An example of a space whose all continuous mappings are almost injective

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Abstract. We show that all continuous maps of a space X onto second countable spaces are pseudo-open if and only if every discrete family of nonempty G_δ-subsets of X is finite. We also prove under CH that there exists a dense subspace X of the real line $\mathbb{R}$, such that every continuous map of X is almost injective and X cannot be represented as $K \cup Y$, where K is compact and Y is countable. This partially answers a question of V.V. Tkachuk in [Tk]. We show that for a compact X, all continuous maps of X onto second countable spaces are almost injective if and only if it is scattered. We give an example of a non-compact space Z such that every continuous map of Z onto a second countable space is almost injective but Z is not scattered.

Keywords: almost compact map, pseudo-open map, almost injective map, discrete family, scattered

Classification: Primary 54C10; Secondary 54D18, 54D20, 54D30, 54E52

0. Introduction

Let X be a topological space. What can be said about the properties of X if we know that all continuous maps on X belong to a given class? Many results of this kind were obtained before 1961 and are now present in textbooks of general topology. It is easy to see that a Tychonoff space X is compact if and only if every continuous map, defined on X, is closed. It is not trivial at all to see that a Hausdorff space X is compact if and only if the projection $X \times Y \to Y$ is closed for any normal space Y ([Ku]).

The first paper which deals systematically with the properties mentioned above is [Tk], where the classes of Tychonoff spaces are considered for which every continuous mapping onto a space with countable base belongs to a given class F. In [Tk] such spaces are called F-projective. A space X is quotient-projective (i.e. every surjective continuous map $f : X \to Y$ is quotient if $w(Y) \leq \omega$) if and only if any discrete family of nonempty G_δ-subsets of X is finite ([Tk]). We extend the same criterion to spaces X such that every continuous map of X onto a second countable space is pseudo-open. In the cited paper Tkachuk proves that

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all continuous maps of a space X onto second countable spaces are m-compact if and only if $|\beta X\backslash X| \leq m$. Starting off from the same idea, we want to find a characterization for more general classes of spaces, for example, the spaces whose continuous mappings onto second countable spaces are almost compact (see 3.19 in [Tk]). A mapping f of a space X onto a space Y is called almost injective if $|\{y \in Y : |f^{-1}(y)| > 1\}| \leq \omega$. We establish that a compact space X is scattered if and only if every continuous map of X onto a second countable space is almost injective. We present an example of a non-compact space Y that is not scattered and every continuous map of Y onto a second countable space is almost injective.

1. Notation and terminology

All spaces are considered to be Tychonoff. If X is a space, then $\tau(X)$ is its topology and $\tau^*(x) = \tau(x)\backslash\{\emptyset\}$. All mappings of the spaces considered here are assumed to be continuous. We denote by $\mathcal{U}$ the class of spaces with a countable base, ω is the first infinite cardinal, $\mathbb{N} = \omega\backslash\{\emptyset\}$ and $\mathbb{Q}$ is the set of rational numbers with the topology induced from $\mathbb{R}$. We denote by $\mathrm{dom}(f)$ the domain of a function f. Let X be a topological space, (Y,d) a metric space and $f : A \to Y$ a continuous mapping defined on a dense subset A of X. We say that the oscillation of the mapping f at the point $x \in X$ is equal to zero if, for every $\epsilon > 0$, there exists a neighborhood U of the point x such that $\delta(f(A \cap U)) < \epsilon$, where $\delta(f(A \cap U))$ is the diameter of the set $f(A \cap U)$ in Y. Given a discrete space D, take a $p \notin D$ and consider the space $L(D) = D \cup \{p\}$ where all points of D are isolated and the neighborhoods of p are the sets $U \subset L(D)$ such that $p \in U$ and $L(D) \setminus U$ is countable. The space $L(D)$ is called the one-point Lindelöfication of the discrete space D. Suppose that (X,τ) is a topological space. The family B_ω of all G_δ-sets in (X,τ) forms a base for some topology τ_ω on X. The topology τ_ω is called the ω-modification of the topology τ. Analogously, the space (X,τ_ω) is called the ω-modification of the space (X,τ). The ω-modification of an arbitrary space X is usually denoted briefly by $(X)_\omega$. We denote with the symbol $\square$ the end of a proof and, for the end of a claim inside of a proof, we use the symbol $\triangleleft$.

2. Second countable spaces whose continuous maps are almost injective

We give, under CH, a negative answer to following problem posed in [Tk]: is it true that, for every almost-compact-projective second countable space X, there exists a decomposition $X = K \cup Z$ such that K is compact and Z is countable? Recall that a space X is called almost compact projective in [Tk] if any continuous map of X onto a second countable space is almost compact.

The following observation answers Question 3.8 from [Tk].

2.1 Proposition. *The following conditions are equivalent for any space X:*

(1) *every continuous map $f : X \to Y \in \mathcal{U}$ is pseudo-open;*

(2) *every continuous map $f : X \to Y \in \mathcal{U}$ is quotient;*
(3) *any discrete family of nonempty G_δ-subsets of X is finite.*

Proof: It is evident that (1) $\Rightarrow$ (2), since each pseudo-open mapping is quotient. Applying [Tk], we conclude that (2) $\Rightarrow$ (3). Let us prove that (3) $\Rightarrow$ (1). Suppose that the claim in (3) is true for every discrete family of nonempty G_δ-subsets of X. Then every continuous $f : X \to Y \in \mathcal{U}$ is quotient [Tk]. Since every quotient map onto a Fréchet-Urysohn space is pseudo-open ([Ok]), the map f is pseudo-open. □

2.2 Definition. A mapping f of a space X onto a space Y is almost compact if $|\{y \in Y : f^{-1}(y) \text{ is not compact}\}| \leq \omega$.

2.3 Proposition. *If $X = Y \cup Z$, where Y is compact and $|Z| \leq \omega$, then every continuous $f : X \to Y \in \mathcal{U}$ is almost compact.*

The above proposition motivates the following question: is it true that in a second countable space X every continuous $f : X \to Y \in \mathcal{U}$ is almost compact only when $X = Y \cup Z$, where Y is compact and $|Z| \leq \omega$? We will show that, under the Continuum Hypothesis, the answer is no. The proof of the following lemma can be found in [En].

2.4 Proposition. *Let X be a topological space. Suppose that A is a dense subset of X, and $f : A \to Y$ is a continuous mapping of the subspace A into a complete metric space (Y, d). Let B be the set of all points of X in which the oscillation of f is equal to zero. Then $A \subset B$, and B is a G_δ-set in X. Furthermore, there exists a continuous mapping $F : B \to Y$ such that $F|A = f$.*

2.5 Lemma. *We have $|P| = \mathfrak{c}$, where P is the family of all G_δ-subsets of $\mathbb{R}$.*

2.6 Proposition. *Let X and Y be separable spaces. Suppose that $f : X \to Y$ is a continuous mapping. Then $f^{-1}(y)$ is nowhere dense for every $y \in Y$ except for a countable set.*

2.7 Lemma. *Let $A = \big\{f : Y \to \mathbb{R}^\omega$, Y is a dense G_δ-set in $\mathbb{R}$ and f is continuous$\big\}$. Then $|A| = \mathfrak{c}$.*

2.8 Definition. A mapping f from a space X onto a space Y is almost injective if $|\{y \in Y : |f^{-1}(y)| > 1\}| \leq \omega$.

2.9 Example. *Under CH, there exists a dense subspace X of $\mathbb{R}$ such that:*

(1) *every continuous $f : X \to Y \in \mathcal{U}$ is almost injective;*
(2) *X cannot be expressed as $K \cup Z$, where K is a compact set and Z is countable.*

Proof: Our proof is a modification of the methods developed in [Tk1]. We will construct a dense $X \subset \mathbb{R}$ such that for every separable metric space Y and every continuous onto map $f : X \to Y$ the fibers of f are singletons for almost all points of Y, i.e., the set $\{y \in Y : |f^{-1}(y)| > 1\}$ is countable.

Since we are assuming CH, there exists an enumeration $\{f_\alpha : \alpha < \omega_1\}$ of all mappings f which have the following properties:

(i) f is continuous and $P = \mathrm{dom}(f)$ is a dense G_δ-subset of $\mathbb{R}$;
(ii) $f : P \to \mathbb{R}^\omega$ and the set $f(P)$ is uncountable.

For each $\alpha < \omega_1$, let $F_\alpha = \mathrm{dom}(f_\alpha)$ and $G_\alpha = f_\alpha(F_\alpha)$. Apply Proposition 2.6 to see that the set $P_\alpha = \{x \in G_\alpha : \mathrm{Int}_{F_\alpha}(f_\alpha^{-1}(x)) \neq \emptyset\}$ is countable.

2.9.1 Claim. *The set $f_\alpha^{-1}(x)$ is nowhere dense in $\mathbb{R}$ for every $x \in G_\alpha \backslash P_\alpha$.*

Proof of the claim: Suppose that the claim is false, i.e., we have $\mathrm{Int}_{\mathbb{R}}(\overline{f_\alpha^{-1}(x)}) \neq \emptyset$. Since F_α is dense in $\mathbb{R}$, we have $F_\alpha \cap \mathrm{Int}_{\mathbb{R}}\, (\overline{f_\alpha^{-1}(x)}) \neq \emptyset$, so

$$\emptyset \neq F_\alpha \cap \mathrm{Int}_{\mathbb{R}}\, (\overline{f_\alpha^{-1}(x)}) \subset F_\alpha \cap \overline{f_\alpha^{-1}(x)} = \overline{f_\alpha^{-1}(x)}^{F_\alpha} = f_\alpha^{-1}(x),$$

and hence $\mathrm{Int}_{F_\alpha}(f_\alpha^{-1}(x)) \neq \emptyset$, which is a contradiction. Therefore $\mathrm{Int}_{\mathbb{R}}(\overline{f_\alpha^{-1}(x)}) = \emptyset$, i.e., $f_\alpha^{-1}(x)$ is nowhere dense in $\mathbb{R}$. ◁

Let $\{I_n : n \in \mathbb{N}\}$ be an enumeration of all nontrivial intervals with rational endpoints. We are going to choose recursively a point $x_\alpha \in \mathbb{R}$ for each $\alpha < \omega_1$. Let $x_0 \in \mathbb{R}$ be an arbitrary point. Suppose that we have chosen points $\{x_\alpha : \alpha < \beta\}$ for some $\beta > 0$, $\beta < \omega_1$. There exists a $k \in \mathbb{N}$ such that $\beta = \beta_0 + k$, where β_0 is a limit ordinal or $\beta_0 = 0$. Observe that claim 2.9.1 implies that $f_\gamma^{-1}(f_\gamma(x_\alpha))$ is nowhere dense in $\mathbb{R}$ for all $\alpha < \beta$ and $\gamma < \beta$ such that $f_\gamma(x_\alpha) \notin P_\gamma$. Since the interval I_k has the Baire property, we can choose a point

$$(*) \qquad \begin{aligned} x_\beta \in I_k - \Big[\bigcup \Big\{ f_\gamma^{-1}(f_\gamma(x_\alpha)) : \alpha < \beta, \gamma < \beta, f_\gamma(x_\alpha) \notin P_\gamma \Big\} \\ \bigcup \{x_\alpha : \alpha < \beta\} \Big], \end{aligned}$$

and therefore our transfinite construction can go on. Let $X = \big\{x_\alpha : \alpha < \omega_1\big\} \cup \mathbb{Q}$. We are going to show that X is as promised. Take any continuous onto map $f : X \to Y$, where Y is a second countable space. Since Y embeds into $\mathbb{R}^\omega$, we can assume that $Y \subset \mathbb{R}^\omega$ and hence $f : X \to \mathbb{R}^\omega$. Apply Proposition 2.4 to find a G_δ-subset $D \subset \mathbb{R}$ such that $X \subset D$ and there is a continuous mapping $F : D \to \mathbb{R}^\omega$ with $F \mid X = f$. If the set $f(X)$ is countable, there is nothing to prove. If $|f(X)| > \omega$ then $|F(D)| > \omega$ and hence F satisfies (i) and (ii). Therefore there exists an $\alpha_0 < \omega_1$ such that $F = f_{\alpha_0}$ and $D = F_{\alpha_0}$. Thus $f = f_{\alpha_0}|X$.

2.9.2 Claim. *Let* $A = \{f(x_\beta) : \beta \leq \alpha_0\} \cup (P_{\alpha_0} \cap Y) \cup f(\mathbb{Q})$. *Then* A *is a countable subset of* Y *and* $|f^{-1}(y)| = 1$ *for any* $y \in Y \setminus A$.

Proof of the claim: It is clear that A is countable. Suppose that $y \in Y \setminus A$. There exists a $\beta > \alpha_0$, such that $y = f(x_\beta) = f_{\alpha_0}(x_\beta)$. If $|f^{-1}(y)| > 1$ then there is a $\beta' > \alpha_0$, $\beta' \neq \beta$ for which $f(x_{\beta'}) = f_{\alpha_0}(x_{\beta'}) = f(x_\beta) = f_{\alpha_0}(x_\beta)$. If $\beta' < \beta$ then for $\alpha = \beta'$ and $\gamma = \alpha_0$ we have $f_\gamma(x_\alpha) \notin P_\gamma$, so the set $f_\gamma^{-1}(f_\gamma(x_\alpha)) = f_{\alpha_0}^{-1}(f_{\alpha_0}(x_{\beta'})) = f_{\alpha_0}^{-1}(f_{\alpha_0}(x_\beta)) = f_{\alpha_0}^{-1}(y)$ enters in the union of the fibers in $(*)$. Hence $x_\beta \in X \setminus f_{\alpha_0}^{-1}(f_{\alpha_0}(x_{\beta'}))$ which is a contradiction. The case $\beta' > \beta$ can be handled analogously if we take $\gamma = \alpha_0$ and $\alpha = \beta$. $\triangleleft$

It follows immediately from Claim 2.9.2 that the map f is almost injective. Since it was taken arbitrarily, every continuous map of X onto a second countable space is almost injective. It remains to verify condition (2). Suppose that $X = Y \cup Z$, where Y is compact and Z is a countable set. Since Y is a compact set in $\mathbb{R}$, it is bounded in $\mathbb{R}$, i.e., $Y \subset (a, b)$ for some $a, b \in \mathbb{R}$. Hence, outside of (a, b), the points of X form a countable set. There exists a k such that $I_k \subset (b+1, b+2)$. The set $X \cap (b+1, b+2) \subset Z$ is countable. On the other hand the set $T = \{x_{\alpha+k} : \alpha \text{ is a limit ordinal}\} \subset I_k \cap X \subset X \cap (b+1, b+2)$ is uncountable and hence $|X \cap (b+1, b+2)| > \omega$ which is a contradiction. $\square$

The following example gives a partial negative answer to the problem 3.19 from [Tk].

2.10 Corollary. *Under CH, there exists a subspace* X *of* $\mathbb{R}$ *such that every continuous onto map* $f : X \to Y \in \mathcal{U}$ *is almost compact and there is no decomposition* $X = K \cup Z$, *such that* K *is compact and* Z *is countable.*

Proof: Example 2.9 is the space we need, because for any second countable space Y and any continuous onto map $f : X \to Y$, we have

$$|\{y \in Y : f^{-1}(y) \text{ is not compact}\}| \leq |\{y \in Y : |f^{-1}(y)| > 1\}| \leq \omega.$$

$\square$

2.11 Corollary. *Under CH, there exists a second countable space* X *such that every continuous onto map of* X *is almost injective and there is no decomposition* $X = K \cup Z$, *such that* K *is compact and* Z *is countable.*

Proof: Let X be the space from Example 2.9 and let $f : X \to Y$ be a surjective continuous map. As X is second countable, the space Y has a countable network and hence there exists a condensation $g : Y \to Z \in \mathcal{U}$. For the map $g \circ f : X \to Z$ we have $|\{z \in Z : |(g \circ f)^{-1}(z)| > 1\}| \leq \omega$ and

$$|\{y \in Y : |f^{-1}(y)| > 1\}| = |\{z \in Z : |(g \circ f)^{-1}(z)| > 1\}| \leq \omega.$$

Consequently, every continuous onto map of X is almost injective. $\square$

Let us establish some characterizations for spaces whose continuous maps are almost injective.

2.12 Definition. A space X is scattered if every subspace $A \subset X$ has an isolated point.

The proof of the following lemma is immediate and is left to the reader.

2.13 Lemma. *Suppose that every continuous map of X onto a second countable space is almost injective. Then every continuous image of X has the same property.*

2.14 Proposition. *Suppose that X is compact. Then X is scattered if and only if every continuous map of X onto a second countable space is almost injective.*

Proof: To prove the sufficiency, take a compact space X such that every continuous map of X onto a second countable is almost injective. If it is not scattered, we would have a continuous surjective function $f : X \to [0,1]$ ([Sh]). Apply Lemma 2.13 to conclude that every continuous map of $[0,1]$ onto a second countable space is almost injective. But it is not true because for the continuous onto function $f : [0,1] \to [0,1/4]$ defined by $f(x) = (x-1/2)^2$, we have $|\{y \in Y : |f^{-1}(y)| > 1\}| > \omega$, which is a contradiction.

To prove the necessity, suppose that X is scattered. Let $f : X \to Y \in \mathcal{U}$. A compact space is scattered if and only if any second countable image of this space is countable ([Sh]). Thus $|Y| \leq \omega$, and consequently $|\{y \in Y : |f^{-1}(y)| > 1\}| \leq |Y| \leq \omega$. $\square$

2.15 Corollary. *If X is a compact metrizable space, then X is countable if and only if every continuous map of X (onto a second countable space) is almost injective.*

2.16 Corollary. *If X is pseudocompact and every continuous $f : X \to Y \in \mathcal{U}$ is almost injective then X is scattered.*

Proof: We establish first that every continuous map of βX onto a second countable space is almost injective. Assume that $F : \beta X \to Z \in \mathcal{U}$ and $F|X = f$. We have $f(X) = Z'$ where Z' is compact because it is pseudocompact and $Z' \in \mathcal{U}$. Since $\overline{X} = \beta X$ we have $Z' = \overline{Z'} = Z$ which implies $Z' = Z$. It turns out that $f(\beta X) = Z'$, where Z' is compact and every continuous map of Z' onto a second countable space is almost injective, so by Corollary 2.15 the space Z' is countable. From the above we can conclude that every continuous map of βX onto a second countable space is almost injective. By Proposition 2.14 the space βX is scattered and hence so is X, because $X \subset \beta X$. $\square$

2.17 Corollary. *If X is countably compact and every continuous $f : X \to Y \in \mathcal{U}$ is almost injective then X is scattered.*

2.18 Proposition. *If X is Lindelöf P or Lindelöf scattered then every continuous map of X onto a second countable space is almost injective.*

PROOF: We consider first the case when X is a Lindelöf P-space. Take any $f : X \to Y \in \mathcal{U}$. The space Y has a countable base, hence every $\{y\}$ is a G_δ-set in Y. Thus, $f^{-1}(y)$ is a G_δ-set and X is a P-space, whence $f^{-1}(y)$ is open. We have $X = \bigcup_{y \in Y} f^{-1}(y)$, i.e., $\{f^{-1}(y) : y \in Y\}$ is an open cover of X. Since X is Lindelöf, there exists a $C \subset Y$ such that $|C| \leq \omega$, and $X = \bigcup_{y \in C} f^{-1}(y)$. Therefore $Y = f(X) = f(\bigcup_{y \in C} f^{-1}(y))$ and $|Y| \leq |C| \leq \omega$. From the above we deduce that $|\{y \in Y : |f^{-1}(y)| > 1\}| \leq |Y| \leq \omega$.

In case when X is Lindelöf scattered, the ω-modification $(X)_\omega$ of X is a Lindelöf P-space ([Us]), and hence every continuous $f : (X)_\omega \to Y \in \mathcal{U}$ is almost injective. Being X a continuous image of its ω-modification $(X)_\omega$, every continuous $f : X \to Y \in \mathcal{U}$ is almost injective. □

The following example shows that we cannot omit the compactness in Theorem 2.14.

2.19 Example. *There exists a (noncompact) space Y which is not scattered and every continuous map of Y onto a second countable is almost injective.*

PROOF: Let X be the Lindelöfication of a discrete uncountable space D. It is known that, if X is Lindelöf P, then the free group $F(X)$ of X is Lindelöf P ([Tka]). Every continuous map of $F(X)$ onto a second countable space is almost injective by Proposition 2.18. Furthermore $F(X)$ is not scattered, because if we assume the opposite then $F(X)$ would have an isolated point. But the topological groups are homogeneous spaces so $F(X)$ would be discrete and countable. This contradiction shows that $F(X)$ is not scattered. □

2.20 Proposition. *Suppose that Y is C-embedded in X.*

(a) *If any continuous map of X onto a second countable space is almost injective then Y has the same property.*
(b) *If any continuous onto map of X is almost injective then every continuous onto map of Y is almost injective.*

PROOF: We consider first the case (a). Let $f : Y \to Z$, where $Z \in \mathcal{U}$. We can consider that $Z \subset \mathbb{R}^\omega$ and hence $f = \triangle_{n \in \omega} f_n$, where $f_n : Y \to \mathbb{R}$. Since Y is C-embedded, there exists a continuous $F_n : X \to \mathbb{R}$ such that $F_n|Y = f_n$. We have $F \mid Y = f$, where $F : X \to \mathbb{R}^\omega$ is defined by $F(x) = \{F_n(x)\}_{n \in \omega}$. The inequality

$$|\{z \in Z : |f^{-1}(z)| > 1\}| \leq |\{z \in Z : |F^{-1}(z)| > 1\}| \leq \omega$$

implies that every continuous map of Y onto a second countable space is almost injective.

It remains to prove (b). It is easy to see that any restriction of an almost injective map is an almost injective map. Let $f : Y \to Z$ be a continuous onto map, where Z is a subspace of $\mathbb{R}^\kappa$ for some cardinal κ. Then there exists a continuous map $F : X \to \mathbb{R}^\kappa$ such that $F|Y = f$. Since the map F is almost injective, so is f. □

2.21 Corollary. *Let X be a space such that every continuous map of X onto a second countable space is almost injective and let Y be a closed subspace of X.*

(1) *If X is normal then every continuous map of Y onto a second countable space is almost injective.*
(2) *If Y is compact then every continuous map of Y onto a second countable space is almost injective.*

Proof: In both cases the subspace Y is C-embedded in X and by Proposition 2.20, every continuous map of Y onto a second countable space is almost injective. □

2.22 Corollary. *In the space X of Example* 2.9*, all compact subspaces are countable.*

Proof: Applying Corollary 2.21, we see that for each compact $K \subset X$, every continuous map of K onto a second countable space is almost injective. Applying Corollary 2.15 we conclude that K is countable. □

2.23 Proposition. *If X_i is compact and every continuous map of X_i onto a second countable space is almost injective for all $i \leq n$, then every continuous map of $\prod_{i=1}^n X_i$ onto a second countable space is almost injective.*

Proof: By Proposition 2.14, each X_i is scattered. It is clear that the product of a finite family of scattered spaces is scattered. Since this product is compact, apply 2.14 to see that every continuous map of this product onto a second countable space is almost injective. □

2.24 Remark. Observe that any two-point space has all its continuous maps almost injective. However, the countable product of two-point spaces is homeomorphic to the Cantor set C which is a non-scattered compact space and therefore not all continuous maps of C are almost injective. Thus, our property is not preserved by countable products.

2.25 Example. *Under CH, there exists a space X such that every continuous map of X onto a second countable space is almost injective, but $X \times X$ does not have this property.*

Proof: Take the space X constructed in Example 2.9. Consider the projection of $X \times X$ onto the first coordinate. Note that $\pi^{-1}(y) = \{y\} \times X$ for any $y \in X$.

Hence $X = \{y \in X : |\pi^{-1}(y)| > 1\}$. If every continuous map of $X \times X$ onto a second countable space were almost injective, the space X would be countable. □

2.26 Example. *There exists pseudocompact scattered spaces which can be continually mapped onto* $[0,1]$. *Hence, not every map of such spaces onto second countable ones is almost injective.*

Proof: In [Mr] it is proved that there exists Mrówka spaces which can be continuously mapped onto $[0,1]$. Recall that the Mrówka spaces are constructed as follows. Let γ be an almost disjoint family of infinite subsets of ω such that $|\gamma| = \mathfrak{c}$. Extending γ if necessary we can suppose that γ is maximal. For each $A \in \gamma$ take a point $x_A \notin \omega$. In the set $\omega \cup \{x_A : A \in \gamma\}$ we introduce the topology τ in the following way: if $x \in \omega$ then $\{x\} \in \tau$. If $x = x_A$ then the base at x consists of the sets $\{x_A\} \cup (A - B)$ where $B \subset \omega$ is finite. The set $\omega \cup \{x_A : A \in \gamma\}$, with the topology described above, is called the space of Mrówka. It is easy to see that any space of Mrówka is pseudocompact and scattered. Hence any Mrówka space, which can be mapped onto $[0,1]$, cannot have the property mentioned above. □

3. Open questions

The following questions seem to outline a natural development of the research done in this paper.

3.1 Question. *Does there exist in ZFC a second countable space* X *such that every continuous* $f : X \to Y \in \mathcal{U}$ *is almost injective and which cannot be expressed as* $K \cup Z$, *where* K *is compact and* Z *is countable?*

3.2 Question. *Does there exist in ZFC a space* X *such that every continuous* $f : X \to Y \in \mathcal{U}$ *is almost injective and such that* $X \times X$ *does not have this property?*

3.3 Question. *Let* X *be a space such that every continuous map of* X *onto a second countable space is almost injective. Let* $F \subset X$ *be a closed subspace. Does the space* F *have this property?*

3.4 Question. *Suppose that a space* X *is countably compact and scattered. Is it true that every continuous map of* X *onto a second countable space is almost injective?*

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