Comment.Math.Univ.Carolin. 42,3 (2001)481–485

A new proof of weighted weak-type inequalities for fractional integrals

D. Cruz-Uribe, SFO

Abstract. We give a new and simpler proof of a two-weight, weak (p,p) inequality for fractional integrals first proved by Cruz-Uribe and Pérez [4].

Keywords: weights, weak-type inequalities, fractional integrals

Classification: 42B20, 42B25

1. Introduction

For $0 < \alpha < n$, the fractional integral operator I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

In [4], Cruz-Uribe and Pérez proved a two-weight, weak-type norm inequality which answered a question posed by Sawyer and Wheeden [9].

Theorem. *Given a pair of weights (u,v), p, $1 < p < \infty$, and α, $0 < \alpha < n$, suppose that for some $r > 1$ and for all cubes Q,*

$$|Q|^{\alpha/n} \left(\frac{1}{|Q|} \int_Q u^r\, dx \right)^{1/rp} \left(\frac{1}{|Q|} \int_Q v^{-p'/p}\, dx \right)^{1/p'} \le C < \infty. \tag{1.1}$$

Then the fractional integral operator I_α satisfies the weak (p,p) inequality

$$u(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \le \frac{C}{t^p} \int_{\mathbb{R}^n} |f|^p v\, dx. \tag{1.2}$$

Their proof of Theorem 1.1 was fairly complex, and depended on a technical lemma resembling a good-λ inequality. The purpose of this paper is to give another, more elementary proof. It is based on three weighted norm inequalities

I want to thank the referee for detecting an error in the first version of this paper.

for the fractional integral operator and the closely related fractional maximal operator,

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{|Q|^{\alpha/n}}{|Q|} \int_Q |f|\, dy.$$

These are stated in Section 2 and the proof of Theorem 1.1 is in Section 3.

Finally, we remark that we believe that condition (1.1) in Theorem 1.1 is not the best possible. In [4], Cruz-Uribe and Pérez proved an analogue of Theorem 1.1 for singular integrals, and in [3] they sharpened this result by replacing the local L^r norm on the left-hand side of (1.1) by the smaller Orlicz space norm $\|\cdot\|_{L(\log L)^{p-1+\delta}}$, $\delta > 0$. (A similar condition is sufficient for a strong (p,p) inequality for fractional integrals. See Pérez [7].) We conjecture that the corresponding weak (p,p) result holds for fractional integrals. For a partial result, see Cruz-Uribe and Fiorenza [2].

2. Preliminary results

The first result we need is due to Muckenhoupt and Wheeden [6, p. 262].

Theorem 2.1. *Given α, $0 < \alpha < n$, and a weight $w \in A_\infty$, there exists a constant C, depending only on α, n and the A_∞ constant of w, such that for all functions f,*

$$\sup_{t>0} t\, w(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \le C \sup_{t>0} t\, w(\{x \in \mathbb{R}^n : M_\alpha f(x) > t\}).$$

The second result is due to Sawyer [8, p. 285]; for a simple proof see [1].

Theorem 2.2. *Given α, $0 < \alpha < n$, and a weight w, there exists a constant C, depending only on α and n, such that for all functions f,*

$$w(\{x \in \mathbb{R}^n : M_\alpha f(x) > t\}) \le \frac{C}{t} \int_{\mathbb{R}^n} |f| M_\alpha w\, dx.$$

The third result is a special case of a theorem due to Pérez [7, p. 668].

Theorem 2.3. *Given α, $0 < \alpha < n$, $r > 1$, and a pair of weights (u, v) such that* (1.1) *holds, then there exists a constant s, $1 < s < \min(n/\alpha, r)$, and a constant C such that for all functions f,*

$$\int_{\mathbb{R}^n} (M_{\alpha s} f)^{p'/s} v^{-p'/p}\, dx \le C \int_{\mathbb{R}^n} |f|^{p'/s} u^{-p'/p}\, dx. \tag{2.1}$$

Proof: Given s, $1 < s < \min(n/\alpha, r)$, condition (1.1) is equivalent to

$$|Q|^{s\alpha/n} \left(\frac{1}{|Q|} \int_Q [v^{-s/p}]^{p'/s}\, dx \right)^{s/p'} \left(\frac{1}{|Q|} \int_Q [u^{s/p}]^{rp/s}\, dx \right)^{s/rp} \le C.$$

Pérez showed that this implies (2.1), provided that $(rp/s)' < p'/s$. This is true for $s = 1$, so by continuity it is true for $s > 1$ sufficiently small. $\square$

3. Proof of Theorem 1.1

The proof requires one lemma.

Lemma 3.1. *Given α, $0 < \alpha < n$, and s, $1 < s < n/\alpha$, then for all non-negative, locally integrable functions g, $M_\alpha(M(g^s)^{1/s})(x) \leq CM_{\alpha s}(g^s)(x)^{1/s}$, where M is the Hardy-Littlewood maximal operator.*

Proof: Our proof is modeled on the proof of a similar result in García-Cuerva and Rubio de Francia [5, p. 158]. It will suffice to show that there exists a constant C such that for any x and any cube Q containing x,

$$\frac{|Q|^{\alpha/n}}{|Q|}\int_Q M(g^s)(y)^{1/s}\,dy \leq CM_{\alpha s}(g^s)(x)^{1/s}.$$

Let $g = g_1 + g_2$, where $g_1 = g\chi_{3Q}$. Then $M(g^s)(x)^{1/s} \leq M(g_1^s)(x)^{1/s} + M(g_2^s)(x)^{1/s}$. Since M is weak $(1,1)$, by Kolmogorov's inequality (cf. [5, p. 485]),

$$\begin{aligned}\frac{|Q|^{\alpha/n}}{|Q|}\int_Q M(g_1^s)(y)^{1/s}\,dy &\leq C|Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n} g_1^s\,dy\right)^{1/s}\\ &\leq C|Q|^{\alpha/n}\left(\frac{1}{|3Q|}\int_{3Q} g^s\,dy\right)^{1/s}\\ &\leq CM_{\alpha s}(g^s)(x)^{1/s}.\end{aligned}$$

Further, $M(g_2^s)^{1/s} \in A_1$ with a constant independent of g, (see [5, p. 158]), so

$$\frac{|Q|^{\alpha/n}}{|Q|}\int_Q M(g_2^s)(y)^{1/s}\,dy \leq C|Q|^{\alpha/n}M(g_2^s)(x)^{1/s}.$$

There exists a cube P containing x such that

$$M(g_2^s)(x) \leq \frac{2}{|P|}\int_P g_2^s\,dy.$$

Since P must intersect $\mathbb{R}^n \setminus 3Q$, $Q \subset 3P$. Therefore,

$$|Q|^{\alpha/n}M(g_2^s)(x)^{1/s} \leq C|3P|^{\alpha/n}\left(\frac{1}{|3P|}\int_{3P} g^s\,dy\right)^{1/s} \leq CM_{\alpha s}(g^s)(x)^{1/s}.$$

□

We can now prove Theorem 1.1. Fix p, $1 < p < \infty$, and a function $f \in L^p(v)$; by a standard argument we may assume that f is non-negative, bounded and has compact support. For each $t > 0$, let $E_t = \{x \in \mathbb{R}^n : I_\alpha f(x) > t\}$. By duality there exists a function $G_t \in L^{p'}$, $\|G_t\|_{p'} = 1$, such that

$$u(E_t)^{1/p} = \|u^{1/p}\chi_{E_t}\|_p = \int_{E_t} u^{1/p} G_t \, dx.$$

Fix $s > 1$ as in Theorem 2.3, and let $w_t = M(u^{s/p}G_t^s)^{1/s}$. Then $w_t \in A_1 \subset A_\infty$, and the A_∞ constant of w_t depends only on s. Hence, by Theorems 2.1 and 2.2, and by Lemma 3.1,

$$\begin{aligned}
\sup_{t>0} t\, u(E_t)^{1/p} &= \sup_{t>0} t \int_{E_t} u^{1/p} G_t \, dx \\
&\leq \sup_{t>0} t\, w_t(E_t) \\
&\leq C \sup_{t>0} t\, w_t(\{x \in \mathbb{R}^n : M_\alpha f(x) > t\}) \\
&\leq C \sup_{t>0} \int_{\mathbb{R}^n} f M_\alpha(w_t)\, dx \\
&\leq C \sup_{t>0} \int_{\mathbb{R}^n} f M_{\alpha s}(u^{s/p}G_t^s)^{1/s}\, dx.
\end{aligned}$$

Then, by Hölder's inequality and Theorem 2.3,

$$\begin{aligned}
\sup_{t>0} t\, u(E_t)^{1/p} &\leq C \sup_{t>0} \left(\int_{\mathbb{R}^n} f^p v\, dx\right)^{1/p} \left(\int_{\mathbb{R}^n} M_{\alpha s}(u^{s/p}G_t^s)^{p'/s} v^{-p'/p}\, dx\right)^{1/p'} \\
&\leq C \sup_{t>0} \left(\int_{\mathbb{R}^n} f^p v\, dx\right)^{1/p} \left(\int_{\mathbb{R}^n} (u^{s/p}G_t^s)^{p'/s} u^{-p'/p}\, dx\right)^{1/p'} \\
&= C \sup_{t>0} \left(\int_{\mathbb{R}^n} f^p v\, dx\right)^{1/p}.
\end{aligned}$$

References

[1] Cruz-Uribe D., SFO, *New proofs of two-weight norm inequalities for the maximal operator*, Georgian Math. J. **7** (2000), 33–42.

[2] Cruz-Uribe D., SFO, Fiorenza A., *The A_∞ property for Young functions and weighted norm inequalities*, Houston J. Math., to appear.

[3] Cruz-Uribe D., SFO, Pérez C., *Sharp two-weight, weak-type norm inequalities for singular integral operators*, Math. Res. Lett. **6** (1999), 417–428.

[4] Cruz-Uribe D., SFO, Pérez C., *Two-weight, weak-type norm inequalities for fractional integrals, Calderón-Zygmund operators and commutators*, Indiana Math. J. **49** (2000), 697–721.

[5] García-Cuerva J., Rubio de Francia J.L., *Weighted Norm Inequalities and Related Topics*, North Holland Math. Studies 116, North Holland, Amsterdam, 1985.

[6] Muckenhoupt B., Wheeden R., *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), 261–274.

[7] Pérez C., *Two weighted inequalities for potential and fractional type maximal operators*, Indiana Math. J. **43** (1994), 663–683.
[8] Sawyer E.T., *Weighted norm inequalities for fractional maximal operators*, 1980 Seminar on Harmonic Analysis, CMS Conf. Proc. 1, pp. 283–309, Amer. Math. Soc., Providence, 1981.
[9] Sawyer E.T., Wheeden R., *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. Math. **114** (1992), 813–874.

Department of Mathematics, Trinity College, Hartford, CT 06106-3100, USA
E-mail: david.cruzuribe@mail.trincoll.edu

(*Received* November 22, 2000, *revised* May 24, 20010)