Comment.Math.Univ.Carolin. 42,3 (2001)451–458

Discontinuous elliptic problems in $\mathbb{R}^N$ without monotonicity assumptions

Silvia Cingolani, Monica Lazzo

Abstract. We prove existence of a positive, radial solution for a semilinear elliptic problem with a discontinuous nonlinearity. We use an approximating argument which requires no monotonicity assumptions on the nonlinearity.

Keywords: free boundary problem, plasma physics

Classification: 35J60, 35J20, 35R05

1. Introduction and statement of the results

In this paper we are concerned with positive radial solutions of the semilinear elliptic problem

$$(\mathrm{P}_a) \qquad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where $N \geq 3$ and f is a nonnegative function with an upward "jump-discontinuity" at some point $a > 0$. Equations with discontinuous nonlinearities arise very naturally from several problems in mathematical physics; for example, see [3], [17] for models in Plasma Physics and [5], [7], [15], [20] for models in Vortex Theory. For discontinuous problems, several notions of solutions have been proposed in literature (e.g., see [1]). In the present paper, by a positive solution of (P_a) we mean a function $u \in C^1(\mathbb{R}^N, \mathbb{R})$ such that

(i) u is C^2 in $\mathbb{R}^N \setminus T(a)$;
(ii) $u > 0$, u solves pointwise $-\Delta u = f(u)$ in $\mathbb{R}^N \setminus T(a)$ and $\lim\limits_{|x|\to\infty} u(x) = 0$,

where $T(a) \equiv \{x \in \mathbb{R}^N : u(x) = a\}$.

Existence of positive radial solutions to (P_a) has been proved in [4], for a nondecreasing, bounded nonlinearity f. In that paper, the authors apply bifurcation arguments and, roughly speaking, perform two limit procedures. First, they prove

Supported by M.U.R.S.T. Research Funds.

existence of a global branch $S(R)$ of radial solutions in a ball B_R as limit of solutions of approximating smooth problems; then they prove that, as $R \to \infty$, $S(R)$ converges in a suitable sense to a global unbounded branch S of radial solutions to (P_a) with $(0,0) \in S$.

In this paper we aim to obtain existence of positive solutions for possibly unbounded nonlinearities, provided they grow subcritically. The main obstruction in applying the topological arguments in [4] to the case of unbounded nonlinearities seems to be the lack of apriori bounds for the solutions of the approximating problems. Furthermore, we aim to drop the monotonicity assumption on the nonlinearity. In [4], such an assumption is crucial: it allows to apply a generalized maximum principle [24] and to prove that the free boundary $T(a)$ of any $u \in S$ has zero measure, thus u is a solution to (P_a). We remark that the monotonicity of the nonlinear term is a common requirement in several papers dealing with elliptic problems with upward or downward "jump-discontinuous" nonlinearities (see [1], [2], [4], [6], [8]–[12], [15], [23], [27]).

Our main result is the following:

Theorem 1.1. *Assume*

(f_1) $f(s) = 0 < f(a+) \equiv \lim_{s\to a^+} f(s) < +\infty$ *for any* $s \le a$;

(f_2) f *is a nonnegative and continuous function on* $(a, +\infty)$;

(f_3) *there exists* $0 \le p < \frac{N+2}{N-2}$ *such that* $\limsup_{s\to+\infty} f(s)\, s^{-p} < +\infty$.

Then (P_a) *has at least one positive radial solution* u_a *for any* $a > 0$. *Furthermore: there exists* $r_a \ge 0$ *such that* $u_a'(r) = 0$ *for* $0 \le r \le r_a$ *and* $u_a'(r) < 0$ *for* $r > r_a$; *finally* $u_a(0) = \|u_a\|_\infty > a$.

Remark 1.2. The solution u_a found in Theorem 1.1 actually solves the equation $-\Delta u = f(u)$ almost everywhere in $\mathbb{R}^N$. Indeed, the monotonicity properties of u_a imply $\operatorname{meas}\big(T(a)\big) = 0$.

In proving our result, we look for a solution of (P_a) as the limit of solutions of approximating problems in which f is replaced by continuous nonlinearities. We shall prove the monotonicity properties of such limit function required in Theorem 1.1 by using a recent result due to Flucher and Müller [19].

Let us point out that a very wide literature is available on discontinuous elliptic problems. Besides the work based on bifurcation theory that we have already mentioned, we recall [2], [6], [18], where Clarke's dual action principle is used, and [8], [9], [16], where Chang's critical point theory for locally Lipschitz functionals is applied. Several other approaches can be pursued in proving existence results for discontinuous problems, see for instance [14], [22], [25], [26], [29]–[31] and references therein. We believe that the main value in our approach is that it works under minimal hypotheses; indeed, we are able to drop the monotonicity requirement on f, as we already noticed, and we can relax regularity assumptions on f: in [4], as in some other papers quoted above, f is Hölder continuous in $(0,+\infty)$, whereas we only assume f continuous (see Remark 4.4 below).

2. Variational ground states

In this section we briefly recall some definitions and results contained in a recent paper by Flucher and Müller [19].

Consider the semilinear elliptic problem

$$\text{(P)} \qquad \begin{cases} -\Delta u = g(u) & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ satisfies $g(0) = g'(0) = 0$. Let $G(u) = \int_0^u g(s)\,ds$ and $\Sigma = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \|u\| \le 1\}$; as usual, $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the energy norm $\|u\|^2 \equiv \int |\nabla u|^2\,dx$ (when no confusion arises, the integration set $\mathbb{R}^N$ in the integrals will be understood).

A *variational ground state solution* of (P) is a solution of the variational problem for the generalized Sobolev constant

$$S^G \equiv \sup\left\{\int G(u(x))\,dx : u \in \Sigma\right\}.$$

Any variational ground state v satisfies

$$\text{(2.1)} \qquad \int G(v(x))\,dx = S^G, \qquad \|v\| = 1;$$

indeed, if $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $\|v\| < 1$ and $\int G(v(x))\,dx = S^G$, then the function defined by $w(x) = v(\|v\|^{2/(N-2)}x)$ satisfies $\|w\| = 1$ and $\int G(w(x))\,dx = S^G/\|v\|^{2^*} > S^G$, a contradiction.

In [19, Lemma 4], the authors prove monotonicity properties of variational ground state solutions to (P) under a mild growth condition on G, the antiderivative of the nonlinearity g. Among other results, they prove the following one:

Theorem 2.1. *Assume that G is upper semicontinuous, $G \ne 0$ in the L^1 sense and $0 \le G(t) \le c|t|^{2^*}$ for some constant c; moreover, G is nondecreasing in $[0,+\infty)$. Then any positive variational ground state v for S^G is radial with respect to the origin and the function $r \to v(r)$ is strictly decreasing in $\{r : v(r) < v(0)\}$.*

In Section 4 we shall use Theorem 2.1 to obtain some qualitative information on a candidate solution to (P_a).

3. Approximating problems and preliminaries

In order to find solutions of (P_a) we first solve suitable approximating "smooth" problems.

Fix $0 < \delta < a$ and set $a_n = a - \frac{\delta}{n}$ for any integer n; let f_n be the continuous function such that

$$f_n(t) = \begin{cases} f(t), & t \le a_n \ \text{ or } \ t \ge a \\ \text{linear}, & a_n < t < a \end{cases}$$

and let $F(u) = \int_a^u f(t)\,dt$, $F_n(u) = \int_{a_n}^u f_n(t)\,dt$, for any u. For any n, let us consider the problem

$$(\mathrm{P}_{n,a}) \qquad \begin{cases} -\Delta u = f_n(u) & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

A positive solution u_n to $(\mathrm{P}_{n,a})$ can be found, for instance, by solving the variational problem for the generalized Sobolev constant S^{F_n} (cf. Section 2). To this aim, we adapt to our setting some arguments from [13, Section 5].

First, as we are interested in radial solutions, we set the variational problem in $\mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$, the space of radial functions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Let us recall the *Radial Lemma* (see [13], [28]): every radial function $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is almost everywhere equal to a function $U(x)$, continuous for $x \ne 0$, such that

$$|U(x)| \le C_N |x|^{1-\frac{N}{2}} \|u\| \qquad \text{for any } \ x \ne 0, \tag{3.1}$$

where $C_N > 0$ only depends on N. We identify any $u \in \mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$ with U.

By (3.1) there exists $R > 0$, depending only on N, such that for any radial $u \in \Sigma$ we have $|u(x)| < a - \delta$ in $\mathbb{R}^N \setminus B_R(0)$. Then, by (f_1) and (f_3) it is easy to see that there exists $w \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $F_n(u) \le w\,\chi_{B_R(0)}$ for any $n \in \mathbb{N}$ and for any radial $u \in \Sigma$, where $\chi_{B_R(0)}$ denotes the characteristic function of $B_R(0)$. As a consequence, for any such u we have $\int F_n(u) \le \int w\,\chi_{B_R(0)}$, whence $0 < S^F \le S^{F_n} \le S^{F_1} < +\infty$ for any n.

Now, let $v_{n,k} \in \mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$ be a maximizing sequence for S^{F_n}, namely $\|v_{n,k}\| \le 1$ and $\int F_n(v_{n,k}) \to S^{F_n}$ as $k \to \infty$. We can assume that $v_{n,k}$ is nonnegative and nonincreasing: indeed, the Schwarz symmetrization $v^*_{n,k}$ of $v_{n,k}$ is again a maximizing sequence for S^{F_n}. Up to a subsequence, $v_{n,k}$ converges to some v_n, weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$. Plainly, v_n is nonnegative, radially symmetric and nonincreasing. Lebesgue's Theorem applies and gives $\int F_n(v_{n,k}) \to \int F_n(v_n)$ as $k \to \infty$, thus S^{F_n} is achieved in v_n, and v_n is a variational ground state for S^{F_n}.

Since any F_n is a smooth function, v_n is a solution of the Euler-Lagrange equation $-\Delta v_n = \left(2^* S^{F_n}\right)^{-1} f_n(v_n)$ (cf. [19], where the Lagrange multiplier is computed). Finally, a standard bootstrap argument yields $v_n \in C^2(\mathbb{R}^N)$ (cf. [13, Section 5c]).

Now, let us define $u_n(x) = v_n\left((2^* S^{F_n})^{\frac{1}{2}} x\right)$; plainly, u_n is a classical solution of $(\mathrm{P}_{n,a})$. As we are interested in performing limit procedures, let us remark that $\|u_n\|^2 = \left(2^* S^{F_n}\right)^{1-\frac{N}{2}}$, hence

$$c_1 \leq \|u_n\| \leq c_2 \tag{3.2}$$

for any n, where $c_1 = \left(2^* S^{F_1}\right)^{\frac{2-N}{4}}$ and $c_2 = \left(2^* S^{F}\right)^{\frac{2-N}{4}}$.

4. Proof of Theorem 1.1 and remarks

We first prove that the sequence $\{u_n\}$ of solutions of the approximating problems, found in the previous section, converges in the C^1_{loc} sense.

Lemma 4.1. *There exists a radial function u such that $u_n \to u$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to \infty$. Furthermore, $u \in C^{1,\alpha}(\mathbb{R}^N)$ for any $0 < \alpha < 1$.*

Proof: By Ascoli-Arzelà's Theorem, it suffices to prove that the sequence $\|u_n\|_{C^2(\mathbb{R}^N)}$ is bounded. By (3.2) and the Radial Lemma, we can fix $\bar{R} > 0$ such that $f_n(u_n) \equiv 0$ in $|x| \geq \bar{R}$. As u_n solves $(r^{N-1} u_n'(r))' = 0$ for $r > \bar{R}$, easy computations yield $\|u_n\|_{C^2(\mathbb{R}^N \setminus B_{\bar{R}})} \leq C$, for some positive C. On the other hand, u_n solves

$$\begin{cases} \Delta u_n + f_n(u_n) = 0 & \text{in } B_{\bar{R}}, \\ u_n = u_n(\bar{R}) & \text{on } \partial B_{\bar{R}}; \end{cases}$$

by the Radial Lemma and (3.2) again, we get $u_n(\bar{R}) \leq C_N\, \bar{R}^{1-N/2} \|u_n\| \leq C$ (the constant is not necessarily the same). By (f_3), a standard boot-strap argument shows that $\|u_n\|_{L^\infty(B_{\bar{R}})} \leq C$; finally, integrating $(r^{N-1} u_n')' = -r^{N-1} f_n(u_n)$ and the previous estimate yield $\|u_n\|_{C^2(B_{\bar{R}})} \leq C$. $\square$

The previous lemma and the monotonicity of u_n imply $u'(r) \leq 0$ for $r > 0$. Actually, we want to prove that u is strictly decreasing with r, below the level $u(0)$. Without monotonicity assumptions on f, u' may fail to be weakly subharmonic, hence generalized maximum principles (e.g. cf. [21], [24]) do not apply. Thus, in order to study the behaviour of u, we turn the problem around: precisely we show that, up to a scaling, u solves a suitable variational problem and then we apply Theorem 2.1.

Lemma 4.2. *Let u be as in Lemma 4.1 and let $v(x) = u\big((2^* S^F)^{-\frac{1}{2}} x\big)$. Then v is a variational ground state, namely*

$$\int F(v)\,dx = S^F, \qquad \|v\| = 1. \tag{4.1}$$

PROOF: Let v_n be the variational ground state found in Section 3; by construction, $S^{F_n} = \int F_n(v_n)$ and $\|v_n\| = 1$. Up to a subsequence, v_n converges to some v weakly in $D^{1,2}(\mathbb{R}^N)$, almost everywhere in $\mathbb{R}^N$; plainly, $\|v\| \le 1$. By definition, F_n converges to F uniformly, thus $F_n(v_n)$ converges to $F(v)$ almost everywhere in $\mathbb{R}^N$; furthermore, we know from Section 3 that $F_n(v_n) \le w\,\chi_{B_R(0)}$, with $w \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then Lebesgue's Theorem applies and we get

$$S^F \le \lim_{n\to\infty} S^{F_n} = \lim_{n\to\infty} \int F_n(v_n)\,dx = \int F(v)\,dx \le S^F.$$

This proves that v is a variational ground state and (4.1) (cf. (2.1)). Now, let u be as in Lemma 4.1; in particular, u is the pointwise limit of the sequence $u_n(x) = v_n\big((2^* S^{F_n})^{\frac{1}{2}} x\big)$; as a consequence, $u(x) = v\big((2^* S^F)^{\frac{1}{2}} x\big)$ almost everywhere in $\mathbb{R}^N$, which proves the lemma. □

At this point, Theorem 2.1 guarantees that u is strictly decreasing below the top level $u(0)$; in other words, there exists $r_0 \ge 0$ such that $u'(r) = 0$ in $[0, r_0]$ and $u'(r) < 0$ in $]r_0, +\infty[$. If we assume $u(r) \le a$ for any r, the same holds for $v(r)$, hence $S^F = \int F(v) = 0$, a contradiction; as a result, $u(r) > a$ for some $r \ge 0$ and there exists $r_1 > r_0$ such that $u(r_1) = a$. Thus the set $T(a) = \{x \in \mathbb{R}^N : u(x) = a\}$ is exactly the boundary of the ball $B_{r_1}(0)$, and $\operatorname{meas}\big(T(a)\big) = 0$. We are now able to conclude the

PROOF OF THEOREM 1.1: It remains to prove that u solves $-\Delta u = f(u)$ in $\mathbb{R}^N \setminus T(a)$. First remark that

$$\lim_{n\to\infty} f_n\big(u_n(x)\big) = f\big(u(x)\big) \qquad \text{in } \mathbb{R}^N \setminus T(a). \tag{4.2}$$

Indeed, if $u(x) > a$ then for n large $u_n(x) > a$, thus $f_n\big(u_n(x)\big) = f\big(u_n(x)\big) \to f\big(u(x)\big)$ as $n \to \infty$, since f is continuous in $]a, +\infty[$. If $u(x) < a$, for n large $u_n(x) < a_n$, thus $f_n\big(u_n(x)\big) = 0 = f\big(u(x)\big)$. This proves (4.2). At this point, by (3.2) and the Radial Lemma, Lebesgue's Theorem applies and gives

$$\lim_{n\to\infty} \int f_n(u_n)\,\varphi = \int f(u)\,\varphi$$

for any test function φ. As $u_n \to u$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $u \in C^{1,\alpha}(\mathbb{R}^N)$ and the Divergence Theorem applies, we get

$$\int -\Delta u\, \varphi = \int f(u)\, \varphi$$

for any test function φ. As a consequence, u solves $-\Delta u = f(u)$ in any point of continuity of f, namely in $\mathbb{R}^N \setminus T(a)$. By the above remarks, u satisfies the second part of the statement in Theorem 1.1. □

Remark 4.3. The assumption $N \geq 3$ in Theorem 1.1 is not restrictive. Indeed, it is easy to prove that for $N = 2$ and $a > 0$ there are no radial solutions of (P_a), whatever f is. In addition, let us observe that (P_a) has no radial solutions for $f = \text{const} > 0$ and $a = 0$.

Remark 4.4. As we already pointed out in Section 1, several authors studied (P_a) under Hölder continuity assumptions on f (for instance, see [4], [10], [15]). Indeed, when $f \in C^{0,\mu}$ classical regularity results for elliptic equations allow to prove apriori bounds in $C^{1,\mu}$ for the solutions of the approximating problems, which in turn yield the existence of a limit in $C^{1,\mu}$ sense. As we assume f only continuous, we get the apriori bounds that we need to perform the limit procedure by direct estimates on the solutions of the approximating problems.

References

[1] Allegretto W., Nistri P., *Elliptic equations with discontinuous nonlinearities*, T.M.N.A. **2** (1993), 233–251.

[2] Ambrosetti A., Badiale M., *The dual variational principle and elliptic problems with discontinuous nonlinearities*, J. Math. Anal. Appl. **140.2** (1989), 363–373.

[3] Ambrosetti A., Calahorrano M., Dobarro F., *Remarks on the Grad-Shafranov equation*, Appl. Math. Letters **3** (1990), 9–11.

[4] Ambrosetti A., Calahorrano M., Dobarro F., *Global branching for discontinuous problems*, Comment. Math. Univ. Carolinae **31** (1990), 213–222.

[5] Ambrosetti A., Struwe M., *Existence of steady vortex rings in an ideal fluid*, Arch. Rational Mech. Anal. **108** (1989), 97–109.

[6] Ambrosetti A., Turner R.E.L., *Some discontinuous variational problems*, Differential Integral Equations **1** (1988), 341–349.

[7] Amick C.J., Turner R.E.L., *A global branch of steady vortex rings*, J. Rein. Angew. Math. **384** (1988), 1–23.

[8] Arcoya D., *Positive solutions for semilinear Dirichlet problems in an annulus*, J. Differential Equations **94** (1991), 217–227.

[9] Arcoya D., Calahorrano M., *Some discontinuous problems with a quasilinear operator*, J. Math. Anal. Appl. **187** (1994), 1059–1072.

[10] Arcoya D., Zertiti A., *Global branching for discontinuous problems in the exterior domain of a ball*, Boll. U.M.I. (7) **10A** (1996), 641–652.

[11] Badiale M., *Semilinear elliptic problems in* $\mathbb{R}^N$ *with discontinuous nonlinearities*, Atti Sem. Fis. Univ. Modena **153** (1995), 293–305.

[12] Badiale M., Tarantello G., *Existence and multiplicity results for elliptic problems with critical growth and discontinuous nonlinearities*, Nonlinear Anal. T.M.A. **29** (1997), 639–677.

[13] Berestycki H., Lions P.L., *Nonlinear scalar field equations, I – Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), 313–375.

[14] Carl S., Heikkilä S., *Elliptic equations with discontinuous nonlinearities in* $\mathbb{R}^N$, Nonlinear Anal. T.M.A. **31** (1998), 217–227.

[15] Cerami G., *Soluzioni positive di problemi con parte non lineare discontinua e applicazione a un problema di frontiera libera*, Boll. U.M.I. (6) **2B** (1983), 321–338.

[16] Chang K.C., *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102–129.

[17] Cimatti G., *A nonlinear elliptic eigenvalue problem for the Elenbaas equation*, Boll. U.M.I. (5) **16B** (1979), 555–565.

[18] Clarke F.H., *Periodic solutions of Hamiltonian inclusions*, J. Differential Equations **40** (1981), 1–6; C.R.A.S. **128** (1978), 951–952.

[19] Flucher M., Müller S., *Radial symmetry and decay rate of variational ground states in the zero mass case*, SIAM J. Math. Anal. **29** (1998), 712–719.

[20] Fraenkel L.E., Berger M.S., *A global theory of steady vortex rings in an ideal fluid*, Acta Math. **132** (1974), 13–51.

[21] Gilbarg D., Trudinger N.S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[22] Heikkilä S., *First order discontinuous implicit differential equations with discontinuous boundary conditions*, Nonlinear Anal. T.M.A. **30** (1997), 1753–1761.

[23] Heikkilä S., Lakshmikantham V., *On the method of upper and lower solutions for discontinuous nonlinearities*, Nonlinear Anal. T.M.A. **23** (1994), 265–273.

[24] Littman W., *Generalized subharmonic functions: monotonic approximations and an improved maximum principle*, Ann. Scuola Norm. Sup. Pisa **17** (1963), 207–222.

[25] Marano S., *Existence theorems for a semilinear elliptic boundary value problem*, Ann. Polonici Math. **LX** (1994), 57–67.

[26] Marano S., *Elliptic boundary-value problems with discontinuous nonlinearities*, Set-Valued Anal. **3** (1995), 167–180.

[27] Massabò I., *Positive eigenvalues for elliptic equations with discontinuous nonlinearities*, Boll. U.M.I. (5) **15B** (1978), 814–827.

[28] Strauss W.A., *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162.

[29] Stuart C.A., *Differential equations with discontinuous nonlinearities*, Arch. Rational Mech. Anal. **63** (1976), 59–75.

[30] Stuart C.A., *Maximal and minimal solutions of elliptic differential equations with discontinuous nonlinearities*, Math. Z. **163** (1978), 239–249.

[31] Stuart C.A., Toland J.F., *A variational approach for boundary value problems with discontinuous nonlinearities*, J. London Math. Soc. **21** (1980), 319–328.

Dipartimento Interuniversitario di Matematica, via E. Orabona 4, 70125 Bari, Italy

(*Received* June 5, 2000, *revised* March 1, 2001)