Distributional chaos on tree maps: the star case

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Abstract. Let $\mathbb{X} = \{z \in \mathbb{C} : z^n \in [0,1]\}$, $n \in \mathbb{N}$, and let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map having the branching point fixed. We prove that f is distributionally chaotic iff the topological entropy of f is positive.

Keywords: distributional chaos, topological entropy, star maps

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1. Introduction

Let (X, d) be a compact metric space and let $C(X)$ be the set of continuous maps $f : X \to X$. The pair (X, f) is called a discrete dynamical system. For any $x \in X$, the sequence $(f^i(x))_{i=0}^{\infty}$ is called the *trajectory of* x (also *orbit of* x). For $x, y \in X$, denote $\delta_{x,y}(i) = d(f^i(x), f^i(y))$ for $i \geq 0$. This paper deals with several notions of chaos for discrete dynamical systems. All these notions are closely related to $\delta_{x,y}(i)$ for $x, y \in X$.

The first notion we introduce is distributional chaos. For any set B we denote by $\mathrm{Card}(B)$ its cardinality. As usual, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, respectively. For any $t \in \mathbb{R}^+$ and any $n \in \mathbb{N}$, let

$$\xi(x, y, t, n, f) = \sum_{i=0}^{n-1} \chi_{[0,t)}(\delta_{x,y}(i)) = \mathrm{Card}(\{i : 0 \leq i \leq n-1 \text{ and } \delta_{x,y}(i) < t\}),$$

where $\chi_{[0,t)}$ is the characteristic function of the interval $[0,t)$. Let us define the upper and lower distribution functions as:

$$F^*_{x,y}(t) = \limsup_{n\to\infty} \frac{1}{n} \xi(x, y, t, n, f)$$

and

$$F_{x,y}(t) = \liminf_{n\to\infty} \frac{1}{n} \xi(x, y, t, n, f).$$

Both $F_{x,y}$ and $F^*_{x,y}$ are non-decreasing functions satisfying $F^*_{x,y}(t) = F_{x,y}(t) = 0$ for all $t < 0$ and $F^*_{x,y}(t) = F_{x,y}(t) = 1$ for all $t > \mathrm{diam}(X)$, where $\mathrm{diam}(X)$

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denotes the diameter of X. We identify distribution functions which are indistinguishable in the L^1 metric. So, we can choose $F^*_{x,y}$ and $F_{x,y}$ to be left-continuous. A function $f \in C(X)$ is said to be *distributionally chaotic* if there are x, $y \in X$ such that $\chi_{(0,\infty)} = F^*_{x,y} > F_{x,y}$ (see [17]). We will use capital letters to denote distribution functions.

A more common definition of chaos can be given as follows (see [13] or [18]). A subset $S \subset X$ is called *scrambled* for f if for any $x, y \in S$, $x \neq y$, it holds that $\limsup_{n\to\infty} \delta_{x,y}(n) > 0$ and $\liminf_{n\to\infty} \delta_{x,y}(n) = 0$. We say that f is *chaotic in the sense of Li-Yorke* if there is an uncountable scrambled set for f.

There is another way of measuring different behavior of trajectories. This measure is given by *topological entropy* (see [1] or [2]). For each positive integer n and for any pair of points x and y we denote $d_n(x,y) = \max\{\delta_{x,y}(i) : 0 \leq i \leq n-1\}$. A finite set E is called (n,ϵ)-separated if for all $x, y \in E$, $x \neq y$, it holds that $d_n(x,y) > \epsilon$. Let $s_n(f,\epsilon)$ be the maximal cardinality of an (n,ϵ)-separated set. The *topological entropy* of f is defined by

$$h(f) := \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(f,\epsilon).$$

When one-dimensional maps ($X = I = [0,1]$) are concerned, the relationship between distributional chaos and topological entropy is stated by the following result (see [17]).

Theorem 1. *Let $f \in C(I)$.*

(a) *If $h(f) = 0$, then $F_{x,y} = F^*_{x,y}$ for all $x, y \in I$. Moreover, if $\liminf_{i\to\infty} \delta_{x,y}(i) = 0$ then $F_{x,y} = \chi_{(0,\infty)}$.*

(b) *If $h(f) > 0$, then there exist x, y and t in I such that $\chi_{(0,\infty)} = F^*_{x,y}(t) > F_{x,y}(t)$.*

Additionally, if $f \in C(X)$ is distributionally chaotic, then it is chaotic in the sense of Li-Yorke (see e.g. [17]). In the interval case, Li-Yorke chaotic maps with zero topological entropy, and hence non-distributionally chaotic, can be found in [18].

In the setting of two-dimensional maps the situation is more complicated. In general, Theorem 1 does not hold. More precisely, it was shown in [10] and [11] that there are distributionally chaotic maps with zero topological entropy. Even more, there is a wider definition of distributional chaos and there is a two-dimensional map which is distributionally chaotic in this new sense and non-chaotic in the sense of Li-Yorke [5] (both definitions of distributional chaos are equivalent for interval maps [17]).

For circle maps Theorem 1 holds (see [15]). Additionally, following [12], there are Li-Yorke chaotic circle maps which are not distributionally chaotic. So, the one dimensional case remains open. More precisely, does Theorem 1 hold for continuous maps defined on finite graphs?

In this paper we consider the n-star $\mathbb{X} := \{z \in \mathbb{C} : z^n \in [0,1]\}$, $n \in \mathbb{N}$. Continuous maps of the n-star have been studied in the literature (see [3], [4] or [6]) from the point of view of topological dynamics. Let $0 \in \mathbb{C}$ be the *branching point* of $\mathbb{X}$. Let $\mathbf{X}_0 := \{f \in C(\mathbb{X}) : f(0) = 0\}$. The aim of this paper is to prove the following result:

Theorem 2. *Let $f \in \mathbf{X}_0$.*

(a) *If $h(f) = 0$, then $F_{x,y} = F^*_{x,y}$ for all $x, y \in \mathbb{X}$. If, in addition, $\liminf_{i\to\infty} \delta_{x,y}(i) = 0$ then $F_{x,y} = \chi_{(0,\infty)}$.*
(b) *If $h(f) > 0$, then there are $x, y \in \mathbb{X}$ and $t \in \mathbb{R}^+$ such that $\chi_{(0,\infty)} = F^*_{x,y}(t) > F_{x,y}(t)$.*

The paper is organized as follows. Below we introduce additional basic notation and definitions. Section 2 is devoted to prove useful technical results which are used in the last section, where the main result is proved.

Recall that a point $x \in X$ is *periodic* if $f^i(x) = x$ for some $i \in \mathbb{N}$. Let $\mathrm{Per}(f)$ denote the set of periodic points of f. For $x \in X$, let $\omega(x, f)$ denote the set of limit points of the sequence $(f^i(x))_{i=0}^{\infty}$. $\omega(x, f)$ is called the *omega limit set* of f at x. Let $\omega(f) = \bigcup_{x\in X} \omega(x, f)$ be the *omega limit set* of f.

Before proving our results, we need some information on n-star maps. The components $\mathbb{X} \setminus \{0\}$ are called *branches* of $\mathbb{X}$. We denote them by $B_1, B_2, \dots, B_n$. Clearly, for $1 \le i \le n$, the closure of B_i fulfills $\overline{B}_i = B_i \cup \{0\}$. For $x \in \mathbb{X}$ we denote its modulus by $|x|$. For $x, y \in \overline{B}_i$, $1 \le i \le n$, $|x| < |y|$, we define the *interval* $[x, y]$ by $[x, y] := \{z \in \overline{B}_i : |x| \le |z| \le |y|\}$. Similarly we define the intervals $[x, y)$, $(x, y]$ and (x, y). If $x, y \in \overline{B}_i$ for some $1 \le i \le n$ and $|x| < |y|$ (resp. $|x| \le |y|$), we will write $x < y$ (resp. $x \le y$). Notice that $\overline{B}_i$ is an interval for $1 \le i \le n$. Consider the following metric on $\mathbb{X}$. For any $x, y \in \mathbb{X}$, let $d(x, y) = |x - y|$ if x and y lie in the same branch and let $d(x, y) = |x| + |y|$ if x and y do not lie in the same branch.

Finally, we recall the notion of horseshoe (see [16]). Let $k \in \mathbb{N}$. We say that f has a *k-horseshoe* if there is a closed interval J and k closed subintervals $J_i \subset J$, $1 \le i \le k$, with pairwise disjoint interiors and such that $J \subseteq f(J_i)$ for $1 \le i \le k$.

2. Preliminary results

We begin with the following lemma partially proved in [15]. For each $x \in \mathbb{R}$, $[x]$ denotes the greatest integer such that $[x] \le x$. Lemma 3 can be found in [9]. Since it is an unpublished reference, we include the proof.

Lemma 3. *Let (X, d) be a compact metric space and let $f \in C(X)$. Fix $k \in \mathbb{N}$ and $x, y \in X$. Then, $F_{x,y} < F^*_{x,y} = \chi_{(0,\infty)}$ iff $(F^k)_{x,y} < (F^k)^*_{x,y} = \chi_{(0,\infty)}$.*

Proof: The sufficiency condition was proved in Lemma 3.3 from [15]. So, we must prove the necessity condition. To this end, fix $t \in \mathbb{R}^+$. Since

$$\xi(x,y,t,n,f) \leq \sum_{i=0}^{k-1} \xi(f^i(x), f^i(y), t, [n/k]+1, f^k),$$

it follows from the definitions that $1 \leq \frac{1}{k}\sum_{i=0}^{k-1}(F^k)^*_{f^i(x),f^i(y)}(t)$. This gives us

$$(F^k)^*_{f^i(x),f^i(y)}(t) = 1$$

for $i = 0,1,\dots,k-1$. On the other hand, assume that $(F^k)_{x,y}(t) = 1$ for all $t > 0$. Since f is uniformly continuous, for any $\varepsilon > 0$ there is $\delta > 0$ ($\delta \leq \varepsilon$) such that $d(x,y) < \delta$ implies $\delta_i(x,y) < \varepsilon$ for $1 \leq i < k$. Then $(F^k)_{x,y}(\delta) = 1$ implies $F_{x,y}(\varepsilon) = 1$, which leads us to a contradiction. Thus $(F^k)_{x,y} < 1$ and the proof concludes. □

We start with the n-star case by formulating the following two results. Their proofs are immediate.

Lemma 4. *Let $f \in C(\mathbb{X})$ and let $[x,y] \subset \overline{B}_i \subset \mathbb{X}$ be an interval, $1 \leq i \leq n$. If $f(x), f(y) \in \overline{B}_i$ and either $f(x) < x$ and $y < f(y)$ or $x < f(x)$ and $f(y) < y$, then there is $z \in [x,y]$ such that $f(z) = z$.*

Lemma 5. *Let $f \in \mathbf{X}_0$. Assume there is $i \in \{1,2,\dots,n\}$ such that $z \leq f(z) \in \overline{B}_i$ for all $z \in \overline{B}_i$. Then there are at least two fixed points in $\overline{B}_i$.*

We assign a code to any $x \in \mathbb{X}$ as follows; let $s(x) := (s_i)_{i=0}^{\infty} \in \{0,1,2,\dots,n\}^{\mathbb{N}}$ be defined by $s_i := j$ iff $f^i(x) \in B_j$ for some $j \in \{1,2,\dots,n\}$, and $s_i := 0$ iff $f^i(x) = 0$. We say that $s(x)$ is *eventually constant* if there is $k \in \mathbb{N}$ such that $s_i = s_k$ for all $i \geq k$; if $k = 0$ then we say that $s(x)$ is *constant*. We say that $f \in \mathbf{X}_0$ has property P if $s(x)$ is a constant code for any $x \in \mathrm{Per}(f)$.

Lemma 6. *Let $f \in \mathbf{X}_0$ have property* P. *Let $x \in \mathbb{X}$ be such that $s(x)$ is not eventually constant. Then $\lim_{k\to\infty} f^k(x) = 0$, that is, $\omega(x,f) = \{0\}$.*

Proof: For $1 \leq i \leq n$, let $\mathcal{A}_i := \{j \in \mathbb{N} : f^j(x) \in B_i\}$. Notice that given $i \in \{1,2,\dots,n\}$, $\mathcal{A}_i$ can be finite or empty for some i.

Let $k, l \in \mathcal{A}_i$, $k < l$, be such that $f^{k+1}(x) \notin B_i$. We claim that $f^l(x) < f^k(x)$. Assume the contrary and denote

$$A := \{z \in \overline{B}_i : z < f^k(x) \text{ and } f(z) \in \overline{B}_i\}$$

and

$$B := \{z \in \overline{B}_i : f^k(x) < z \text{ and } f(z) \in \overline{B}_i\}.$$

Since $0 \in A \neq \emptyset$, let $a := \sup A$. It is clear that $a < f^k(x)$ and $f(a) = 0$. We distinguish two cases: $a \neq 0$ and $a = 0$. First assume that $a \neq 0$. Since $f^{l-k}(a) = 0 < a$ and $f^k(x) < f^l(x) = f^{l-k}(f^k(x))$, by Lemma 4, there is $z \in (a, f^k(x))$ such that $f^{l-k}(z) = z$. Since $f((a, f^k(x))) \cap \overline{B}_i = \emptyset$, $f(z) \notin \overline{B}_i$ and this leads us to a contradiction because z would be a periodic point with $s(z)$ not constant. Secondly, assume that $a = 0$. Again we distinguish two cases: $B \neq \emptyset$ and $B = \emptyset$. If $B \neq \emptyset$ let $b := \inf B$. Similarly to the previous case, $f^k(x) < b$ and $f(b) = 0$. Since $f^{l-k}(b) = 0 < b$ and $f^k(x) < f^l(x) = f^{l-k}(f^k(x))$, again by Lemma 4, there is $z \in (f^k(x), b)$ such that $f^{l-k}(z) = z$, which also provides a contradiction. Finally, assume $B = \emptyset$, which implies $\mathrm{Per}(f) \cap B_i = \emptyset$. If $f^{l-k}(y) < y$ for some $y \in B_i$, we get again by Lemma 4 a fixed point $z \in (y, f^k(x))$ or $z \in (f^k(x), y)$, a contradiction. So $y \leq f^{l-k}(y)$ for all $y \in B_i$. Now, by Lemma 5, there is $z \in B_i$ such that $f^{l-k}(z) = z$, which again provides a contradiction.

Now, let $i \in \{1, 2, \ldots, n\}$ be such that $\mathcal{A}_i$ is infinite. Let $(a_j^i)_{j=1}^{\infty} \subset \mathcal{A}_i$ be such that $f^{a_j^i+1}(x) \notin \overline{B}_i$. Then $(f^{a_j^i}(x))_{j=1}^{\infty}$ is decreasing and therefore it converges. Let $y_i = \lim_{j\to\infty} f^{a_j^i}(x)$ (notice also that $y_i = \lim_{\substack{j \in A_i \\ j\to\infty}} f^j(x)$). Clearly, $\omega(x, f) = \{y_i : \mathcal{A}_i \text{ is infinite}\}$. Then $\omega(x, f)$ is a periodic orbit. Since f has no periodic orbits contained in more than one branch, we conclude that $\omega(x, f) = \{0\}$, which completes the proof. $\square$

Let $f \in \mathbf{X}_0$. Define $f_i : \overline{B}_i \to \overline{B}_i$, $i \in \{1, 2, \ldots, n\}$, by

$$f_i(x) = \begin{cases} f(x) & \text{if } f(x) \in \overline{B}_i; \\ 0 & \text{if } f(x) \notin \overline{B}_i. \end{cases}$$

Note f_i is conjugate to an interval map $p : [0,1] \to [0,1]$ such that $p(0) = 0$ (recall that f_i is conjugate to p if there is a homeomorphism $\phi : \overline{B}_i \to [0,1]$ such that $p \circ \phi = \phi \circ f_i$). For $i, j \in \{1, 2, \ldots, n\}$, $i < j$, define $f_{i,j} : \overline{B}_i \cup \overline{B}_j \to \overline{B}_i \cup \overline{B}_j$ by

$$f_{i,j}(x) = \begin{cases} f_i(x) & \text{if } x \in \overline{B}_i; \\ f_j(x) & \text{if } x \in \overline{B}_j. \end{cases}$$

Notice that $f_{i,j}$ is conjugate to an interval map $g : [-1,1] \to [-1,1]$ with $g(0) = 0$. Define $\widetilde{f} \in \mathbf{X}_0$ by $\widetilde{f}(x) = f_i(x)$ if $x \in \overline{B}_i$.

Lemma 7. *Let $f \in \mathbf{X}_0$ have property* P. *Then for all $j \in \{1, 2, \ldots, n\}$, $\omega(f) \cap B_j = \omega(f_j)$. In particular, $\omega(f) = \bigcup_{j=1}^n \omega(f_j)$.*

Proof: Let $y \in \mathbb{X}$. If $s(y)$ is not eventually constant, then $\omega(y, f) = \{0\}$ and there is nothing to prove. So, assume that $s(y)$ is eventually constant, that is, there are $k \in \mathbb{N}$ and $j \in \{1, 2, \ldots, n\}$ such that $s_i = s_k = j$ for any integer $i \geq k$.

Let $y_k := f^k(y) \in B_j$. Clearly $(f^i(y_k))_{i=0}^{\infty} \subset B_j$ and hence $f^i(y_k) = f_j^i(y_k)$ for all $i \in \mathbb{N}$. Then $\omega(y, f) = \omega(y_k, f) = \omega(y_k, f_j)$. □

Lemma 8. *Let $f \in \mathbf{X}_0$ have property* P. *Let $f_{i,j}$ and f_i be the maps defined above. If $h(f) = 0$, then $h(f_{i,j}) = h(f_i) = 0$ for all $i, j \in \{1, 2, \ldots, n\}$.*

Proof: By [2, Chapter 4], it holds that $h(f_{i,j}) = \max\{h(f_i), h(f_j)\}$ for all $i, j \in \{1, 2, \ldots, n\}$. So, we must prove that $h(f_i) = 0$ for all $i \in \{1, 2, \ldots, n\}$. On the other hand, it follows by [8, Corollary DG2] and Lemma 7 that

$$h(f_i) = \sup_{x \in B_i} h(f_i|_{\omega(x,f_i)}) \le \sup_{x \in \mathbb{X}} h(f|_{\omega(x,f)}) = h(f) = 0,$$

which completes the proof. □

3. Main result

Proof of Theorem 2. (a). Assume that $h(f) = 0$. Following the proof of Theorem 1.5 from [4] we see that f^N has property P for $N = n!(n-1)!\ldots 2!$. Additionally, by [2, Chapter 4], $h(f^N) = Nh(f) = 0$. Due to Lemma 3, we can assume without loss of generality that f has any periodic orbit contained in one branch, that is, f has property P.

Now, fix $x, y \in \mathbb{X}$, $x \neq y$. According to Lemmas 6 and 7, we distinguish three cases: (a1) $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^n(y) = 0$; (a2) $\lim_{n\to\infty} f^n(x) = 0$ and there is an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $f^n(y) \in B_i$, $i \in \{1, 2, \ldots, n\}$; (a3) there is an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $f^n(y) \in B_i$ and $f^n(x) \in B_j$ for some $i, j \in \{1, 2, \ldots, n\}$. If (a1) happens, then clearly $F_{x,y} = \chi_{(0,\infty)}$. If (a3) happens, then it is easy to check that $F_{x,y} = F_{f^{n_0}(x), f^{n_0}(y)} = (F_{i,j})_{f^{n_0}(x), f^{n_0}(y)}$ and $F^*_{x,y} = F^*_{f^{n_0}(x), f^{n_0}(y)} = (F_{i,j})^*_{f^{n_0}(x), f^{n_0}(y)}$. This, together with Lemma 8 and Theorem 1, give us $F_{x,y} = F^*_{x,y}$. So, assume that (a2) holds and fix $\varepsilon > 0$. By Lemma 8, $h(f_i) = 0$. Then by [17, Lemma 4.2], there is a periodic point of f_i, $p \in \overline{B}_i$, such that $F_{y,p}(t) = F_{f^{n_0}(y),p}(t) > 1 - \varepsilon$ for $t \ge \varepsilon$. On the other hand, it is clear that $F^*_{0,x}(t) = F_{0,x}(t) = 1 > 1 - \varepsilon$. Then, following the proof of Proposition 4.3 from [17], we see that $F_{x,y} = F^*_{x,y}$.

Now assume that $\liminf_{i\to\infty} \delta_{x,y}(i) = 0$ and let us prove $F_{x,y} = \chi_{(0,\infty)}$. By Lemmas 6 and 7, we distinguish two possibilities: (p1) $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^n(y) = 0$; (p2) there is an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ $f^n(y) \in B_i$, and $f^n(x) \in B_i$ with $i \in \{1, 2, \ldots, n\}$. If (p1) happens, then clearly $F_{x,y} = \chi_{(0,\infty)}$. If (p2) happens, then notice that $F_{x,y} = F_{f^{n_0}(x), f^{n_0}(y)} = (F_i)_{f^{n_0}(x), f^{n_0}(y)} = \chi_{(0,\infty)}$ by Lemma 8 and Theorem 1.

(b). Now, assume that $h(f) > 0$. By [16], there is an $l \in \mathbb{N}$ such that f^l has a k-horseshoe. Since $h(f^l) = lh(f) > 0$, by Lemma 3 we may assume that $l = 1$. There is an invariant compact subset Y included in at most two branches such that $f|_Y$ is semiconjugate to a shift map defined on $\Sigma = \{(x_n)_{n=1}^{\infty} : x_n \in \{0,1\}\}$ (see e.g. [7, Chapter 2]). Then, following [14] or [15], it is easy to see that $f|_Y$ is distributionally chaotic. □

Corollary 9. *Let $f : \mathbb{X} \to \mathbb{X}$ be such that $0 \in \mathrm{Per}(f)$. Then f is distributionally chaotic iff $h(f) > 0$.*

Proof: Just apply Lemma 3 and Theorem 2. □

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