\omega_1}$ and X is a separable, α-normal space, then every closed discrete subset A of X is countable. In addition, if X is a separable, α-normal space, then for any closed discrete subset A of X we have $|A| < 2^\omega$.*

Example 1.7. With the help of Proposition 1.4, it is easy to verify that the following spaces are **not** α-normal (the standard proofs of their non-normality show this. Only in the case of Mrowka space we have to slightly expand the argument).

1. *The deleted Tychonoff Plank.*
2. *The Dieudonné Plank.*
3. *Any nontrivial Mrowka Space M.*

Recall that M is constructed with the help of a maximal almost disjoint infinite family of infinite subsets of ω (see [10]). This space is pseudocompact (due to maximality of the family) and not countably compact (since it contains an infinite closed discrete subspace). Therefore, M is not normal (this is an elementary well known fact: every pseudocompact normal space is countably compact [10]). So there exist closed disjoint subsets A and B of M that cannot be separated by open sets.

Suppose M is α-normal. Let

$$\begin{aligned}
A_1 &= A \cap (M \setminus \omega), \\
B_1 &= B \cap (M \setminus \omega), \\
A_2 &= A \cap \omega, \\
B_2 &= B \cap \omega.
\end{aligned}$$

Then A_1 and B_1 are disjoint closed discrete subsets of M. Hence, there exists open disjoint subsets U_1 and V_1 of M such that $U_1 \cap A_1 = \overline{U_1 \cap A_1} = A_1 \subseteq U_1$ and $V_1 \cap B_1 = \overline{V_1 \cap B_1} = B_1 \subseteq V_1$. Now $(U_1 \setminus B) \cup A_2$ and $(V_1 \setminus A) \cup B_2$ are disjoint open subsets of M which contain A and B respectively. This contradicts the fact that A and B cannot be separated by open sets.

4. *The square of the Sorgenfrey Line.*

5. *The Niemytzki Plane.*

A natural step now is to take a classical theorem about normality and to try to replace normality with α-normality or with β-normality. Sometimes this turns out to be possible. Corollary 1.6 is just one of such results.

Here is a generalization of the famous theorem of F.B. Jones [12]:

Corollary 1.8. *If $2^\omega < 2^{\omega_1}$, then every separable α-normal Moore space X is metrizable.*

PROOF: Every Moore space has a σ-discrete network. Therefore, it suffices to show that each closed discrete subset of X is countable (this would imply that X is Lindelöf). It remains to apply Corollary 1.6. □

Another natural question to consider is: what additional properties, when added to α-normality, result in normality? Here is a result of this kind. Recall that a Hausdorff space X is *extremally disconnected* if the closure of every open set in X is open. The proof of the next result is straightforward.

Theorem 1.9. *Every extremally disconnected, α-normal space X is normal.*

§2. First results on β-normality

Proposition 2.1. *A space X is β-normal if and only if, for each closed $A \subseteq X$ and for all open $U \subseteq X$ with $A \subseteq U$, there exists an open $V \subseteq X$ such that $\overline{V \cap A} = A \subseteq \overline{V} \subseteq U$.*

PROOF: $(\rightarrow)$ Let X be β-normal, A a closed subset of X, and U an open subset of X such that $A \subseteq U$. Put $B = X \setminus U$. Since B is closed and X is β-normal, there exist open sets W_1, W_2 in X with disjoint closures and such that $W_1 \cap A$ and $W_2 \cap B$ are dense in A and B, respectively. Thus, $\overline{W_1} \cap B = \emptyset$ which implies that $\overline{W_1} \subseteq U$.

$(\leftarrow)$ Let A and B be closed disjoint subsets of X. Since $X \setminus B$ is open in X and $A \subseteq X \setminus B$, by hypothesis there exist W_1 open in X such that $\overline{W_1} \subseteq X \setminus B$ and $\overline{W_1 \cap A} = A$. Moreover, $X \setminus \overline{W_1}$ is open in X and contains B, so, by the hypothesis, there exists an open subset W_2 of X such that $\overline{W_2} \subseteq X \setminus \overline{W_1}$ and $\overline{W_2 \cap B} = B$. Since $\overline{W_1} \cap \overline{W_2} = \emptyset$, the space X is β-normal. □

Every normal pseudocompact space is countably compact (see [10]). The same is true for β-normal spaces.

Theorem 2.2. *Every β-normal, pseudocompact space X is countably compact.*

Proof: Suppose that X is not countably compact. This implies that there exists a countable, closed, discrete subset $A = \{a_n : n \in \omega\}$ in X. Since X is regular, there exists a family $\{U_n : n \in \omega\}$ of open subsets of X such that $a_n \in U_n$ for all $n \in \omega$ and $U_i \cap U_j = \emptyset$ for all $i \neq j$. So, $\bigcup_{n\in\omega} U_n$ is open in X, hence $F = X \setminus \left(\bigcup_{n\in\omega} U_n\right)$ and A are closed disjoint subsets of X. Since X is β-normal, there exist open subsets U and V of X such that $\overline{U} \cap \overline{V} = \emptyset$, $\overline{U \cap F} = F$ and $\overline{V \cap A} = A$. Now $\gamma = \{V \cap U_n : n \in \omega\}$ is a discrete family of nonempty open subsets of X. But this contradicts X being pseudocompact. □

Theorem 2.3. *Let X be a dense subspace of a product of metrizable spaces. Then X is normal if and only if X is β-normal.*

Proof: E.V. Ščepin [19] defined a space to be *κ-normal* if any two disjoint canonically closed sets in X have disjoint neighborhoods. Ščepin and R. Blair independently showed that every dense subspace of any product of metrizable spaces is κ-normal ([8], [20]). Therefore, since X is κ-normal and β-normal, it remains to apply the next theorem. □

Theorem 2.4. *A space X is normal if and only if it is κ-normal and β-normal.*

Proof: $(\rightarrow)$ Clear.

$(\leftarrow)$ Let A and B be any two closed disjoint subsets of X. Since X is β-normal, there exist open subsets U and V of X such that $\overline{U} \cap \overline{V} = \emptyset$, $\overline{U \cap A} = A$, and $\overline{V \cap B} = B$. So, $\overline{U}$ and $\overline{V}$ are disjoint canonical closed sets containing A and B, respectively. Since X is κ-normal, there exist disjoint, open subsets W_1 and W_2 of X such that $A \subseteq \overline{U} \subseteq W_1$ and $B \subseteq \overline{V} \subseteq W_2$. □

Recall that $\mathbb{R}^\tau$ is not normal if $\tau > \omega$ ([10]). So, we have the following corollary:

Corollary 2.5. *If $\tau > \omega$, then $\mathbb{R}^\tau$ is not β-normal.*

Corollary 2.6. *For any Tychonoff space X, the space $C_p(X)$ is normal if and only if it is β-normal.*

Proof: Indeed, $C_p(X)$ is always a dense subspace of $\mathbb{R}^X$ (see [3]). □

In the above statement, β-normality cannot be replaced by α-normality. We will see this in the next section.

§3. Examples of α-normal not β-normal spaces

We will now present two examples of α-normal, not normal spaces.

Theorem 3.1. *Let X be a regular space such that for every closed subspace Y of X there exists a Lindelöf subspace Z of Y that is dense in Y. Then X is α-normal.*

Proof: Let A and B be closed disjoint subsets of X. By the hypothesis, there exist Lindelöf subspaces W, Z, and Y that are dense subsets of A, B, and X, respectively. Put $D = Y \cup W \cup Z$. Then D is a dense Lindelöf subspace of X. Hence, D is normal. So, there exist open disjoint subsets U_A and U_B of D separating the closed disjoint subsets $A \cap D$ and $B \cap D$ of D. Moreover, since D is dense in X, there exist open disjoint subsets V_A and V_B of X such that $U_A = V_A \cap D$ and $U_B = V_B \cap D$. Since W is dense in A, from $W \subseteq A \cap D \subseteq V_A \cap A \subseteq A$ it follows that $\overline{V_A \cap A} = A$. Similarly, $\overline{V_B \cap B} = B$. Hence, X is α-normal. $\square$

Corollary 3.2. *If every closed subspace of a regular space X is separable, then X is α-normal.*

Now we are ready to present one of the main results of this section:

Theorem 3.3 (CH). *There exists a locally compact, Tychonoff, first countable, not normal space Y such that Y^n is hereditarily α-normal, for each $n \in \omega$.*

Proof: In his classic 1948 article [14], M. Katětov showed that a compact Hausdorff space X is metrizable if and only if $X \times X \times X$ is hereditarily normal. On the other hand, K. Kunen constructed, under the continuum hypothesis, a non-metrizable, compact, perfectly normal space K of cardinality continuum such that all finite powers of K are hereditarily separable (see [17]). Now, $K \times K \times K$ is compact and hereditarily separable, so by Corollary 3.2, $K \times K \times K$ is hereditarily α-normal. However, by Katětov's result, $K \times K \times K$ is not hereditarily normal, since K is not metrizable. It is well known and easy to prove, that a space X is hereditarily normal if and only if every open subspace of X is normal. So, $K \times K \times K$ has an open subspace Y such that Y is not normal. Clearly, Y is locally compact, and Y^n is hereditarily separable and therefore, hereditarily α-normal, for each $n \in \omega$. $\square$

This example not only answers Question 3 (consistently), but also helps to answer Question 2 (also consistently). In [13], F.B. Jones showed that if there exists a non-normal, hereditarily separable, regular Hausdorff space X, then there exists a non-Tychonoff, hereditarily separable, regular Hausdorff space. So under CH, we can create a new example of a space via Jones' machine that is α-normal by Corollary 3.2 but not Tychonoff. This answers in positive Question 2 under CH.

Example 3.4. Assume CH. Then there exists a space X such that $C_p(X)$ is hereditarily α-normal but not normal. Indeed, from Katětov's and Kunen's results cited and applied above we know that under CH there exists a non-normal Tychonoff space X such that every finite power X^n of X is hereditarily separable. Fix such X and consider the second C_p-space over X, that is, the space $Z = C_p(C_p(X))$. Then X is homeomorphic to a closed subspace of Z (see [3]). Therefore, Z is not normal. On the other hand, the space Z is κ-normal (see the proof of Corollary 2.6). Now from Theorem 2.4 it follows that Z is not β-normal. We claim that Z is hereditarily α-normal. Indeed, Z is hereditarily separable, since every finite power of X is hereditarily separable (see Corollary 2.5.29 in [3]). Hence, Z is hereditarily α-normal by Corollary 3.2.

Notice, that Z is a "very nice" space: it is a linear topological space, therefore Z is homogeneous; also the Souslin number of Z is countable.

Question 6. Is there a ZFC example of an α-normal space $C_p(X)$ which is not normal?

It is well known that, for every Tychonoff space X, the space $C_p(X)$ is a dense subspace of $\mathbb{R}^X$. Therefore, Example 3.4 shows, under CH, that in Theorem 2.3 it is not possible to replace β-normality with α-normality, even when all factors are separable metrizable spaces. However, the situation becomes more delicate when we consider the products of not more than ω_1 of separable metrizable spaces (in particular, $\mathbb{R}^{\omega_1}$).

We need the following lemma, the proof of which is contained in the proof of Theorem 1 in D.P. Baturov's paper [7], and is therefore omitted.

Lemma 3.5. *Suppose that Y is a dense subspace of the product $X = \Pi\{X_\alpha : \alpha < \omega_1\}$ of separable metrizable spaces X_α. Suppose further that Z is an uncountable discrete subspace of Y. Then there exist disjoint subsets A and B of Z such that, for each countable subset S of ω_1, the images $\pi_S(A)$ and $\pi_S(B)$ of the sets A and B under the natural projection of X onto the space $X_S = \Pi\{X_\alpha : \alpha \in S\}$ are not separated in X_S (that is, the closure of at least one of them intersects the other set).*

Now we need the next version of M.F. Bockstein's lemma [9] which is a part of the folklore. We briefly sketch the proof of it, for the sake of completeness.

Lemma 3.6. *Let $X = \Pi\{X_\alpha : \alpha \in A\}$ be the product of separable metrizable spaces X_α. Suppose further that Y is a dense subspace of X, and U, V are disjoint open subsets of Y. Then there exists a countable subset S of A such that the images $\pi_S(U)$ and $\pi_S(V)$ are separated in the space $X_S = \Pi\{X_\alpha : \alpha \in S\}$.*

Proof: Since Y is dense in X, we can expand U and V to disjoint open sets in X. Therefore, we can assume that $Y = X$. Now, by the classic Bockstein's lemma [9] (see also 2.7.12 b) in [10]), there exists a countable subset S of A such that the images $\pi_S(U)$ and $\pi_S(V)$ are disjoint. Since $\pi_S(U)$ and $\pi_S(V)$ are open sets in X_S, they are separated in X_S. $\square$

The least upper bound of the cardinalities of closed discrete subsets of a space X is called the *extent* of X. From Lemmas 3.5 and 3.6 we immediately obtain:

Theorem 3.7. *The extent of each α-normal dense subspace Y of the product $X = \Pi\{X_\alpha : \alpha < \omega_1\}$ of separable metrizable spaces X_α is countable.*

J. Mycielski showed [16] that the extent of any uncountable power ω^τ of the discrete space ω is uncountable. Combining this with Theorem 3.7, and also using the fact that α-normality is closed hereditary, we obtain a new proof of the following recent result of D. Burke, announced at the Ben Fitzpatrick Memorial Conference at Auburn University in February 2001:

Corollary 3.8 (D.K. Burke). *The product of uncountably many of non-compact metrizable spaces is never α-normal.*

In particular, the space $\mathbb{R}^{\omega_1}$ is not α-normal (D.K. Burke). Here is a curious application of Theorem 3.7, based on some delicate results of C_p-theory. The next theorem improves a result in [4].

Theorem 3.9. *The space $C_p(\omega_1 + 1)$ is not α-normal. Moreover, no dense subspace of $C_p(\omega_1 + 1)$ is α-normal.*

Proof: Assume the contrary, and fix an α-normal dense subspace Y of $C_p(\omega_1+1)$. Since $C_p(\omega_1 + 1)$ is a dense subspace of $\mathbb{R}^{\omega_1}$ ([3]), Y is also a dense subspace of $\mathbb{R}^{\omega_1}$. Therefore, by Theorem 3.7, the extent of Y is countable. Now it follows from Baturov's theorem [6] (see also Theorem 3.6.1 in [3]), that the space Y is Lindelöf. Obviously, the set $Y \subseteq C_p(\omega_1 + 1)$ separates points of the space $\omega_1 + 1$. This contradicts Corollary 4.11.10 in [3]. $\square$

In connection with Theorem 3.9, we state the following question:

Question 7. Suppose X is a compact space such that $C_p(X)$ is α-normal. Is then $C_p(X)$ normal? Is then the tightness of X countable?

We now present an example of an α-normal, pseudocompact, not normal space in ZFC.

Let $\mathcal{D} = \{0,1\}$ be the two-point discrete space. Fix a set A of cardinality $\aleph_\omega$. Put $X = \prod_{\alpha\in A} \mathcal{D}_\alpha$ and take the subspace $H = \{\chi_B : B \subseteq A, |B| < \aleph_\omega\}$ of X, where χ_B is the characteristic function of B (on A). It is well known that the space H has the following properties: H is dense in X, H is pseudocompact, but H is neither countably compact, nor Lindelöf. It follows that H is not normal (since every pseudocompact normal space is countably compact, see [10]).

G. Gruenhage and R. Buzyakova independently established that H is also linearly Lindelöf (see [2]). This again implies that H is not countably compact, since every linearly Lindelöf countably compact space is compact.

Our interest in the space H lies in the following fact established by R. Buzyakova. We are grateful to her for allowing us to present her argument in this article.

Theorem 3.10 (R.Z. Buzyakova). *The space H is α-normal.*

Proof: Let F be a closed subset of H. We will show that there exists a subset M of F such that $\overline{M} = F$ and M is σ-compact. Then, by Theorem 3.1, H will be α-normal. Since $w(X) = \aleph_\omega$, we have $w(F) \leq \aleph_\omega$. Hence, $d(F) \leq \aleph_\omega$. Fix $L \subseteq F$ such that L is dense in F and $|L| \leq \aleph_\omega$. Then $L = \{\chi_{C_\alpha} : \alpha < \aleph_\omega\}$.

Put $L_n = \{\chi_{C_\alpha} : |C_\alpha| \leq \aleph_n, \alpha \leq \aleph_n\}$ for all $n \in \omega$. Then $L = \bigcup_{n\in\omega} L_n$ and $|L_n| \leq \aleph_n$, for all $n \in \omega$. Next, put $C_n = \bigcup\{C \subset A : \chi_C \in L_n\}$. Clearly, $|C_n| \leq \aleph_n$. Finally, put $W_n = \prod_{\alpha\in C_n} D_\alpha \times \prod_{\alpha\in A\setminus C_n} O_\alpha$, where $O_\alpha = \{0\}$, for each $\alpha \in A \setminus C_n$. Now $L_n \subseteq W_n \subseteq H$, and W_n is compact. Therefore, W_n is closed in H. Hence, the closure of L_n in H is compact, and the subspace $M = \bigcup_{n\in\omega} \overline{L_n}^H$ is σ-compact. Since $L \subseteq M \subseteq F$ and L is dense in F, we have $\overline{M} = F$. □

By the above argument, we have actually proved the following more general statement:

Proposition 3.11. (R.Z. Buzyakova)**.** *If F is a closed subset of H, then, for every subset L of F such that $|L| \leq \aleph_\omega$, there exists a subspace $M \subseteq F$ such that $L \subseteq M$ and M is σ-compact.*

Since H is pseudocompact and not countably compact, it follows from Theorem 2.2 that H is not β-normal. Thus, we have the next result:

Corollary 3.12 (R.Z. Buzyakova)**.** *There exists a pseudocompact α-normal not β-normal space* (*which is, therefore, not normal*).

This answers positively Questions 3 and 5 in ZFC. Notice, that the property of H established in Proposition 3.11 can be regarded as a natural weakening of the requirement that every closed subspace of a space should be separable. It is much more difficult to construct a pseudocompact space in which every closed subspace is separable (see [2], [10]).

A space X is said to be *weakly normal* ([1]) if for every pair A, B of disjoint closed subsets of X there exists a continuous mapping f of X onto a separable metrizable space Y such that the sets $f(A)$ and $f(B)$ are disjoint.

Question 8. Is the space H weakly normal?

It was observed in [1] that every countably compact weakly normal T_1-space is normal. However, we do not know if the same is true for pseudocompact weakly normal spaces. Thus we have the following question which is obviously related to Question 8.

Question 9. Is every pseudocompact weakly normal space normal?

§4. On hereditary β-normality and the property wD

A space X is called *weakly perfect* if for all closed subsets F of X there exists a G_δ set A in X such that $\overline{A} = F$. Interesting results on weakly perfect spaces were obtained by R.W. Heath [11] and L. Kočinac [15].

Proposition 4.1. *Suppose that X is a countably compact Hausdorff space and Y a countable space with exactly one non-isolated point e. Suppose also that $X \times Y$ is hereditarily β-normal. Then X is weakly perfect.*

PROOF: Let F be a closed subset of X. Put $E = Y \setminus \{e\}$, and enumerate E in a one-to-one way: $E = \{a_i : i \in \omega\}$. Consider the subsets $A = F \times E$ and $B = (X \setminus F) \times \{e\}$ of the product space $X \times Y$. Clearly, $\overline{A} = F \times Y$. Therefore, $\overline{A} \cap B = \emptyset$. We also have $A \cap \overline{B} = \emptyset$, since $\overline{B} \subseteq X \times \{e\}$ and $A \cap (X \times \{e\}) = \emptyset$. Hence, the subspace $Z = (X \times X) \setminus (\overline{A} \cap \overline{B})$ contains the sets A and B, and the closures of A and B in Z are disjoint.

By the assumption, the space Z is β-normal. It is also open in $X \times Y$. Therefore, there exist open sets U and V in Z such that $A \subseteq \overline{U \cap A}$, $B \subseteq \overline{V \cap B}$, and the closures of U and V in Z are disjoint. Then, in particular, $B \cap \overline{U} = \emptyset$.

Now put $U_i = \{x \in X : (x, a_i) \in U\}$, for each $i \in \omega$. The set U_i is open in X, since a_i is isolated in X and U is open in $X \times Y$.

Clearly, the set $F_i = F \times \{a_i\}$ is an open and closed subset of A. Since $U \cap A$ is dense in A, it follows that $U \cap F_i$ is open and dense in F_i. Hence, $U_i \cap F$ is open and dense in F.

Finally, let us show that the set $M = \cap\{U_i : i \in \omega\}$ is contained in F. Assume the contrary. Then we can fix $x \in M \setminus F$. By the definition of U_i, we have $(x, a_i) \in U$, for each $i \in \omega$. It follows that $(x, e) \in \overline{U}$, since $(x, e) \in \overline{\{x\} \times E}$. However, $(x, e) \in B$, since $x \notin F$. Therefore, $B \cap \overline{U} \neq \emptyset$, a contradiction. Hence, $M \subseteq F$.

The space F is countably compact, since F is closed in X. Therefore, M is dense in F, by the Baire Category Theorem. Hence, X is weakly perfect. $\square$

Theorem 4.2. *If X and Z are infinite countably compact Hausdorff spaces such that $X \times Z$ is hereditarily β-normal, then X and Z are first countable and weakly perfect.*

PROOF: By Proposition 1.1, the spaces X and Z are regular. Therefore, there exists an infinite countable discrete subspace E of Z. Since Z is countably compact, the set E accumulates to some point $e \in Z \setminus E$. Put $Y = E \cup \{e\}$. Clearly, the space $X \times Y$ is hereditarily β-normal and X, Y satisfy all assumptions in Proposition 4.1. It follows that X is weakly perfect. Since every point in a weakly perfect space is a G_δ, and every regular countably compact space satisfies the first axiom countability at every G_δ-point, it follows that the space X is first countable. Similarly, Z is weakly perfect and first countable. $\square$

A space X is said to be a space *with a dense G_δ-diagonal* if the diagonal in the product $X \times X$ contains a dense G_δ-subset of $X \times X$. In [5] Arhangel'skii and Kočinac showed that if X is a Čech-complete space with a dense G_δ-diagonal, then in every closed subspace of X there exists a dense subspace metrizable by a complete metric. Thus, since a compact Hausdorff space is Čech-complete, we have the following:

Corollary 4.3. *Let X be a compact Hausdorff space such that $X \times X \times X$ is hereditarily β-normal. Then in every closed subspace of X there exists a dense subspace metrizable by a complete metric.*

Notice that, under CH, Proposition 4.1 and Theorem 4.2 do not generalize to hereditarily α-normal spaces.

For example, take the one point compactification $a(Y)$ of the locally compact Hausdorff space Y constructed in the proof of Theorem 3.3. Since Y is not normal, the space $a(Y)$ is not first countable at the point a. Clearly, $(a(Y))^n$ is hereditarily separable, for each $n \in \omega$, since Y^n is hereditarily separable. Therefore, by Corollary 3.2, $(a(Y))^n$ is hereditarily α-normal, for each $n \in \omega$.

A space X is said to satisfy *Property wD* ([18]) if for every infinite closed discrete subspace C of X, there exists a discrete collection $\{U_n : n \in \omega\}$ of open subsets of X such that each U_n meets C in exactly one point.

Proposition 4.4. *Every β-normal space X has Property wD.*

PROOF: Let A be any infinite closed discrete subspace of X. We may assume that A is countable. Fix a one-to-one enumeration of A: $A = \{a_n : n \in \omega\}$. By Proposition 1.1, the space X is regular. Using this, it is easy to construct (by induction) a disjoint family of open neighborhoods U_n of the points a_n in X. Put $U = \bigcup\{U_n : n \in \omega\}$. Then U is an open set and U contains A. By Proposition 2.1, we can find an open set V such that $A \cap V$ is dense in A and $\overline{V} \subseteq U$. Since A is discrete, it follows that $A \subseteq V$. Put $V_n = V \cap U_n$, for each $n \in \omega$. Then $\{V_n : n \in \omega\}$ is, obviously, a discrete collection of open sets witnessing that X has Property wD. □

Note, that the proof of Proposition 4.4 shows actually more: if X is a β-normal space and $\{x_n : n \in \omega\}$ is a sequence of pairwise distinct points in X such that $\{x_n : n \in \omega\}$ is a closed discrete subspace of X, then there exists a discrete family $\{V_n : n \in \omega\}$ of open sets in X such that $x_n \in V_n$, for each $n \in \omega$. This property is sometimes called *Property D*, it obviously implies Property wD.

Theorem 4.5. *Let X be a compact Hausdorff space and x a non-isolated point in X. Then the following conditions are equivalent:*

1. *$(X \times X) \setminus \{(x, x)\}$ has Property wD, and there exists at least one sequence in $X \setminus \{x\}$ converging to x;*
2. *X is first countable at x.*

PROOF: ($\rightarrow$) Let $S = \{x_n : n \in \omega\}$ be a sequence in $X \setminus \{x\}$ converging to x. Put $A = \{x\} \times S$ and $B = (X \setminus \{x\}) \times \{x\}$. We also put $z_i = (x, x_i)$, for each $i \in \omega$.

Note, that $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$, and $\overline{A} \cap \overline{B} = \{(x,x)\}$. Put $Z = (X \times X) \setminus (\overline{A} \cap \overline{B})$. The set Z is open in $X \times X$, and A and B are closed in Z. Since Z has Property wD, there exists $\{U_i : i \in \omega\}$, a discrete family of open subsets of Z (also open in X) such that U_i meets exactly one element z_{n_i} of A. Obviously, U_i can be chosen to be disjoint from B. Since the space Z is regular, there exists a discrete (in Z) family $\gamma = \{V_i : i \in \omega\}$ of open subsets of Z such that $z_{n_i} \in V_i \subseteq \overline{V_i} \subseteq U_i$, for each $i \in \omega$. Since $B \cap U_i = \emptyset$ and γ is discrete in Z, we have $B \cap \overline{\bigcup_{i\in\omega} V_i} = \emptyset$. Now, put $W_i = \{y \in X : (y, x_{n_i}) \subseteq V_i\}$, for each $i \in \omega$. The set W_i is open in X, for each $i \in \omega$.

Claim: $\bigcap_{i\in\omega} W_i = \{x\}$.

Suppose not. Then there exists $w \in \bigcap_{i\in\omega} W_i \setminus \{x\}$. Thus, for all $i \in \omega$, $(w, x_{n_i}) \in V_i \subseteq \bigcup_{i\in\omega} V_i$ and $x \in \overline{S} \setminus S$. Hence, $(w,x) \in \overline{\{x\} \times S}$, and therefore, $(w,x) \in \overline{\bigcup_{i\in\omega} V_i}$. But this is a contradiction, since $(w,x) \in B$.

($\leftarrow$) Since X is first countable at x and x is not isolated, there exists a sequence $S = \{x_i : i \in \omega\}$ in $X \setminus \{x\}$ converging to x. Clearly, $(X \times X) \setminus \{(x,x)\} = Y$ is σ-compact. Therefore, Y is normal and has Property wD. $\square$

Remark 4.6. After this paper was submitted to Commentationes Mathematicae Universitatis Carolinae, its results were presented and discussed at seminars on Topology in Prague, and in Oxford, Ohio. E. Murtinová from Prague and D. Burke from Oxford independently answered Question 1 (by providing relevant examples). L. Ludwig and P. Szeptycki showed consistency of the existence of a β-normal non-normal Tychonoff space, and E. Murtinová constructed such an example in ZFC. Thus, only Questions 6, 7, 8 and 9 remain unanswered at present.

References

[1] Arhangel'skii A.V., *Divisibility and cleavability of spaces*, in: W. Gähler, H. Herrlich, G. Preuss, ed-s, Recent Developments in General Topology and its Applications, pp. 13–26. Mathematical Research 67, Akademie Verlag, 1992.

[2] Arhangel'skii A.V., *Some recent advances and open problems in general topology*, Uspekhi Mat. Nauk. **52:5** (1997), 45–70; English translation in Russian Math. Surveys **52:5** (1997), 929–953.

[3] Arhangel'skii A.V., *Topological Function Spaces*, Dordrecht; Boston: Kluwer Academic Publishers, 1992.

[4] Arhangel'skii A.V., *Normality and dense subspaces*, to appear in Proc. Amer. Math. Soc. 2001.

[5] Arhangel'skii A.V., Kočinac L., *On a dense G_δ-diagonal*, Publ. Inst. Math. (Beograd) (N.S.) **47** (61) (1990), 121–126.

[6] Baturov D.P.,, *Subspaces of function spaces*, Vestnik Moskov. Univ. Ser. Mat. Mech. **4** (1987), 66–69.

[7] Baturov D.P., *Normality in dense subspaces of products*, Topology Appl. **36** (1990), 111–116.

[8] Blair R.L., *Spaces in which special sets are z-embedded*, Canad. J. Math. **28:4** (1976), 673–690.

[9] Bockstein M.F., *Un theoréme de séparabilité pour les produits topologiques*, Fund. Math. **35** (1948), 242–246.

[10] Engelking R., *General Topology*, Heldermann-Verlag, Berlin, 1989.

[11] Heath R.W., *On a question of Ljubiša Kočinac*, Publ. Inst. Math. (Beograd) (N.S.) **46** (60) (1989), 193–195.

[12] Jones F.B., *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. **43** (1937), 671–677.

[13] Jones F.B., *Hereditarily separable, non-completely regular spaces*, in: Topology Conference (Virginia Polytech. Inst. and State Univ., Blacksburg, Va., 1973), pp. 149–152. Lecture Notes in Math. 375, Springer, Berlin, 1974.

[14] Katětov M., *Complete normality of Cartesian products*, Fund. Math. **35** (1948), 271–274.

[15] Kočinac L., *An example of a new class of spaces*, Mat. Vesnik **35:2** (1983), 145–150.

[16] Mycielski J., *α-incompactness of N^α*, Bull. Acad. Polon. Sci. Ser. Math. Astr. Phys. **12** (1964), 437–438.

[17] Negrepontis S., *Banach spaces and topology*, in: The Handbook of Set Theoretic Topology, North Holland, 1984, pp. 1045–1142.

[18] Nyikos P., *Axioms, theorems, and problems related to the Jones lemma*, General topology and modern analysis (Proc. Conf., Univ. California, Riverside, Calif., 1980), pp. 441–449, Academic Press, New York-London, 1981.

[19] Ščepin E.V., *Real functions and spaces that are nearly normal*, Siberian Math. J. **13** (1972), 820–829.

[20] Ščepin E.V., *On topological products, groups, and a new class of spaces more general than metric spaces*, Soviet Math. Dokl. **17:1** (1976), 152–155.

[21] Singal M.K., Shashi Prabha Arya, *Almost normal and almost completely regular spaces*, Glasnik Mat. Ser. III **5** (25) (1970), 141–152.

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