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An answer to a question of Arhangel'skii

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Abstract. We prove that there exists an example of a metrizable non-discrete space X, such that $C_p(X \times \omega) \approx_l C_p(X)$ but $C_p(X \times S) \not\approx_l C_p(X)$ where $S = (\{0\} \cup \{\frac{1}{n+1} : n \in \omega\})$ and $C_p(X)$ is the space of all continuous functions from X into reals equipped with the topology of pointwise convergence. It answers a question of Arhangel'skii ([2, Problem 4]).

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1. Introduction

All spaces under consideration are completely regular ($T_{3\frac{1}{2}}$). Symbol $\mathbb{R}$ stands for the real numbers. For a space X define $C_p(X)$ as the space of all continuous functions from X to $\mathbb{R}$ equipped with the topology inherited from the Tychonoff product $\mathbb{R}^X$. We say that a space X has *the Baire property*, if the intersection of a countable family of open dense sets in X is dense in X. For spaces X and Y the symbol $C_p(X) \approx_l C_p(Y)$ means that there exists a linear homeomorphism between the spaces $C_p(X)$ and $C_p(Y)$.

In his paper [2] Arhangel'skii investigates spaces X with the properties that $C_p(X) \approx_l C_p(X \times \omega)$ or $C_p(X) \approx_l C_p(X \times S)$, where $S = (\{0\} \cup \{\frac{1}{n+1} : n \in \omega\})$. He asks a question ([2, Problem 4]) whether the first of these properties implies the second. In other words: does there exist a non-discrete space X such that $C_p(X) \approx_l C_p(X \times \omega)$ but $C_p(X) \not\approx_l C_p(X \times S)$? In this note we give two examples of such spaces X.

2. A non-metrizable example

First, we give a relatively simple example of a non-metrizable X with the property. The example has even a stronger property that $C_p(X) \approx_l C_p(X \times \omega)$ but $C_p(X)$ and $C_p(X \times S)$ are not homeomorphic. Its construction is based on a result of Lutzer and McCoy ([5]) and it appeared in other context in a paper by W. Marciszewski and J. van Mill ([6]).

According to Theorem 1.3.4 of [1], if a space Y contains an infinite compact subspace, then $C_p(Y)$ does not have the Baire property. It implies in particular that for any space X the function space $C_p(X \times S)$ does not have the Baire property.

In the paper by Lutzer and McCoy ([5]) a countable, non-metrizable space Y with one non-isolated point is constructed such that $C_p(Y)$ has the Baire property. The space $C_p(Y)$ is separable and metrizable as a subspace of the countable Tychonoff product of the real line.

Let us define $X = \omega \times Y$. Space $C_p(X)$ has the Baire property, because it is linearly homeomorphic to $C_p(Y)^\omega$ and the Baire property is preserved under the countable Tychonoff products of metrizable separable spaces ([7]).

The space $X \times \omega$ is homeomorphic to X and, in particular, $C_p(X) \approx_l C_p(X \times \omega)$. Since $C_p(X)$ has the Baire property and $C_p(X \times S)$ does not have it, there is no homeomorphism between $C_p(X)$ and $C_p(X \times S)$. This finishes the presentation of the non-metrizable example.

3. The main example

The rest of the paper is devoted to the description of a non-discrete metrizable space X such that $C_p(X) \not\approx_l C_p(X \times S)$ but $C_p(X) \approx_l C_p(X \times \omega)$. Arhangel'skii's paper [2] concerns mainly metrizable spaces and it seems that it is more natural and important to answers his question by giving a metrizable example.

A map $x \mapsto S(x)$ which assigns to points in X nonempty subsets of Y is called *lower-semicontinuous*, if for each point $y \in S(x)$ and its neighborhood U there exists a neighborhood V of x such that $S(z) \cap U \neq \emptyset$ for all $z \in V$.

We shall call a set $C \subset \omega_1$ *closed and unbounded* if it is closed in the sense of ordinal topology on ω_1 and unbounded in ω_1 in the sense of natural order on ω_1. Countably intersection of closed and unbounded sets is closed and unbounded ([4, Lemma 7.4]).

A subset $A \subset \omega_1$ is *stationary* if A intersects all closed and unbounded subsets of ω_1. A subset $A \subset \omega_1$ is *non-stationary* if there exists a closed and unbounded $C \subset \omega_1$ such that $C \cap A = \emptyset$.

Let A be a subset of ω_1. A function $f : A \to \omega_1$ is called *regressive* if for every $0 \neq \xi \in A$ it holds $f(\xi) < \xi$. *Pressing Down Lemma* ([4, Theorem 22]) says that for every stationary set A and regressive function $f : A \to \omega_1$ there exists a stationary set $B \subset A$ such that f is constant on B.

Let $\Phi : C_p(X) \to C_p(Y)$ be a linear surjection between the function spaces on metrizable spaces X and Y. Then we may associate (Chapter 1.4 of [3]) with each $y \in Y$ a nonempty finite set $\operatorname{supp}(y) \subset X$ and non-zero real numbers $\{a_x\}_{x \in \operatorname{supp}(y)}$ such that for every function $f \in C_p(X)$ it holds $\Phi(f)(y) = \sum_{x \in \operatorname{supp}(y)} a_x f(x)$. We call $\operatorname{supp}(y)$ the *support* of the point y; for every $A \subset Y$ symbol $\operatorname{supp}(A)$ denotes the union of supports of points in A.

The map $y \mapsto \operatorname{supp}(y)$ has the following properties:

- it is lower semicontinuous and, in particular, if $A \subset Y$ and $y \in \operatorname{cl}_Y(A)$

then (Proposition 1.4.4 of [3])

$$\operatorname{supp}(y) \subset \operatorname{cl}_X(\operatorname{supp}(A)); \tag{1}$$

- if $S \subset Y$ is a compact subset then (Lemma 1.5.6 of [3])

$$\operatorname{cl}_X(\operatorname{supp}(S)) \text{ is a compact subset of } X; \tag{2}$$

- if Φ is a linear homeomorphism and $x \mapsto \operatorname{supp}(x)$ is the map associated with the inverse Φ^{-1}, then (Proposition 1.4.3 of [3]) for all $y \in Y$

$$y \in \operatorname{supp}(\operatorname{supp}(y)). \tag{3}$$

For every limit countable ordinal $\alpha < \omega_1$ we fix an increasing sequence $(x_n^\alpha)_{n\in\omega}$, such that $x_n^\alpha < \alpha$ and $\sup_{n\in\omega} x_n^\alpha = \alpha$. We define the *Stone space* (comp. Chapter 5.1 of [10]) as

$$E = \{(x_n^\alpha)_{n\in\omega} : \alpha < \omega_1,\ \alpha \text{ limit}\} \subset \omega_1^\omega$$

where the distance between two distinct points $x, y \in E$ is $\frac{1}{n+1}$ if $n \in \omega$ is the minimal natural number such that $x(n) \neq y(n)$.

Let X be a metrizable space of weight $\aleph_1$. We shall call an increasing sequence

$$A_1 \subset A_2 \subset \ldots \subset A_\xi \ldots \subset X \quad \text{with} \quad |A_\xi| \le \aleph_0 \quad \text{and} \quad \xi < \omega_1,$$

admissible, if

$$\operatorname{cl}(\bigcup\{A_\xi : \xi < \omega_1\}) = X, \quad \text{and} \quad A_\xi = \bigcup\{A_\alpha : \alpha < \xi\} \quad \text{for limit } \xi.$$

We shall call the set

$$\widehat{A}_\xi = \operatorname{cl}(A_\xi) \setminus \bigcup\{\operatorname{cl} A_\alpha : \alpha < \xi\}, \quad \text{for limit } \xi,$$

the *layer at the level* ξ determined by the admissible sequence.

We shall need the following theorem proved by R. Pol.

Theorem 1 (R. Pol, [8]). *For any two admissible sequences of subsets of a metrizable space X of weight $\aleph_1$, the layers determined by these sequences coincide at all levels, apart from a non-stationary set in ω_1.* □

Let us fix an admissible sequence for the space E:

$$E_\xi = \{(x_n^\alpha)_{n\in\omega} : \alpha < \xi\}.$$

Repeating Stone's arguments ([10, Chapter 5]) one can verify that all layers $\widehat{E}_\xi$, except possibly a non-stationary set of levels, are singletons of the form $\{(x_n^\xi)_{n\in\omega}\}$ for some $\xi < \omega_1$.

For the reader's convenience we give a **proof of this fact**. Firstly let us observe that for every $\alpha < \xi$ there exists $n_0 \in \omega$ such that $x_n^\xi > \alpha$ for all $n > n_0$. It proves that $(x_n^\xi)_{n\in\omega} \notin \mathrm{cl}(E_\alpha)$ for every $\alpha < \xi$. Hence $\widehat{E_\xi}$ is empty or contains exactly one element, namely the sequence $(x_n^\xi)_{n\in\omega}$.

Let us assume, on the contrary, that there exists a closed and unbounded set $C \subset \omega_1$ such that for all $\xi \in C$ layers $\widehat{E}_\xi$ are empty. We may assume that the set C is a subset of the limit ordinals. With every number $\xi \in C$ we may associate a natural number $n_\xi \in \omega$ such that the open ball with radius $\frac{1}{n_\xi}$ around the sequence $(x_n^\xi)_{n\in\omega}$ does not contain any sequence $(x_n^\alpha)_{n\in\omega}$ for $\alpha < \xi$. Since the ball contains exactly those sequences which coincide with $(x_n^\xi)_{n\in\omega}$ on at least the first n_ξ places, it means that for every $\alpha < \xi$ there exists $n < n_\xi$ such that $x_n^\alpha \neq x_n^\xi$.

Let us define $C_n = \{\xi \in C : n_\xi = n\}$. Since the union of countably many non-stationary sets is non-stationary, there exists $n_0 \in \omega$ such that C_{n_0} is stationary.

Let us define $A_0 = C_{n_0}$ and $f_n : C \to \omega_1$ by the formula $f_n(\xi) = x_n^\xi$ $(n \in \omega)$. The function f_0 is regressive and the set A_0 is stationary. According to the Pressing Down Lemma there exists a stationary set $A_1 \subset A_0$ such that f_0 is constant on A_1. Inductively we may construct a decreasing sequence $\{A_n\}_{0\le n\le n_0}$ of stationary subsets of C such that f_n is constant on A_{n+1} $(0 \le n < n_0)$.

Let us fix some $\alpha, \xi \in A_{n_0}$, $\alpha < \xi$. Since the functions f_n $(0 \le n < n_0)$ are constant on the set A_{n_0} it holds

$$x_n^\xi = f_n(\xi) = f_n(\alpha) = x_n^\alpha$$

for every $0 \le n < n_0$. Since $n_\xi = n_0$, it is a contradiction with the fact that the ball with radius $\frac{1}{n_\xi}$ around the sequence $(x_n^\xi)_{n\in\omega}$ does not contain $(x_n^\alpha)_{n\in\omega}$. It finishes the proof of the fact.

Remark. Stone's arguments quoted above together with Pol's theorem give that for every admissible decomposition of E all layers, except possible a non-stationary set of levels, are singletons. Therefore, for all $\xi < \omega_1$ except possibly a non-stationary subset of ω_1, the layers of $E \times \omega$ and the layers of $E \times \omega \times S$ are of the form $\{(x_n^\xi)_{n\in\omega}\} \times \omega$, $\{(x_n^\xi)_{n\in\omega}\} \times \omega \times S$ respectively.

This remark shall be used in the course of the proof of the following

Theorem 2. *There is no linear homeomorphism between* $C_p(E\times\omega)$ *and* $C_p(E\times\omega\times S)$.

Before the proof of the theorem let us observe that the space $E \times \omega$ gives the example mentioned in the abstract and in the beginning of Section 3. We have $C_p(E \times \omega \times \omega) \approx_l C_p(E \times \omega)$ because $E \times \omega \times \omega$ is homeomorphic to $E \times \omega$. The second property of the example is expressed in the statement of the theorem.

Proof of Theorem 2: To obtain a contradiction, suppose that there exists a linear homeomorphism Φ from $C_p(X)$ onto $C_p(Y)$, where X denotes $E \times \omega$ and Y denotes $E \times \omega \times S$. Let

$$A_1^0 \subset A_2^0 \subset \ldots \subset A_\xi^0 \ldots \subset X \ (\xi < \omega_1)$$

be any admissible sequence for the space X.
We define inductively for $n \in \omega$

$$B_\xi^n = \bigcup\{\mathrm{supp}\,(x) : x \in A_\xi^n\}$$

and

$$A_\xi^{n+1} = \bigcup\{\mathrm{supp}\,(y) : y \in B_\xi^n\}.$$

Finally, for every $\xi < \omega_1$ let

$$C_\xi = \bigcup_{n \in \omega} A_\xi^n$$

and

$$D_\xi = \bigcup_{n \in \omega} B_\xi^n.$$

Then the sequences

$$C_0 \subset C_1 \subset \ldots \subset C_\xi \subset \ldots \subset X,\ \xi < \omega_1,$$

and

$$D_0 \subset D_1 \subset \ldots \subset D_\xi \subset \ldots \subset Y,\ \xi < \omega_1,$$

are admissible and have the property that $C_\xi = \mathrm{supp}\,(D_\xi)$ and $D_\xi = \mathrm{supp}\,(C_\xi)$ for every $\xi < \omega_1$. The only fact which requires an explanation is that the union of the sequence $(D_\xi)_{\xi<\omega_1}$ is dense in Y. Let us fix a point $y \in Y$. According to the property (3) of support maps there exists some $x \in \mathrm{supp}\,(y)$ such that $y \in \mathrm{supp}\,(x)$. The sequence $\{C_\xi\}_{\xi<\omega_1}$ is admissible. In particular, $x \in \mathrm{cl}_X(\bigcup_{\xi<\omega_1} C_\xi)$. According to the property (1) of support maps it holds $\mathrm{supp}\,(x) \subset \mathrm{cl}_Y(\mathrm{supp} \bigcup_{\xi<\omega_1} C_\xi)$. Together with the fact that $\mathrm{supp}(\bigcup_{\xi<\omega_1} C_\xi) = \bigcup_{\xi<\omega_1} D_\xi$ it gives that $y \in \mathrm{supp}\,(x) \subset \mathrm{cl}_Y(\bigcup_{\xi<\omega_1} D_\xi)$. It finishes the proof of the fact that $\bigcup_{\xi<\omega_1} D_\xi$ is dense in Y.

According to the Remark formulated before the proof of Theorem 2, there exists $\xi < \omega_1$ such that $\widehat{C}_\xi = (x_n^\xi)_{n\in\omega} \times \omega$ and $\widehat{D}_\xi = (x_n^\xi)_{n\in\omega} \times \omega \times S$. We fix a copy of S in $\widehat{D}_\xi$. Due to the property (2) of support maps we know that the space $T = \mathrm{cl}_X(\mathrm{supp}(S))$ is a compact subset of X. Moreover, the property (1) of support maps together with the fact that $\mathrm{supp}(D_\xi) = C_\xi$ imply that the set T is a subspace of $\mathrm{cl}_X(C_\xi)$. The intersection of T with $\widehat{C}_\xi$ is finite, because $\widehat{C}_\xi$ is discrete and T is compact. This implies that we can find $y \in S \setminus \mathrm{supp}(T \cap \widehat{C}_\xi)$.

We strive to obtain a contradiction with the fact that $y \in \bigcup\{\mathrm{supp}(x) : x \in \mathrm{supp}(y)\}$ (property (3) of support maps). We can represent the set $\mathrm{supp}(y)$ as a union of two subsets $\mathrm{supp}(y) = X_1 \cup X_2$, where $X_1 = \mathrm{supp}(y) \cap \widehat{C_\xi}$ and $X_2 = \mathrm{supp}(y) \setminus X_1 \subset \bigcup_{\alpha<\xi} \mathrm{cl}_X(C_\alpha)$; it implies that exists $\alpha < \xi$ such that $X_2 \subset \mathrm{cl}_X(C_\alpha)$.

According to our choice of the point y, we have $y \notin \mathrm{supp}(X_1)$. On the other hand

$$\mathrm{supp}(X_2) \subset \mathrm{supp}(\mathrm{cl}_X(C_\alpha)) \subset \mathrm{cl}_Y(D_\alpha),$$

thanks to the property (1) of support maps and the equality $\mathrm{supp}(C_\alpha) = D_\alpha$. In particular $\mathrm{supp}(X_2) \cap \widehat{D_\xi} = \emptyset$. Finally we obtain $y \notin \mathrm{supp}(X_1) \cup \mathrm{supp}(X_2) = \bigcup\{\mathrm{supp}(x) : x \in \mathrm{supp}(y)\}$, a contradiction. $\square$

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