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Isotype subgroups of mixed groups

Charles Megibben, William Ullery

Abstract. In this paper, we initiate the study of various classes of isotype subgroups of global mixed groups. Our goal is to advance the theory of Σ-isotype subgroups to a level comparable to its status in the simpler contexts of torsion-free and p-local mixed groups. Given the history of those theories, one anticipates that definitive results are to be found only when attention is restricted to global k-groups, the prototype being global groups with decomposition bases. A large portion of this paper is devoted to showing that primitive elements proliferate in Σ-isotype subgroups of such groups. This allows us to establish the fundamental fact that finite rank Σ-isotype subgroups of k-groups are themselves k-groups.

Keywords: global k-group, Σ-isotype subgroup, $*$-isotype subgroup, knice subgroup, primitive element, $*$-valuated coproduct

Classification: 20K21, 20K27

1. Primitive elements, $*$-valuated coproducts and knice subgroups

This section is mainly expository. Here we establish notation and review facts concerning primitive elements and knice subgroups that will be needed in the sequel. Those readers familiar with [HM2] and [HM4] may wish to skip this section and return only as necessary.

Let $\mathcal{O}_\infty$ denote the class of ordinals with the symbol ∞ adjoined as a maximal element, with the convention that $\infty < \infty$. If $\mathbb{P}$ denotes the set of rational primes, by a *height matrix* we mean a $\mathbb{P} \times \omega$ matrix $M = [m_{p,i}]$, where $m_{p,i} \in \mathcal{O}_\infty$ and $m_{p,i} < m_{p,i+1}$ for all $p \in \mathbb{P}$ and $i < \omega$. A *height sequence* $\alpha = \{\alpha_i\}_{i<\omega}$ is any sequence in $\mathcal{O}_\infty$ with $\alpha_i < \alpha_{i+1}$ for all i. Thus, the *p-row* M_p of a height matrix M is a height sequence. Note that the set of positive integers acts multiplicatively on the classes of height matrices and height sequences in the usual way; for example, if $|n|_p = j$ is the height in $\mathbb{Z}$ of the positive integer n at the prime p, then the height matrix nM has p-row $\{m_{p,i+j}\}_{i<\omega}$. Furthermore, the ordering of $\mathcal{O}_\infty$ induces in a pointwise manner the lattice relations $\leq$ and $\wedge$ on the classes of height matrices and sequences.

If x is an element of an abelian group G, we write $|x|_p$ for the height of x at the prime p. That is, $|x|_p = \sigma$ where σ is the smallest ordinal with $x \notin p^{\sigma+1}G$; if no such σ exists, set $|x|_p = \infty$. With each $x \in G$ we associate the height matrix $\|x\|$ whose (p, i)-entry is $|p^i x|_p$. Note that $\|nx\| = n\|x\|$ for each positive integer n.

When necessary to avoid confusion, we at times affix superscripts to indicate the group in which heights are computed. For example, if H is a subgroup of G, $\|x+H\|^{G/H}$ denotes the height matrix of the coset $x+H$ as computed in G/H. If $x \in H$ and $p \in \mathbb{P}$, the meaning of the expressions $|x|_p^H$, $\|x\|_p^H$ and $\|x\|^H$ should be clear.

We now assume once and for all that G is an additively written (possibly mixed) abelian group. For every height matrix M, we let $G(M)$ denote the subgroup of G consisting of all x with $\|x\| \geq M$. Two height matrices M and N are said to be *quasi-equivalent*, and we write $M \sim N$, if there are positive integers k and l such that $kM \geq N$ and $lN \geq M$. It is important yet completely elementary to observe that $M \sim N$ implies that $M_p = N_p$ for almost all primes p. If M is now any height matrix not quasi-equivalent to $\overline{\infty}$, the height matrix with all entries ∞, we define

$$G(M^*) = \langle x \in G(M) : \|x\| \nsim M \rangle.$$

On the other hand, if $M \sim \overline{\infty}$ we let $G(M^*)$ be the maximal torsion subgroup of $G(M)$. For every height sequence $\alpha = \{\alpha_i\}_{i<\omega}$ and $p \in \mathbb{P}$, $G(\alpha^*, p)$ is the subgroup generated by those x such that $\|x\|_p \geq \alpha$ but $|p^i x|_p \neq \alpha_i$ for infinitely many $i < \omega$. Finally, we set

$$G(M^*, p) = G(M) \cap [G(M^*) + G(M_p^*, p)].$$

Definition 1.1. Call an element $x \in G$ *primitive* if for each height matrix M, prime p and positive integer n, $nx \in G(M^*, p)$ implies that either $\|x\| \nsim M$ or $|p^i nx|_p \neq m_{p,i}$ for infinitely many $i < \omega$.

It is clear that a primitive element must have infinite order. Recall that a direct sum $A = \bigoplus_{i \in I} A_i$ of subgroups of G is a *valuated coproduct* if $A \cap G(M) = \bigoplus_{i \in I}(A_i \cap G(M))$ for all height matrices M. The following refinement of this concept is necessary for our purposes.

Definition 1.2. A valuated coproduct $A = \bigoplus_{i \in I} A_i$ in G is called $*$-*valuated* if $A \cap F = \bigoplus_{i \in I}(A_i \cap F)$ for every subgroup F of the form $G(M^*)$, $G(\alpha^*, p)$ or $G(M^*, p)$.

Knice subgroups were introduced in [HM2] and were used in [HM4] to give an Axiom 3 characterization of global Warfield groups. For their definition, a version of niceness more suited to the global setting is needed. Here and throughout the remainder of this paper, a subgroup N of G is called a *nice* subgroup if for each prime p and ordinal σ, the cokernel of the natural map

$$(p^\sigma G + N)/N \rightarrowtail p^\sigma (G/N)$$

contains no element of order p.

Definition 1.3. A subgroup N of G is a *knice subgroup* if the following conditions are satisfied.

(a) N is a nice subgroup of G.
(b) To each finite subset S of G there corresponds a (possibly empty) set of primitive elements $\{x_1, x_2, \dots, x_n\}$ such that

$$N' = N \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$$

is a $*$-valuated coproduct for which $m\langle S \rangle \subseteq N'$ for some positive integer m.

We say that G is a (global) *k-group* if the trivial subgroup 0 is a knice subgroup. Since nonzero multiples of primitive elements are primitive, it follows that if G is a k-group and if $x \in G$ has infinite order, then there exist a positive integer m and primitive elements $x_1, x_2, \dots, x_n \in G$ such that

$$mx = x_1 + x_2 + \cdots + x_n$$

and

$$\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$$

is a $*$-valuated coproduct.

We conclude this section with three lemmas upon which much of our subsequent work rests. Our first is a useful characterization of knice subgroups.

Lemma 1.4 ([HM4]). *A subgroup N of the group G is a knice subgroup of G if and only if the following conditions are satisfied.*

(1) *N is a nice subgroup of G.*
(2) *G/N is a k-group.*
(3) *To each $g \in G$ there corresponds a positive integer m such that the coset $mg + N$ contains an element x with $\|x\|^G = \|mg + N\|^{G/N}$.*

It will be convenient to have the following notation. If $x, y \in G$ and if p is a prime, we say $\|x\|_p$ and $\|y\|_p$ are *quasi-equal*, and write $\|x\|_p \approx \|y\|_p$, if $\|p^e x\|_p = \|p^e y\|_p$ for some nonnegative integer e. Note that $\approx$ is a transitive relation in the sense that if $\|x\|_p \approx \|y\|_p$ and $\|y\|_p \approx \|z\|_p$, then $\|x\|_p \approx \|z\|_p$.

Lemma 1.5 ([HM2]). (1) *Suppose $A = \langle x \rangle \oplus B$ is a $*$-valuated coproduct in G with x a primitive element. If $y = x + z$ for some $z \in B$ and if $\|y\| = \|x\|$, then y is primitive and $A = \langle y \rangle \oplus B$ is a $*$-valuated coproduct.*

(2) *If $x = x_1 + x_2 + \cdots + x_n$ where $\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is a $*$-valuated coproduct, and if the x_i's are primitive elements with mutually quasi-equivalent height matrices, then x is primitive if and only if, for each prime p, there is some $i \leq n$ such that $\|x\|_p \approx \|x_i\|_p$.*

(3) If $N' = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle \oplus N$ is a $*$-valuated coproduct in G where N is a knice subgroup of G and all the x_i's are primitive, then N' is a knice subgroup of G.

Finally, a slight modification of the second half of the proof of Proposition 1.7 in [HM4] yields the following.

Lemma 1.6. *Suppose that N is a knice subgroup of G and $y \in G$ is such that $y + N$ is primitive in G/N. If $\|y\|^G = \|y + N\|^{G/N}$, then y is primitive and $\langle y \rangle \oplus N$ is a $*$-valuated coproduct.*

2. Σ-isotype and $*$-isotype subgroups

Recall that a subgroup H of a group G is an *isotype subgroup* if $H \cap p^\sigma G = p^\sigma H$ for all primes p and ordinals σ; or equivalently, $H \cap G(M) = H(M)$ for all height matrices M. Note that H is isotype in G if and only if $\|h\|^H = \|h\|^G$ for all $h \in H$. We say that H is Σ-*isotype* in G if

$$H \cap \sum_{i=1}^{n} G(M_i) = \sum_{i=1}^{n} H(M_i)$$

for each finite collection of height matrices $M_1, M_2, \ldots, M_n$. Σ-isotype subgroups were introduced in the torsion-free setting by [HM3], and considered at length in the local setting by [HMU].

In this section, we begin by establishing some basic properties of isotype and Σ-isotype subgroups that culminate in Theorem 2.5 below. Our first lemma and its corollary will prove to be very useful.

Lemma 2.1. *Suppose H is an isotype subgroup of the group G and that $h \in H$ has finite order. If $h \in \sum_{i=1}^{n} G(M_i)$ for some height matrices $M_1, M_2, \ldots, M_n$, then $h \in \sum_{i=1}^{n} H(M_i)$. Therefore, every isotype torsion subgroup of G is Σ-isotype.*

Proof: Decompose the torsion subgroup of H into its primary components and write $h = h_1 + h_2 + \cdots + h_r$, where each $h_i \in H$ has finite order $p_i^{\alpha_i}$ for distinct primes $p_1, p_2, \ldots, p_r$. Set $\lambda_j = \prod_{i \neq j} p_i^{\alpha_i}$ for $j = 1, 2, \ldots, r$. Note that $(\lambda_j, p_j) = 1$ and that $\lambda_j h = \lambda_j h_j$. Thus, $\lambda_j h_j \in \sum_{i=1}^{n} G(\lambda_j M_i)$.

To complete the proof, it is enough to show that $h_j \in \sum_{i=1}^{n} H(M_i)$ for each j. So, temporarily fix j and, for convenience of notation, set $p = p_j$. Since $|q^k h_j|_q = \infty$ for all primes $q \neq p$ and $k < \omega$,

$$(*) \qquad \|h_j\|_q \geq (M_i)_q \ \text{ for all primes } q \neq p \ \text{ and } \ i = 1, 2, \ldots, n.$$

For each i, let m_i denote the leading term of the height sequence $(\lambda_j M_i)_p = (M_i)_p$ and select l so that

$$m_l = \min\{m_1, m_2, \ldots, m_n\}.$$

Then, by the triangle inequality, $|\lambda_j h_j|_p = |h_j|_p \geq m_l$. If the order of h_j is p, this last inequality and $(*)$ imply that $\|h_j\| \geq M_l$. In this case,

$$h_j \in H \cap G(M_l) = H(M_l) \subseteq \sum_{i=1}^{n} H(M_i).$$

Proceeding by induction on the order of h_j, we may assume that

$$ph_j = x_1 + x_2 + \cdots + x_n$$

where $x_i \in H(pM_i)$ for $i = 1, 2, \ldots, n$. But H is isotype in G, so there exist elements $h'_1, h'_2, \ldots, h'_n$ such that $h'_i \in H(M_i)$ and $ph'_i = x_i$ for all i. Clearly then

$$z = h_j - (h'_1 + h'_2 + \cdots + h'_n) \in H$$

has order p and $|z|_p \geq m_l$. From what we have just shown above, $z \in H(M_l)$. If we now set $h''_l = z + h'_l$ and $h''_i = h'_i$ for $i \neq l$, then $h''_i \in H(M_i)$ for all i and

$$h_j = h''_1 + h''_2 + \cdots + h''_n \in \sum_{i=1}^{n} H(M_i).$$

As remarked above, this completes the proof. $\square$

Corollary 2.2. *Let H be an isotype subgroup of G. If $h \in H \cap \sum_{i=1}^{n} G(M_i)$ for some height matrices $M_1, M_2, \ldots, M_n$ and if $mh \in \sum_{i=1}^{n} H(mM_i)$ for some positive integer m, then $h \in \sum_{i=1}^{n} H(M_i)$.*

Proof: By hypothesis,

$$mh = h_1 + h_2 + \cdots + h_n$$

where $h_i \in H(mM_i)$ for $i = 1, 2, \ldots, n$. Since H is isotype in G, it follows that $h_i = mh'_i$ for some $h'_i \in H(M_i)$. Then,

$$h' = h - (h'_1 + h'_2 + \cdots + h'_n) \in H$$

is an element of finite order in $\sum_{i=1}^{n} G(M_i)$. By Lemma 2.1, $h' \in \sum_{i=1}^{n} H(M_i)$. Therefore, $h = h' + (h'_1 + h'_2 + \cdots + h'_n) \in \sum_{i=1}^{n} H(M_i)$. $\square$

Lemma 2.3. *Suppose N is a nice subgroup of G and that $a + N \in p^\sigma(G/N)$ for some prime p and ordinal σ. If $a + N$ has finite order p^n in G/N for some $n < \omega$, then $a + N \in p^\sigma G + N/N$.*

Proof: We induct on n. The result is clear if $n = 0$, so assume that $n \geq 1$. Then, $pa + N \in p^{\sigma+1}(G/N)$ has order p^{n-1}. By induction, $pa + N \in p^{\sigma+1}G + N/N$. Thus, $pa + N = pg + N$ for some $g \in p^\sigma G$. Then, $(a - g) + N \in p^\sigma(G/N)$ and $p(a - g) \in N$ certainly implies that $p(a - g) + N \in p^\sigma G + N/N$. Hence, because N is nice, $(a - g) + N \in p^\sigma G + N/N$ so that $(a - g) + N = g' + N$ for some $g' \in p^\sigma G$. Therefore, $a + N = (g + g') + N \in p^\sigma G + N/N$. $\square$

Our next result is a key ingredient in the proof of Theorem 2.5.

Proposition 2.4. *Suppose N is a knice subgroup of G and that H is an isotype subgroup of G that contains N. Then, H/N is isotype in G/N.*

PROOF: Suppose $h \in H$ and $h + N \in (G/N)(M)$ for some height matrix M. To complete the proof, we need to show that $\|h + N\|_p^{H/N} \geq M_p$ for all primes p. Since N is a knice subgroup of G, condition (3) of Lemma 1.4 says that there exist a positive integer m and an element $x \in mh+N$ such that $\|x\|^G = \|mh+N\|^{G/N}$. Thus, since $N \subseteq H$ and H is isotype in G, $x \in H \cap G(mM) = H(mM) = mH(M)$. We conclude therefore that $x = mh'$ for some $h' \in H(M)$ and that $mh + N = mh' + N$.

Temporarily fix p and write $M_p = \{\alpha_0, \alpha_1, \alpha_2, \dots\}$. Then, $p^i(h - h') + N \in p^{\alpha_i}(G/N)$ for all $i < \omega$, and if p^k is the largest power of p that divides m, $p^i(m/p^k)(h - h') + N \in p^{\alpha_i}(G/N)$. Moreover, $p^i(m/p^k)(h - h') + N$ has order a power of p. Since N is nice, we conclude from Lemma 2.3 that

$$p^i(m/p^k)(h - h') + N \in p^{\alpha_i}G + N/N.$$

We can now write $p^i(m/p^k)(h - h') + N = g_i + N$ with $g_i \in p^{\alpha_i}G$. Recalling that $N \subseteq H$, $g_i \in H \cap p^{\alpha_i}G = p^{\alpha_i}H$. Therefore,

$$|p^i(h - h') + N|_p^{H/N} = |p^i(m/p^k)(h - h') + N|_p^{H/N} = |g_i + N|_p^{H/N} \geq \alpha_i.$$

Thus, $\|(h - h') + N\|_p^{H/N} \geq M_p$. But $\|h' + N\|_p^{H/N} \geq M_p$ so that $\|h + N\|_p^{H/N} \geq M_p$, as desired. □

We are now in position to establish the following result. As we shall see, this will play an important role in an inductive proof of Theorem 4.5 below.

Theorem 2.5. *Suppose $N \subseteq H$ where H is an isotype subgroup of G and N is a knice subgroup of G. Then, H is a Σ-isotype subgroup of G if and only if H/N is Σ-isotype in G/N.*

PROOF: Suppose first that H is Σ-isotype in G and that $h+N \in \sum_{i=1}^n (G/N)(M_i)$ for some $h \in H$ and height matrices $M_1, M_2, \dots, M_n$. Write

$$h + N = (g_1 + N) + (g_2 + N) + \dots + (g_n + N)$$

with $g_i \in G$ and $g_i + N \in (G/N)(M_i)$ for $i = 1, 2, \dots, n$. The fact that N is knice together with condition (3) of Lemma 1.4 implies that there exist a positive integer m and $x_1, x_2, \dots, x_n \in G$ such that $x_i \in mg_i + N$ and $\|x_i\| = \|mg_i + N\|$. Therefore, there is an $x \in N \subseteq H$ such that

$$mh + x = x_1 + x_2 + \dots + x_n \in H \cap \sum_{i=1}^n G(mM_i) = \sum_{i=1}^n H(mM_i)$$

and so $mh+N \in \sum_{i=1}^{n}(H/N)(mM_i)$. Because H/N is isotype in G/N by Proposition 2.4, Corollary 2.2 yields $h+N \in \sum_{i=1}^{n}(H/N)(M_i)$.

Conversely, assume that H/N is Σ-isotype in G/N and suppose that

$$h = g_1 + g_2 + \cdots + g_n$$

where $h \in H$ and $g_i \in G(M_i)$ for some height matrices $M_1, M_2, \ldots, M_n$. Then, since N is a knice subgroup, there is a positive integer r and a $*$-valuated coproduct $N' = N \oplus A$ that contains each rg_i. Thus, for each i, we can write $rg_i = x_i + a_i$, with $x_i \in N$ and $a_i \in A$. Notice that $x_i \in H \cap G(rM_i) = H(rM_i)$ since $N \subseteq H$, and $a_i \in G(rM_i)$. Furthermore, $a = a_1 + a_2 + \cdots + a_n \in A \cap H$ and, since H/N is Σ-isotype,

$$a + N = (h_1 + N) + (h_2 + N) + \cdots + (h_n + N)$$

where $h_i + N \in (H/N)(rM_i)$. But A can also be chosen so that N' is a knice subgroup. So, by enlarging A if necessary, there exists a positive multiple l of r such that, for each i, $lh_i = y_i + b_i$ with $y_i \in N$ and $b_i \in A$. Moreover, it follows from Lemma 1.4 that these choices can be arranged so that $\|b_i\| = \|lh_i + N\| \geq mM_i$ where $m = lr$. Now select $z \in N$ such that $a + z = h_1 + h_2 + \cdots + h_n$ and observe that

$$l(a+z) = (y_1 + y_2 + \cdots + y_n) + (b_1 + b_2 + \cdots + b_n).$$

Consequently, $la = b_1 + b_2 + \cdots + b_n$, where $b_i \in H \cap G(mM_i) = H(mM_i)$, and

$$mh = l(x_1 + x_2 + \cdots + x_n) + la = (lx_1 + b_1) + (lx_2 + b_2) + \cdots + (lx_n + b_n)$$

with $lx_i + b_i \in H$ and $\|lx_i + b_i\| = \|lx_i\| \wedge \|b_i\| \geq mM_i$ for all i. Therefore, $mh \in \sum_{i=1}^{n} H(mM_i)$ and $h \in \sum_{i=1}^{n} H(M_i)$ by Corollary 2.2. □

Corollary 2.6. *If H is an isotype knice subgroup of G, then H is a Σ-isotype subgroup.*

PROOF: Since the trivial subgroup is a Σ-isotype subgroup, the conclusion follows by taking $H = N$ in Theorem 2.5. □

It may be of interest to note that an isotype knice subgroup H of a k-group G is more than just a Σ-isotype subgroup; in fact, by Theorem 2.8 in [HM4], H is also a k-group.

We conclude this section with the introduction of a new type of isotypeness that is closely related to the notion of Σ-isotype.

Definition 2.7. An isotype subgroup H of G is called a $*$-*isotype subgroup* if for all height matrices M, height sequences α and primes p, the intersections of H with $G(M^*)$, $G(\alpha^*, p)$ and $G(M^*, p)$ are $H(M^*)$, $H(\alpha^*, p)$ and $H(M^*, p)$, respectively.

Proposition 2.8. *A Σ-isotype subgroup is $*$-isotype.*

PROOF: Suppose that H is a Σ-isotype subgroup of G and that $h \in H \cap G(M^*, p)$ for some height matrix M and prime p. Then certainly $h \in H(M)$. Assuming without loss that $M \not\sim \overline{\infty}$,

$$h = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_m y_m + \beta_1 z_1 + \beta_2 z_2 + \cdots + \beta_n z_n$$

where $\alpha_i, \beta_j \in \mathbb{Z}$ for all i and j, and the y_i's and z_j's are elements of G which satisfy the following properties: $\|y_i\| \not\sim M$ and $\|y_i\| \geq M$ for $i = 1, 2, \ldots, m$, and for $j = 1, 2, \ldots, n$, $\|z_j\|_p \geq M_p$ and $|p^e z_j|_p \neq m_{p,e}$ for infinitely many $e < \omega$. Since H is Σ-isotype, there exist elements $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ in H such that

$$h = a_1 + a_2 + \cdots + a_m + b_1 + b_2 + \cdots + b_n,$$

$\|a_i\| \geq \|\alpha_i y_i\| \geq \|y_i\|$ for all i, and $\|b_j\| \geq \|\beta_j z_j\| \geq \|z_j\|$ for all j. Hence, $\|a_i\| \not\sim M$ and $\|a_i\|^H \geq M$ for all i and we conclude that $a_1 + a_2 + \cdots + a_m \in H(M^*)$. Moreover, it is equally clear that $b_1 + b_2 + \cdots + b_n \in H(M_p^*, p)$. Therefore, $H \cap G(M^*, p) = H(M^*, p)$. The proofs that $H \cap G(M^*) = H(M^*)$ and $H \cap G(\alpha^*, p) = H(\alpha^*, p)$ are similar. $\square$

Proposition 2.9. *Suppose that G is a k-group and that H is a $*$-isotype subgroup of G. If H itself is a k-group, then H is a Σ-isotype subgroup of G.*

PROOF: From the fact that H is $*$-isotype, it follows easily that an element of H is primitive in G if and only if it is primitive in H. Moreover, a direct sum of subgroups of H is $*$-valuated in G if and only if it is $*$-valuated in H. These observations will be used below without further mention.

Now suppose that $h \in H \cap \sum_{i=1}^n G(M_i)$ for some height matrices M_i. Write $h = g_1 + g_2 + \cdots + g_n$ where $g_i \in G(M_i)$ for all i. To show that H is Σ-isotype in G, Lemma 2.1 allows us to assume that h has infinite order. Therefore, since H is a k-group, there is a positive integer m such that $mh = x_1 + x_2 + \cdots + x_r$, where each $x_i \in H$ is primitive and

$$N = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_r \rangle$$

is a $*$-valuated coproduct. By Lemma 1.5(3), N is a knice subgroup of G. So, we have a $*$-valuated coproduct $N' = N \oplus A$ and a positive integer k with $kg_i = y_i + a_i$, where $y_i \in N \subseteq H$ and $a_i \in A$ for each i. But since we may take k to be a multiple of m, $kh \in N$ and $\sum_{i=1}^n a_i \in N \cap A = 0$. Furthermore, $y_i \in H \cap G(kM_i) = H(kM_i)$ for each i and therefore,

$$kh = y_1 + y_2 + \cdots + y_n \in \sum_{i=1}^n H(kM_i).$$

Since H is an isotype subgroup, Corollary 2.2 yields the desired conclusion that $h \in \sum_{i=1}^n H(M_i)$. $\square$

3. Lemmas on Σ-isotype subgroups of k-groups

In this section we prove several results that will be needed in the next section for the construction of primitive elements in Σ-isotype subgroups of k-groups. First, however, we require a few technical preliminaries provided by Lemmas 3.1 through 3.3 below.

Define a relation $\prec$ on the class $\mathcal{M}$ of all height matrices by decreeing that if $M, N \in \mathcal{M}$, then $M \prec N$ means $M \leq kN$ for some positive integer k. Then, $\mathcal{M}$ is "quasi" partially ordered by $\prec$ in the following sense: for all $M, N, K \in \mathcal{M}$,

(1) $M \prec M$;
(2) if $M \prec N$ and $N \prec M$, then $M \sim N$;
(3) if $M \prec N$ and $N \prec K$, then $M \prec K$.

Now, if $\mathcal{M}'$ is a finite collection of height matrices, we say that $M \in \mathcal{M}'$ is a *minimal element* if whenever $N \in \mathcal{M}'$ is such that $N \prec M$, then $N \sim M$. The proof of our first lemma is a routine induction using properties (2) and (3).

Lemma 3.1. *Every finite collection of height matrices contains a minimal element.*

Lemma 3.2. *Suppose*

$$B = G_1 \oplus G_2 \oplus \cdots \oplus G_r \oplus C$$

is a valuated coproduct in a group G such that each G_i is torsion free and every nonzero element of G_i has height matrix quasi-equivalent to M_i for $i = 1, 2, \ldots, r$. Further suppose that $M_i \nsim M_j$ whenever $i \neq j$ and that the G_i's are arranged so that each M_i is minimal in $\{M_i, M_{i+1}, \ldots, M_r\}$. If $a \in B$ is such that $l\|a\| \geq M_i$ for some $i \geq 2$ and positive integer l, then a has no nonzero component in $G_1 \oplus \cdots \oplus G_{i-1}$.

PROOF: Suppose to the contrary that a has a nonzero component $c_j \in G_j$ for some $j \leq i-1$. Then, $M_i \leq l\|a\| \leq l\|c_j\|$ and $l\|c_j\| \sim M_j$. Therefore, there is a positive integer m such that $M_i \leq mM_j$; that is, $M_i \prec M_j$. However, M_j is minimal in $\{M_j, M_{j+1}, \ldots, M_r\}$ and $i \geq j+1$. We conclude that $M_i \sim M_j$, which contradicts the hypothesis $M_i \nsim M_j$. $\square$

Lemma 3.3. *Suppose $N = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is a valuated coproduct where the x_i's are elements of infinite order in G with mutually quasi-equivalent height matrices. Then, each nonzero element of N has height matrix quasi-equivalent to $\|x_1\|$.*

PROOF: Select positive integers k and l such that $\|x_i\| \leq k\|x_1\|$ and $\|x_1\| \leq l\|x_i\|$ for all $i = 1, 2, \ldots, n$. Thus, if $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ is a nonzero element of N with each $\alpha_i \in \mathbb{Z}$, then there is some j with $\alpha_j \neq 0$. Without loss we assume that $\alpha_j > 0$ and obtain

$$\|x\| \leq \|\alpha_j x_j\| \leq k\|\alpha_j x_1\| = k\alpha_j\|x_1\|.$$

Moreover, since $\|x_1\| \le l\|\alpha_i x_i\|$ for all i, it is clear that $\|x_1\| \le l\|x\|$. Therefore, $\|x\| \sim \|x_1\|$. □

To establish the notation for our next result, we assume that G is a global k-group and that H is a Σ-isotype subgroup of G; also, we assume that H is not torsion. If $h \in H$ has infinite order, we can replace h by a suitable nonzero multiple and write $h = x_1 + x_2 + \cdots + x_m$ where each x_i is a primitive element in G and $N = \langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_m\rangle$ is a $*$-valuated coproduct. Because H is Σ-isotype, there are elements $a_1, a_2, \ldots, a_m \in H$ such that

$$h = a_1 + a_2 + \cdots + a_m = x_1 + x_2 + \cdots + x_m$$

and $\|a_i\| \ge \|x_i\|$ for $i = 1, 2, \ldots, m$. Since N is knice in G by Lemma 1.5(3), there is a $*$-valuated coproduct $B = N \oplus C$ and a positive integer k such that $ka_i \in B$ for all i. By Definition 1.3 and Lemma 1.5(3), we may also assume that B is a knice subgroup of G.

Grouping together the x_i's that have quasi-equivalent height matrices, we can write B as the $*$-valuated coproduct

$$B = G_1 \oplus G_2 \oplus \cdots \oplus G_r \oplus C.$$

By Lemma 3.3, each nonzero element of G_i has height matrix quasi-equivalent to a fixed height matrix M_i, and $M_i \not\sim M_j$ for all $i \ne j$. Moreover, Lemma 3.1 allows us to arrange the G_i's so that M_i is minimal in $\{M_i, M_{i+1}, \ldots, M_r\}$ for $i = 1, 2, \ldots, r$. Also, after reindexing if necessary, we may assume that

$$G_1 = \langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_n\rangle$$

for some $n \le m$. Note that if $i \ge n+1$, then $\|a_i\| \ge \|x_i\|$ and $\|x_i\| \sim M_j$ for some $j \in \{2, 3, \ldots, r\}$. Thus, there is a positive integer l such that $l\|ka_i\| \ge M_j$. Since $ka_i \in B$, Lemma 3.2 allows us to conclude that no ka_i has a nonzero component in G_1 whenever $i \ge n+1$. As a consequence,

$$k(a_1 + a_2 + \cdots + a_n) = k(x_1 + x_2 + \cdots + x_n) + z$$

for some $z \in G_2 \oplus \cdots \oplus G_r \oplus C$. Finally, setting $W = G_2 \oplus \cdots \oplus G_r \oplus C$, we arrive at the following result.

Lemma 3.4. *If H is a nontorsion Σ-isotype subgroup of a global k-group G, then there is a $*$-valuated coproduct $B = \langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_n\rangle \oplus W$ such that B is knice in G and the following conditions are satisfied.*

(a) *Each x_i is primitive and $\|x_1\| \sim \|x_2\| \sim \cdots \sim \|x_n\|$.*
(b) *There exist $a_1, a_2, \ldots, a_n \in H$ such that $\|a_i\| \ge \|x_i\|$ for all i and $ka_i \in B$ for some positive integer k.*
(c) *$k(a_1 + a_2 + \cdots + a_n) = k(x_1 + x_2 + \cdots + x_n) + z$ for some $z \in W$.*

Moreover, if we write each ka_i *as*

$$ka_i = c_{i,1}x_1 + c_{i,2}x_2 + \cdots + c_{i,n}x_n + w_i$$

with $c_{i,j} \in \mathbb{Z}$ *and* $w_i \in W$*, then condition* (c) *implies that*

$$c_{1,j} + c_{2,j} + \cdots + c_{n,j} = k \quad \textit{for} \quad j = 1, 2, \ldots, n.$$

Eventually, we will return to consider the consequences of the above result. However, at this juncture, it will be convenient to deal with groups G that satisfy conditions that are slightly weaker than the conclusions of Lemma 3.4.

Definition 3.5. We say that an abelian group G satisfies the *special hypotheses* if it contains a valuated coproduct $B = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle \oplus W$ with the following properties.

(1) Each x_i has infinite order and $\|x_1\| \sim \|x_2\| \sim \cdots \sim \|x_n\|$.
(2) There exist $a_1, a_2, \ldots, a_n \in G$ such that $\|a_i\| \geq \|x_i\|$ for all i.
(3) There is a positive integer k such that, for all $i = 1, 2, \ldots, n$,

$$ka_i = c_{i,1}x_1 + c_{i,2}x_2 + \cdots + c_{i,n}x_n + w_i$$

where $c_{i,j} \in \mathbb{Z}$, $w_i \in W$ and

$$(\dagger) \qquad c_{1,j} + c_{2,j} + \cdots + c_{n,j} = k \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Lemma 3.6. *If* G *satisfies the special hypotheses, then, for each prime* p*, there exist* $i, j \in I = \{1, 2, \ldots, n\}$ *such that* $c_{i,j} \neq 0$ *and*

$$\|ka_i\|_p \approx \|ka_i - w_i\|_p \approx \|c_{i,j}x_j\|_p.$$

Proof: First observe that if $\|ka_i\|_p \approx \|c_{i,j}x_j\|_p$ for some $c_{i,j} \neq 0$, then $\|ka_i\|_p \approx \|ka_i - w_i\|_p$. Indeed,

$$\|ka_i\|_p \leq \|ka_i - w_i\|_p \leq \|c_{i,j}x_j\|_p \approx \|ka_i\|_p$$

implies that $\|p^e ka_i\|_p = \|p^e(ka_i - w_i)\|_p$ for some nonnegative integer e.

The proof that $\|ka_i\|_p \approx \|c_{i,j}x_j\|_p$ for some $c_{i,j} \neq 0$ divides into two different cases depending on the nature of the prime p. Write $k = p^r k'$ where $(p, k') = 1$, set $y = x_1 + x_2 + \cdots + x_n$, and first consider the case where

$$(*) \qquad \|p^t y\|_p \neq \|p^t x_i\|_p$$

for all nonnegative integers t and $i \in I$. Select a subset J of I maximal with respect to the condition:

> If $j_1, j_2 \in J$, there is a corresponding nonnegative integer s such that either $\|p^s x_{j_1}\|_p \leq \|p^s x_{j_2}\|_p$ or $\|p^s x_{j_2}\|_p \leq \|p^s x_{j_1}\|_p$.

Note that J enjoys the following properties.

(i) There is a nonnegative integer e such that $\{\|p^e x_j\|_p : j \in J\}$ is a totally ordered set of height sequences.

(ii) With e as in item (i), there exists $j_0 \in J$ such that $\|p^e x_{j_0}\|_p \leq \|p^e x_j\|_p$ for all $j \in J$.

(iii) J is a nonempty proper subset of I. (J is certainly nonempty since the case $j_1 = j_2$ is not excluded. That J is a proper subset follows from (ii) and $(*)$, together with the fact that $\langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_n\rangle$ is a valuated coproduct.)

Observe that $p^{r+1} \mid c_{i,j_0}$ whenever $i \in I \setminus J$. Indeed, if this were not the case, then $c_{i,j_0} \neq 0$ and, for all $j \in J$,

$$\|p^{e+r} x_j\|_p \geq \|p^{e+r} x_{j_0}\|_p \geq \|p^e c_{i,j_0} x_{j_0}\|_p \geq \|p^e k a_i\|_p = \|p^{e+r} a_i\|_p \geq \|p^{e+r} x_i\|_p.$$

However, by the maximality of J, this contradicts $i \notin J$. We conclude from condition (†) in Definition 3.5 that $p^{r+1} \nmid c_{j,j_0}$ for some $j \in J$. Selecting such a j, $c_{j,j_0} \neq 0$ and

$$\|p^e k a_j\|_p \leq \|p^e c_{j,j_0} x_{j_0}\|_p \leq \|p^e c_{j,j_0} x_j\|_p \leq \|p^e c_{j,j_0} a_j\|_p \leq \|p^e k a_j\|_p,$$

and we have $\|p^e k a_j\|_p = \|p^e c_{j,j_0} x_{j_0}\|_p$. Therefore, $\|k a_j\|_p \approx \|c_{j,j_0} x_{j_0}\|_p$.

It remains to consider the case where $\|p^t y\|_p = \|p^t x_j\|_p$ for some $t < \omega$ and $j \in I$. In this case, $\|p^t x_j\|_p \leq \|p^t x_i\|_p$ for all $i \in I$. From (†), we know that there is an $i \in I$ such that $p^{r+1} \nmid c_{i,j}$. With i so chosen, $c_{i,j} \neq 0$ and

$$\|p^t k a_i\|_p \leq \|p^t c_{i,j} x_j\|_p \leq \|p^t c_{i,j} x_i\|_p \leq \|p^t k x_i\|_p \leq \|p^t k a_i\|_p.$$

Therefore, $\|p^t k a_i\|_p = \|p^t c_{i,j} x_j\|_p$ so that $\|k a_i\|_p \approx \|c_{i,j} x_j\|_p$. □

For our next result, we again assume that G satisfies the special hypotheses and consider elements of the form

$$h = k(t_1 a_1 + t_2 a_2 + \cdots + t_n a_n)$$

where $t_1, t_2, \ldots, t_n \in \mathbb{Z}$ are not all 0. Thus, $h = g + w$ where

$$g = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

with

$$(**) \qquad c_i = t_1 c_{1,i} + t_2 c_{2,i} + \cdots + t_n c_{n,i} \quad \text{for} \quad i = 1, 2, \ldots, n$$

and

$$w = t_1 w_1 + t_2 w_2 + \cdots + t_n w_n.$$

By the relation (†), we may assume that h has been chosen so that not all c_i's are 0. Indeed, one such choice is where all the t_i's are 1 so that all the c_i's are k.

Lemma 3.7. *Assume that G satisfies the special hypotheses and that $c_{i,j} \neq 0$ for some i and j in $\{1, 2, \ldots, n\}$. Then, in the above notation, $\|g\| \sim \|ka_i - w_i\|$, $\|h\| \sim \|ka_i\|$ and $\|h\| \sim \|g\|$.*

Proof: By Lemma 3.3, every nonzero element of the valuated coproduct $\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ has height matrix quasi-equivalent to $\|x_1\|$. In particular, $\|g\| \sim \|x_1\|$ and, since $c_{i,j} \neq 0$, $\|ka_i - w_i\| \sim \|x_1\|$. Therefore, $\|g\| \sim \|ka_i - w_i\|$.

To see that $\|h\| \sim \|ka_i\|$, observe that $\|x_1\| \sim \|kx_i\| \leq \|ka_i\| \leq \|c_{i,j}x_j\| \sim \|x_1\|$ implies that $\|ka_i\| \sim \|x_1\|$. On the other hand, the relations $\|h\| \leq \|g\| \sim \|x_1\|$ and

$$\begin{aligned}\|h\| \geq \|kt_1a_1\| \wedge \|kt_2a_2\| \wedge \cdots \wedge \|kt_na_n\| &\geq \|kt_1x_1\| \wedge \|kt_2x_2\| \wedge \cdots \wedge \|kt_nx_n\| \\ &= \|k(t_1x_1 + t_2x_2 + \cdots + t_nx_n)\| \sim \|x_1\|\end{aligned}$$

imply that $\|h\| \sim \|x_1\|$. Therefore, $\|h\| \sim \|ka_i\|$.

Finally, we show that $\|h\| \sim \|g\|$. First, as observed above $\|g\| \sim \|x_1\|$ and, since $t_i \neq 0$ for some i, Lemma 3.3 implies that $M = \|t_1x_1 + t_2x_2 + \cdots + t_nx_n\| \sim \|x_1\| \sim \|g\|$. But $\|x_i\| \leq \|kx_i\| \leq \|ka_i\| \leq \|w_i\|$ for $i = 1, 2, \ldots, n$ and we have that

$$N = \|w\| = \|t_1w_1 + t_2w_2 + \cdots + t_nw_n\| \geq \|t_1x_1\| \wedge \|t_2x_2\| \wedge \cdots \wedge \|t_nx_n\| = M.$$

It follows that $M \wedge mN = M$ for every positive integer m. So selecting m such that $m\|g\| \geq M$, we get

$$m\|h\| = \|mh\| = \|mg + mw\| = \|mg\| \wedge \|mw\| \geq M \wedge mN = M.$$

But from $M \sim \|g\|$, there is also a positive integer l with $lM \geq \|g\| \geq \|h\|$. We conclude that $\|h\| \sim M \sim \|g\|$. □

Observe that the equations (∗∗) (in the discussion preceding the statement of Lemma 3.7) can be reformulated as the matrix equation $A\mathbf{t} = \mathbf{c}$ where $A = [c_{i,j}]^T$, and

$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

are in $\mathbb{Z}^n$. In order to gain better control over this relationship between the vectors $\mathbf{t}$ and $\mathbf{c}$, we require the following version of Cramer's Rule.

Lemma 3.8. *Associated with each nonzero $n \times n$ matrix A with integer entries there is a positive integer d with the following property. Whenever $\mathbf{c} =$*

$[c_1\ c_2\ \dots\ c_n]^T \in \mathbb{Z}^n$ is such that the matrix equation $A\mathbf{y} = \mathbf{c}$ has a solution $\mathbf{y} = [t_1\ t_2\ \dots\ t_n]^T \in \mathbb{Z}^n$, then there is a solution $\mathbf{y} = [y_1\ y_2\ \dots\ y_n]^T$ where

$$y_1 = t'_1/d,\ y_2 = t'_2/d,\ \dots\ ,\ y_n = t'_n/d$$

and each t'_i is an integral linear combination of $c_1, c_2, \dots, c_n$.

Proof: Let R be the (unique) row-reduced echelon form of A. Then, there is a finite sequence of row-equivalent matrices $A_0 = A, A_1, A_2, \dots, A_r = R$ and vectors $\mathbf{v}_0 = \mathbf{c}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ such that the sequence of augmented matrices

$$[A_0 \mid \mathbf{v}_0] = [A \mid \mathbf{c}],\ [A_1 \mid \mathbf{v}_1],\ [A_2 \mid \mathbf{v}_2],\ \dots\ ,\ [A_r \mid \mathbf{v}_r] = [R \mid \mathbf{v}_r]$$

satisfies the following condition: For $0 \le i < r$, $[A_{i+1} \mid \mathbf{v}_{i+1}]$ is obtained from $[A_i \mid \mathbf{v}_i]$ by a single elementary row operation of one of the three types:

(a) two rows of $[A_i \mid \mathbf{v}_i]$ are interchanged;
(b) a row of $[A_i \mid \mathbf{v}_i]$ is multiplied by the reciprocal of a nonzero entry of A_i;
(c) some multiple of a row of $[A_i \mid \mathbf{v}_i]$ is added to another row of $[A_i \mid \mathbf{v}_i]$, where the multiplier is an entry of A_i.

A routine induction reveals that $\mathbf{v}_r = [s_1/d\ s_2/d\ \dots\ s_n/d]^T$, where each s_i is an integral linear combination of $c_1, c_2, \dots, c_n$, and the fixed integer $d > 0$ depends only on the choice of the sequence of the A_i's and is independent of $\mathbf{c}$. Further note that each solution of $A\mathbf{y} = \mathbf{c}$ can be written as

$$\mathbf{y} = \mathbf{v}_r^* - \alpha_1 \mathbf{z}_1 - \alpha_2 \mathbf{z}_2 - \cdots - \alpha_m \mathbf{z}_m,$$

where $\mathbf{v}_r^*$ has the same entries as $\mathbf{v}_r$ in those positions corresponding to leading variables and 0's elsewhere, $\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_m\}$ is a basis over $\mathbb{Q}$ for the nullspace of A determined by the collection of nonleading variables, and $\alpha_1, \alpha_2, \dots, \alpha_m$ are arbitrary rational parameters. Thus $\mathbf{y} = \mathbf{v}_r^*$ is a solution with the desired properties. □

Call the positive integer d in Lemma 3.8 a *pseudo-determinant* for A.

4. Construction of primitive elements

In this section, we establish a theorem that exhibits, with appropriate hypotheses, the existence of primitive elements in certain finitely generated subgroups of global k-groups. In particular, this theorem in conjunction with Lemma 3.4 implies that every nontorsion Σ-isotype subgroup H of a global k-group G contains a primitive element. In turn, the latter result provides an essential ingredient in the proof of the fundamental Theorem 4.5 below.

We begin by making an important observation that will be utilized repeatedly without further mention. Suppose that x and y are elements of G with quasi-equivalent height matrices. Thus, for any $p \in \mathbb{P}$, there is a nonnegative integer e such that $\|p^e x\|_p \ge \|py\|_p$. Under these circumstances, $\|p^e x + ty\|_p = \|ty\|_p$ whenever $t \in \mathbb{Z}$ and $p \nmid t$. The effect of the factor p in py is that the latter equality holds even when ∞ is involved in the height sequence $\|p^e x + ty\|_p$.

Theorem 4.1. *Let B be a knice subgroup of a global k-group G such that $B = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle \oplus W$ is a $*$-valuated coproduct in G, and where the x_i's are primitive elements with mutually quasi-equivalent height matrices. Suppose also that there are elements $a_1, a_2, \ldots, a_n$ in G with $\|x_i\| \le \|a_i\|$ for $i = 1, 2, \ldots, n$ and such that $k(a_1 + a_2 + \cdots + a_n) = k(x_1 + x_2 + \cdots + x_n) + z$ for some positive integer k and $z \in W$. Then, there exists a primitive element y of G with $y \in \langle a_1, a_2, \ldots, a_n \rangle$.*

In the special case where $k = 1$ and $W = 0$, this theorem has precursors in the simpler contexts of p-local and torsion-free k-groups. In the p-local setting, the x_i's may be selected so that each a_i is primitive (see Lemma 1.2 in [HMU]). When G is torsion-free, $y = a_1 + a_2 + \cdots + a_n$ is itself primitive (see Proposition 2.7 in [HM1]). But for global groups G, these stronger conclusions do not follow.

We begin our discussion of Theorem 4.1 by singling out certain sets of primes. First, since $\|x_1\| \sim \|x_2\| \sim \cdots \sim \|x_n\|$, the set P consisting of all primes p such that $\|x_1\|_p = \|x_2\|_p = \cdots = \|x_n\|_p$ is cofinite in $\mathbb{P}$. The same applies to $P \setminus \Lambda$, where Λ consists of the prime factors of a positive integer d to be specified in the proof of Proposition 4.3 below. Therefore, the complement Δ of $P \setminus \Lambda$ in $\mathbb{P}$ is a finite set of primes.

The proof of Theorem 4.1 is technically difficult and requires two quite different constructions to establish the existence of the desired primitive element in $\langle a_1, a_2, \ldots, a_n \rangle$. The first relies heavily on Lemmas 3.6 and 3.7 and yields a first approximation $h \in \langle a_1, a_2, \ldots, a_n \rangle$ with $\|h\|_p = \|g\|_p$ for all $p \in \Delta$, where g is a primitive element in $\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ constructed simultaneously with h and such that $h - g \in W$. This h is not primitive unless $\|h\| = \|g\|$. By applying Lemma 3.8 with a relevant pseudo-determinant d, we introduce a second $h' \in \langle a_1, a_2, \ldots, a_n \rangle$ with $h' - dg \in W$ and $\|h'\|_p = \|dg\|_p$ for all primes p in $P \setminus \Lambda$. The proof will then be completed by showing that an appropriate linear combination y of h and h' is primitive.

Proposition 4.2. *Given the hypotheses of Theorem* 4.1, *there exist a primitive element $g \in \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ and an $h \in \langle a_1, a_2, \ldots, a_n \rangle$ with $h - g \in W$ and $\|h\|_p = \|g\|_p$ for all $p \in \Delta$.*

Proof: Since $B = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle \oplus W$ is a knice subgroup, by enlarging W and increasing k if necessary, we may assume that B contains each ka_i. In particular, we have, for each $i = 1, 2, \ldots, n$, the equation

$$ka_i = c_{i,1}x_1 + c_{i,2}x_2 + \cdots + c_{i,n}x_n + w_i$$

where $c_{i,j} \in \mathbb{Z}$ and $w_i \in W$ with

$$(\dagger) \qquad c_{1,j} + c_{2,j} + \cdots + c_{n,j} = k \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Thus, G satisfies the special hypotheses (Definition 3.5) so that Lemmas 3.6 and 3.7 are available in the present context.

We shall consider those elements

$$g = c_1x_1 + c_2x_2 + \cdots + c_nx_n \in \langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_n\rangle$$

which are of the special form

$$g = t_1(ka_1 - w_1) + t_2(ka_2 - w_2) + \cdots + t_n(ka_n - w_n)$$

where $t_1, t_2, \ldots, t_n \in \mathbb{Z}$ are not all zero. In view of (†), we may begin our construction with

$$g = k(x_1 + x_2 + \cdots + x_n) = (ka_1 - w_1) + (ka_2 - w_2) + \cdots + (ka_n - w_n)$$

and thus assume that initially that all the integers $c_1, c_2, \ldots, c_n$ and $t_1, t_2, \ldots, t_n$ are nonzero. We associate with each such g the companion element

$$h = k(t_1a_1 + t_2a_2 + \cdots + t_na_n)$$

where clearly $h - g \in W$. Indeed, $h = g + w$ where

$$w = t_1w_1 + t_2w_2 + \cdots + t_nw_n.$$

It is also noteworthy that each c_i can be expressed explicitly in terms of the integers $t_1, t_2, \ldots, t_n$ and the $c_{i,j}$'s. In fact, as noted in Section 3,

$$(**)\qquad c_i = t_1c_{1,i} + t_2c_{2,i} + \cdots + t_nc_{n,i} \ \text{ for } \ i = 1, 2, \ldots, n.$$

We wish to show that it is possible to choose the integers $t_1, t_2, \ldots, t_n$ in such a manner that g is primitive and $\|h\|_p = \|g\|_p$ for all $p \in \Delta$. Without loss of generality, we assume that $\Delta \neq \emptyset$. Indeed if Δ were empty, then $g = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ would be primitive by Lemma 1.5(2) and $\|h\|_p = \|g\|_p$ would be vacuously satisfied for all $p \in \Delta$.

Beginning with g and h as in the previous paragraph, we define

$$Q(g) = \{p \in \Delta : \|h\|_p \approx \|g\|_p \approx \|c_ix_i\|_p \ \text{ for some } \ i = 1, 2, \ldots, n\}.$$

Observe that if $Q(g) = \Delta$, then g is primitive by Lemma 1.5(2) and that, replacing h and g by multiples, we also have $\|h\|_p = \|g\|_p$ for all $p \in \Delta$.

So suppose that $Q(g)$ is a proper subset of Δ and select any prime q with

$$q \in \Delta \ \text{ and } \ q \notin Q(g).$$

From Lemma 3.6 we have $i_0, j_0 \in \{1, 2, \ldots, n\}$ where

$$(1)\qquad \|ka_{i_0}\|_q \approx \|ka_{i_0} - w_{i_0}\|_q \approx \|c_{i_0,j_0}x_{j_0}\|_q$$

and $c_{i_0,j_0} \neq 0$. By Lemma 3.7, $\|g\| \sim \|ka_{i_0} - w_{i_0}\|$ and $\|h\| \sim \|ka_{i_0}\|$. Therefore, we may choose $r < \omega$ such that both

$$\|q^r g\|_q \geq \|q(ka_{i_0} - w_{i_0})\|_q \text{ and } \|q^r h\|_q \geq \|qka_{i_0}\|_q. \tag{2}$$

Similarly, there is a positive integer s with all its prime factors in the finite set $Q(g)$ such that

$$\|s(ka_{i_0} - w_{i_0})\|_p \geq \|pg\|_p \text{ and } \|ska_{i_0}\|_p \geq \|ph\|_p \tag{3}$$

for all $p \in Q(g)$. (In the exceptional case where $Q(g) = \emptyset$, we simply set $s = 1$ and do not require condition (3).)

We now define new elements $\bar{g}$ and $\bar{h}$ by taking

$$\bar{g} = q^r g + s(ka_{i_0} - w_{i_0}) \text{ and } \bar{h} = q^r h + ska_{i_0}.$$

Note that $\bar{g}$ has the same special form as g. Indeed,

$$\bar{g} = \bar{c}_1 x_1 + \bar{c}_2 x_2 + \cdots + \bar{c}_n x_n = \bar{t}_1(ka_1 - w_1) + \bar{t}_2(ka_2 - w_2) + \cdots + \bar{t}_n(ka_n - w_n)$$

where $\bar{t}_{i_0} = q^r t_{i_0} + s$, $\bar{t}_i = q^r t_i$ for $i \neq i_0$, and

$$\bar{c}_i = \bar{t}_1 c_{1,i} + \bar{t}_2 c_{2,i} + \cdots + \bar{t}_n c_{n,i} = q^r c_i + sc_{i_0,i} \text{ for } i = 1, 2, \ldots, n.$$

Also, $\bar{h}$ is the associated companion element for $\bar{g}$ since

$$\bar{h} = k(\bar{t}_1 a_1 + \bar{t}_2 a_2 + \cdots + \bar{t}_n a_n),$$

where $\bar{h} = \bar{g} + \bar{w}$ with

$$\bar{w} = \bar{t}_1 w_1 + \bar{t}_2 w_2 + \cdots + \bar{t}_n w_n = q^r w + sw_{i_0} \in W.$$

Moreover, because all c_i's and t_i's are nonzero, r can be increased if necessary so that all of the $\bar{c}_i$'s and $\bar{t}_i$'s are nonzero.

As is the case with g and h, $\bar{g}$ and $\bar{h}$ also have the associated finite set

$$Q(\bar{g}) = \{p \in \Delta : \|\bar{g}\|_p \approx \|\bar{h}\|_p \approx \|\bar{c}_i x_i\|_p \text{ for some } i = 1, 2, \ldots, n\}.$$

We now proceed to show that $\{q\} \cup Q(g) \subseteq Q(\bar{g})$.

From (1) and (2), it follows that

$$\|\bar{g}\|_q = \|q^r g + s(ka_{i_0} - w_{i_0})\|_q = \|s(ka_{i_0} - w_{i_0})\|_q \approx \|sc_{i_0,j_0} x_{j_0}\|_q. \tag{4}$$

Similarly,

$$\|\bar{h}\|_q = \|q^r h + ska_{i_0}\|_q = \|ska_{i_0}\|_q \approx \|sc_{i_0,j_0}x_{j_0}\|_q. \tag{5}$$

From (1), there is an f with $\|q^f s(ka_{i_0} - w_{i_0})\|_q = \|q^f sc_{i_0,j_0}x_{j_0}\|_q$. Also, by the form of g, $\|c_{j_0}x_{j_0}\| \geq \|g\|$. Thus, from (2)

$$\|q^r(q^f c_{j_0}x_{j_0})\|_q \geq \|q^r(q^f g)\|_q \geq \|q(q^f s(ka_{i_0} - w_{i_0}))\|_q = \|q(q^f sc_{i_0,j_0}x_{j_0})\|_q$$

and so

$$\|q^f s(c_{i_0,j_0}x_{j_0})\|_q = \|q^f(q^r c_{j_0} + sc_{i_0,j_0})x_{j_0}\|_q = \|q^f \bar{c}_{j_0}x_{j_0}\|_q.$$

Therefore, $\|\bar{c}_{j_0}x_{j_0}\|_q \approx \|sc_{i_0,j_0}x_{j_0}\|_q$, and from (4), $\|\bar{g}\|_q \approx \|\bar{c}_{j_0}x_{j_0}\|_q$. Likewise, (5) implies that $\|\bar{h}\|_q \approx \|\bar{c}_{j_0}x_{j_0}\|_q$ and we conclude that $q \in Q(\bar{g})$.

On the other hand, suppose $Q(g) \neq \emptyset$ and $p \in Q(g)$ with $\|g\|_p \approx \|c_l x_l\|_p$. Then, since $p \neq q$, it follows from (3) that

$$\|\bar{g}\|_p = \|q^r g + s(ka_{i_0} - w_{i_0})\|_p = \|q^r g\|_p \approx \|q^r c_l x_l\|_p \tag{6}$$

and similarly

$$\|\bar{h}\|_p = \|q^r h + ska_{i_0}\|_p = \|q^r h\|_p \approx \|q^r c_l x_l\|_p. \tag{7}$$

Now choose e with $\|p^e q^r g\|_p = \|p^e q^r c_l x_l\|_p$ and note that $\|c_{i_0,l}x_l\| \geq \|ka_{i_0} - w_{i_0}\|$. Thus, from (3),

$$\|p^e(sc_{i_0,l}x_l)\|_p \geq \|p^e s(ka_{i_0} - w_{i_0})\|_p \geq \|p^e(pg)\|_p = \|p(p^e c_l x_l)\|_p$$

and so

$$\|p^e q^r c_l x_l\|_p = \|p^e(q^r c_l + sc_{i_0,l})x_l\|_p = \|p^e \bar{c}_l x_l\|_p.$$

Therefore, $\|\bar{c}_l x_l\|_p \approx \|q^r c_l x_l\|_p$, and from (6) and (7), $\|\bar{g}\|_p \approx \|\bar{c}_l x_l\|_p \approx \|\bar{h}\|_p$. Consequently, $p \in Q(\bar{g})$ and hence we have shown that

$$\{q\} \cup Q(g) \subseteq Q(\bar{g}).$$

Since Δ is finite, repetitions of the foregoing construction yield g,h and w of the appropriate forms with $h = g + w$ and $Q(g) = \Delta$. As mentioned previously, such a g must be primitive. Moreover, $\|g\|_p \approx \|h\|_p$ for all $p \in \Delta$ and replacing g and h by suitable nonzero multiples, we have proved Proposition 4.2. □

With g primitive as above, $\langle g \rangle \oplus W$ is a $*$-valuated coproduct and therefore, if we had $\|h\| = \|g\|$, Lemma 1.5(1) would imply that $y = h \in \langle a_1, a_2, \ldots, a_n \rangle$ is the primitive element required to complete the proof of Theorem 4.1. In other words, we need $\|h\|_p = \|g\|_p$ for those primes $p \in P \setminus \Lambda$ in order for the h constructed in Proposition 4.2 to be primitive. Interestingly enough, if the matrix $C = [c_{i,j}]$ is nonsingular, then a simple application of Cramer's Rule shows that h does satisfy this condition. Unfortunately, C may be singular and we find it necessary to construct an auxiliary $h' \in \langle a_1, a_2, \ldots, a_n \rangle$ with a positive integer d such that $\|h'\|_p = \|dg\|_p$ for all $p \in P \setminus \Lambda$. This is achieved in our next proposition by an application of the weak version of Cramer's Rule established in Lemma 3.8, d arising as a pseudo-determinant for the matrix $A = C^T$.

Proposition 4.3. *Assume the hypotheses of Theorem* 4.1 *and let*

$$g \in \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$$

be the primitive element constructed in the proof of Proposition 4.2*. Thus,*

$$g = t_1(ka_1 - w_1) + t_2(ka_2 - w_2) + \cdots + t_n(ka_n - w_n)$$

with $t_1, t_2, \ldots, t_n \in \mathbb{Z}$ *all nonzero, and* $w_1, w_2, \ldots, w_n \in W$*. Then, there exist an*

$$h' \in \langle a_1, a_2, \ldots, a_n \rangle$$

and a positive integer d *such that* $h' - dg \in W$ *and* $\|h'\|_p = \|dg\|_p$ *for all* $p \in P \setminus \Lambda$*, with* Λ *being the set of prime divisors of* d*.*

Proof: As in the proof of the preceding proposition, $g = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ where

$$(**) \qquad c_i = t_1c_{1,i} + t_2c_{2,i} + \cdots + t_nc_{n,i} \ \text{ for } \ i = 1, 2, \ldots, n.$$

As mentioned in Section 3, $(**)$ can be reformulated as a matrix equation $A\mathbf{t} = \mathbf{c}$ where $A = [c_{i,j}]^T$, $\mathbf{t} = [t_1 \ t_2 \ \ldots \ t_n]^T$ and $\mathbf{c} = [c_1 \ c_2 \ \ldots \ c_n]^T$. Now take d to be a pseudo-determinant of A. By Lemma 3.8, $d\mathbf{c} = A\mathbf{t}'$ where $\mathbf{t}' = [t'_1 \ t'_2 \ \ldots \ t'_n]^T \in \mathbb{Z}^n$ and each t'_i is an integral linear combination of $c_1, c_2, \ldots c_n$. Consequently, $dc_i = t'_1c_{1,i} + t'_2c_{2,i} + \cdots + t'_nc_{n,i}$ for each i and

$$dg = t'_1(ka_1 - w_1) + t'_2(ka_2 - w_2) + \cdots + t'_n(ka_n - w_n).$$

It follows then that $h' = dg + w'$ where

$$h' = k(t'_1a_1 + t'_2a_2 + \cdots + t'_na_n) \ \text{ and } \ w' = t'_1w_1 + t'_2w_2 + \cdots + t'_nw_n \in W.$$

We claim that $h' \in \langle a_1, a_2, \ldots, a_n \rangle$ is the required element. Indeed let p be any prime in $P \setminus \Lambda$ and select j such that

$$\alpha = |c_j|_p^{\mathbb{Z}} = \min\{|c_1|_p^{\mathbb{Z}}, |c_2|_p^{\mathbb{Z}}, \ldots, |c_n|_p^{\mathbb{Z}}\}.$$

Then, since $p \nmid d$ and $\|x_1\|_p = \|x_2\|_p = \cdots = \|x_n\|_p$, it follows that

$$\|dg\|_p = \|g\|_p = \|c_1x_1\|_p \wedge \|c_2x_2\|_p \wedge \cdots \wedge \|c_nx_n\|_p = \|c_jx_j\|_p = \|p^\alpha x_j\|_p.$$

From our choice of α, all of $c_1, c_2, \ldots, c_n$ are divisible by p^α. Therefore, since each t'_i is an integral linear combination of $c_1, c_2, \ldots, c_n$, it follows that p^α also divides each of the integers $t'_1, t'_2, \ldots, t'_n$. We then have

$$\|p^\alpha x_j\|_p = \|p^\alpha x_i\|_p \leq \|p^\alpha kx_i\|_p \leq \|p^\alpha ka_i\|_p \leq \|p^\alpha w_i\|_p \leq \|t'_iw_i\|_p$$

for $i = 1, 2, \ldots, n$. Hence,

$$\|w'\|_p = \|t'_1w_1 + t'_2w_2 + \cdots + t'_nw_n\|_p \geq \|p^\alpha x_j\|_p = \|dg\|_p$$

and

$$\|h'\|_p = \|dg + w'\|_p = \|dg\|_p \wedge \|w'\|_p = \|dg\|_p.$$

This completes the proof of Proposition 4.3. $\square$

Proof of Theorem 4.1. We shall now show that an appropriate linear combination of the elements h and h' constructed in the proofs of Proposition 4.2 and Proposition 4.3, respectively, yields the desired primitive element y in $\langle a_1, a_2, \ldots, a_n\rangle$. First, if g is the primitive element constructed in Proposition 4.2, recall that $h = g + w$ and $h' = dg + w'$, where d is the positive integer specified in the proof of Proposition 4.3, and $w, w' \in W$.

From Lemma 3.7, $\|h\| \sim \|g\|$. Note that the proof given there also can be adapted to show that $\|h'\| \sim \|dg\|$; simply replace each t_i by t_i', h by h', g by dg and w by w'. Since clearly $\|g\| \sim \|dg\|$, it follows that $\|h\| \sim \|h'\|$. Moreover, as in the proof of Proposition 4.2, we may assume that $\Delta \neq \emptyset$. Thus, since Δ is finite, there is a positive integer v with all its prime factors in Δ and

$$\text{for all } \ p \in \Delta, \ \|vh'\|_p \geq \|pdh\|_p \ \text{ and } \ p \mid v.$$

Set $Q = \{p \in P \setminus \Lambda : \|g\|_p = \|h\|_p = \|h'\|_p\}$ and let Q' denote the complement of Q in $P \setminus \Lambda$. Note that Q' is finite by virtue of the fact that $\|g\| \sim \|h\| \sim \|h'\|$ and that $\mathbb{P}$ is the disjoint union of Δ, Q' and Q. Now select a positive integer u with the following properties: if $Q' = \emptyset$ we simply set $u = 1$; otherwise we select u such that all its prime factors are in Q' and

$$\text{for all } \ p \in Q', \ \|duh\|_p \geq \|pvh'\|_p \ \text{ and } \ p \mid u.$$

Because u and v are relatively prime, we can select nonzero integers α and β so that $\alpha u + \beta v = 1$. We now introduce the element

$$y = d\alpha uh + \beta vh' \in \langle a_1, a_2, \ldots, a_n\rangle.$$

Note that

$$y = d\alpha u(g+w)+\beta v(dg+w') = d(\alpha u+\beta v)g+(d\alpha uw+\beta vw') = dg+(d\alpha uw+\beta vw').$$

Therefore, since $dg \in \langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_n\rangle$, $d\alpha uw + \beta vw' \in W$, and

$$\langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_n\rangle \oplus W$$

is a $*$-valuated coproduct, $\|dg\| \geq \|y\|$.

We maintain that $\|dg\| = \|y\|$ and consequently, since dg is primitive, y is primitive by Lemma 1.5(1). Thus, we need to verify that $\|dg\|_p = \|y\|_p$ for all p in the disjoint union $\mathbb{P} = \Delta \cup Q' \cup Q$. In order to do this, we consider the three natural cases.

Case (i) $p \in \Delta$. Recall that $\|vh'\|_p \geq \|pdh\|_p$ for all $p \in \Delta$. Moreover, since $p \mid v$, $p \nmid \alpha u$ and we have that $\|\beta vh'\|_p \geq \|vh'\|_p \geq \|pd\alpha uh\|_p$. Therefore,

$$\|y\|_p = \|d\alpha uh + \beta vh'\|_p = \|d\alpha uh\|_p = \|dh\|_p = \|dg\|_p,$$

with the latter equality holding by virtue of Proposition 4.2.

Case (ii) $Q' \neq \emptyset$ and $p \in Q'$. Recall that $\|duh\|_p \geq \|pvh'\|_p$ for all $p \in Q'$. Since $p \mid u$, $p \nmid \beta v$ and we have that $\|d\alpha uh\|_p \geq \|duh\|_p \geq \|p\beta vh'\|_p$. Therefore,

$$\|y\|_p = \|d\alpha uh + \beta vh'\|_p = \|\beta vh'\|_p = \|h'\|_p = \|dg\|_p,$$

with the latter equality holding by Proposition 4.3 since $Q' \subseteq P \setminus \Lambda$.

Case (iii) $p \in Q$. Since $p \nmid d$, $\|dg\|_p = \|h\|_p = \|h'\|_p$ for all $p \in Q$. We now have that $\|y\|_p = \|d\alpha uh + \beta vh'\|_p \geq \|d\alpha uh\|_p \wedge \|\beta vh'\|_p \geq \|h\|_p \wedge \|h'\|_p = \|dg\|_p$. Also, as observed above, $\|dg\| \geq \|y\|$. Therefore, $\|dg\|_p = \|y\|_p$ for all $p \in Q$. $\square$

Assuming now that H is a nontorsion Σ-isotype subgroup of the global k-group G, observe that hypotheses of Theorem 4.1 are made available by Lemma 3.4. Moreover, $\langle a_1, a_2, \dots, a_n\rangle \subseteq H$. Therefore, the following is an immediate consequence of Theorem 4.1.

Corollary 4.4. *Every nontorsion Σ-isotype subgroup of a global k-group contains a primitive element.*

Since Σ-isotype subgroups are $*$-isotype by Proposition 2.8, our final theorem may be viewed as a partial converse of Proposition 2.9. (At this point, the assumption that H is not torsion is unnecessary.)

Theorem 4.5. *Suppose H is a Σ-isotype subgroup of a global k-group G. If H has finite torsion-free rank, then H is a k-group.*

Proof: Since a torsion subgroup is obviously a k-group, we assume that H is not torsion. Suppose that for some integer $n \geq 1$ we have constructed a $*$-valuated coproduct

$$N_n = \langle y_1\rangle \oplus \langle y_2\rangle \oplus \cdots \oplus \langle y_n\rangle$$

where each y_i is a primitive element in H. That this can be done is a consequence of Corollary 4.4. Then, N_n is a knice subgroup of G by Lemma 1.5(3). Moreover, N_n is contained in H. Thus, H/N_n is Σ-isotype in G/N_n by Theorem 2.5 and the latter is a k-group by Lemma 1.4(2). If H/N_n is not torsion, we may again apply Corollary 4.4 to obtain a primitive element $y + N_n \in H/N_n$. By Lemma 1.4(3), there is a positive integer m and an element $y_{n+1} \in my + N_n$ such that $\|y_{n+1}\|^G = \|my + N_n\|^{G/N_n}$. Thus, $\|y_{n+1}\|^G = \|y_{n+1} + N_n\|^{G/N_n}$ and $y_{n+1} + N_n = my + N_n$ is primitive because nonzero multiples of primitive elements are primitive. Now by Lemma 1.6, $y_{n+1} \in H$ is primitive and

$$N_{n+1} = N_n \oplus \langle y_{n+1}\rangle = \langle y_1\rangle \oplus \langle y_2\rangle \oplus \cdots \oplus \langle y_n\rangle \oplus \langle y_{n+1}\rangle$$

is a $*$-valuated coproduct in G, and hence in H. Since H has finite torsion-free rank, repetitions of this construction eventually yield a $*$-valuated coproduct

$$N_r = \langle y_1\rangle \oplus \langle y_2\rangle \oplus \cdots \oplus \langle y_r\rangle$$

where each $y_i \in H$ is primitive and H/N_r is torsion. Therefore, H is a k-group. $\square$

Observe that the set $\{y_1, y_2, \ldots, y_r\}$ constructed in the proof of Theorem 4.5 is a *decomposition basis* for H; that is, each y_i has infinite order and $N_r = \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_r \rangle$ is a valuated coproduct with H/N_r torsion. Therefore, Theorem 3.2(vi) in [HM4] and Theorem 4.5 immediately yield the following.

Corollary 4.6. *Suppose H is a Σ-isotype subgroup of a global k-group. If H is countable and has finite torsion-free rank, then H is a global Warfield group.*

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Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240, USA

E-mail: megibben@math.vanderbilt.edu

Department of Mathematics, Auburn University, Auburn, Alabama 36849, USA

E-mail: ullery@math.auburn.edu

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