Complete $\aleph_0$-bounded groups need not be $\mathbb{R}$-factorizable

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Abstract. We present an example of a complete $\aleph_0$-bounded topological group H which is not $\mathbb{R}$-factorizable. In addition, every G_δ-set in the group H is open, but H is not Lindelöf.

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1. Introduction

A topological group G is called $\mathbb{R}$-*factorizable* ([5], [6]) if for every continuous function $g: G \to \mathbb{R}$, one can find a continuous homomorphism $p: G \to H$ onto a second countable topological group H and a continuous function $h: H \to \mathbb{R}$ such that $g = h \circ p$. The class of $\mathbb{R}$-factorizable groups includes all totally bounded groups, all Lindelöf groups, arbitrary subgroups of Lindelöf Σ-groups ([5]), and many more.

By [6, Proposition 5.3], every $\mathbb{R}$-factorizable group G is $\aleph_0$-*bounded*, i.e., G can be covered by countably many translates of any neighborhood of the identity. The notion of an $\aleph_0$-bounded group was introduced in [1] and since then it has been intensively studied. It is known that $\aleph_0$-bounded groups need not be $\mathbb{R}$-factorizable ([4]). However, all examples of $\aleph_0$-bounded not $\mathbb{R}$-factorizable groups constructed so far are essentially incomplete, being proper dense subgroups of special $\aleph_0$-bounded groups.

In this note we present an example of a complete $\aleph_0$-bounded group H which fails to be $\mathbb{R}$-factorizable. In addition, H is a P-*group*, i.e., every countable intersection of open sets in H is open.

1.1 Notation and terminology. If A is a subset of a group G, we use $\langle A\rangle$ to denote the subgroup of G generated by A. We say that A is *independent* in an Abelian group G if a linear combination $k_1a_1 + \cdots + k_na_n$ with $k_1, \dots, k_n \in \mathbb{Z}$ and pairwise distinct $a_1, \dots, a_n \in A$ is equal to the neutral element of G iff $k_1 = \cdots = k_n = 0$.

2. The example

Given a topological group L, we denote by $(L)_\omega$ the topological group with the underlying group L (and the same group operation) whose base consists of

G_δ-sets in L. It clear that the identity map $\mathrm{id}_L\colon (L)_\omega \to L$ is continuous, and id_L is a homeomorphism iff L is a P-group.

Our construction is based on three simple lemmas.

Lemma 2.1. *Let H be an $\mathbb{R}$-factorizable P-group. Then the image $f(H)$ is countable for each continuous real-valued function f on H.*

Proof: Consider a continuous function $f\colon H \to \mathbb{R}$. Since H is $\mathbb{R}$-factorizable, one can find a continuous homomorphism $\pi\colon H \to K$ onto a second countable topological group K and a continuous function $g\colon K \to \mathbb{R}$ such that $f = g \circ \pi$. Denote by K_d the group K endowed with the discrete topology. Since H is a P-group, the homomorphism $\pi\colon H \to K_d$ remains continuous. In addition, the group H is $\aleph_0$-bounded by [6, Proposition 5.3]. Therefore, K_d is $\aleph_0$-bounded being a continuous homomorphic image of H. It is easy to see that every $\aleph_0$-bounded discrete group is countable, so $f(H) = g(K_d)$ is also countable. □

Lemma 2.2. *The following conditions are equivalent for a P-group H with $w(H) \le \aleph_1$:*

(1) *H is $\mathbb{R}$-factorizable;*
(2) *H is Lindelöf.*

Proof: It is well known that (2) implies (1) for an arbitrary topological group (see [5, Assertion 1.1] or [3, Assertion 10]). Let H be a P-group of weight $\aleph_1$. Then H is zero-dimensional and paracompact ([7]). Suppose that H is not Lindelöf. Then there exists a disjoint cover $\gamma = \{U_\alpha : \alpha < \omega_1\}$ of H by non-empty open sets U_α. Let $\{r_\alpha : \alpha < \omega_1\}$ be a sequence of pairwise distinct real numbers. Define a function $f\colon H \to \mathbb{R}$ by $f(x) = r_\alpha$ if $x \in U_\alpha$, $\alpha < \omega_1$. Then f is continuous and $|f(H)| > \omega$, so Lemma 2.1 implies that H is not $\mathbb{R}$-factorizable. □

Lemma 2.3. *Let $G = \prod_{i\in I} G_i$ be the direct product of discrete groups endowed with the Tychonoff topology. Then the group $(G)_\omega$ is complete.*

Proof: For every countable subset J of I, put $G_J = \prod_{j\in J} G_j$. Then the projection $\pi_J\colon (G)_\omega \to G_J$ onto the discrete group G_J is continuous and open. Denote by e_J the neutral element of G_J.

Let ξ be a Cauchy filter in $(G)_\omega$. If J is a countable subset of I, then there exists an element $F_J \in \xi$ such that $F_J^{-1} \cdot F_J \subseteq \pi_J^{-1}(e_J)$. Pick a point $a_J \in F_J$. Clearly, $\pi_J(x) = a_J$ for each $x \in F_J$. In addition, if K is a countable subset of I and $J \subseteq K$, then the corresponding point $a_K \in F_K$ satisfies $\pi^K_J(a_K) = a_J$, where $\pi^K_J\colon G_K \to G_J$ is the projection. Indeed, if $F_K \in \xi$ and $F_K^{-1} \cdot F_K \subseteq \pi_K^{-1}(e_K)$, then $\pi_K(x) = a_K$ for each $x \in F_K$. Choose a point $z \in F_J \cap F_K$. Then $\pi_J(z) = a_J$ and $\pi_K(z) = a_K$, whence it follows that

$$a_J = \pi_J(z) = \pi^K_J(\pi_K(z)) = \pi^K_J(a_K).$$

We conclude, therefore, that there exists a point $a \in G$ such that $\pi_J(a) = a_J$ for every countable set $J \subseteq I$. It remains to verify that the filter ξ converges to a in $(G)_\omega$.

Let U be a neighborhood of a in $(G)_\omega$. Then there exists a countable set $J \subseteq I$ such that $V = \pi_J^{-1}\pi_J(a) \subseteq U$. Note that $\pi_J(x) = a_J = \pi_J(a)$ for each $x \in F_J$, so $F_J \subseteq V \subseteq U$. This proves that ξ converges to a. Thus, the group $(G)_\omega$ is complete. $\square$

Let $\mathbb{Q}$ be the set of rationals. Denote by K the free Abelian group $A(\mathbb{Q})$ endowed with the discrete topology. It is clear that the group K is countable. We shall use the additive notation for the group operations in K and K^{ω_1}.

Theorem 2.4. *There exists a closed $\aleph_0$-bounded subgroup H of $(K^{\omega_1})_\omega$ which fails to be Lindelöf. In particular, H is a complete $\aleph_0$-bounded P-group which is not $\mathbb{R}$-factorizable.*

Proof: Our aim is to define a subgroup H of $G = (K^{\omega_1})_\omega$ satisfying the following conditions:

(a) H is closed in G;
(b) H is not Lindelöf;
(c) $|\pi_\alpha(H)| \leq \omega$ for each $\alpha < \omega_1$, where $\pi_\alpha\colon K^{\omega_1} \to K^\alpha$ is the projection.

Suppose that the subgroup H of G satisfying (a)–(c) has been defined. Note that G is a P-group, and so is H. The group G is complete by Lemma 2.3, so (a) implies that H is also complete. Let us verify that $\aleph_0$-boundedness of H follows from (c). Suppose that U is a neighborhood of the neutral element of H. Then there exists $\alpha < \omega_1$ such that $H \cap \pi_\alpha^{-1}(0_\alpha) \subseteq U$, where 0_α is the neutral element of K^α. By (c), there exists a countable subset A of H such that $\pi_\alpha(A) = \pi_\alpha(H)$. Then $U + A = H$, and hence H is $\aleph_0$-bounded. In addition, (c) implies that $w(H) \leq \aleph_1$. Indeed, the family

$$\mathcal{B} = \{H \cap \pi_\alpha^{-1}(y) : y \in \pi_\alpha(H),\ \alpha < \omega_1\}$$

is a base for H and $|\mathcal{B}| \leq \aleph_1$. Finally, H is P-group with $w(H) \leq \aleph_1$, so (b) and Lemma 2.2 together imply that H is not $\mathbb{R}$-factorizable.

Our first step is to define a closed non-Lindelöf subset X of $\mathbb{Q}^{\omega_1}$ which generates a closed subgroup $H = \langle X \rangle$ of G once $\mathbb{Q}^{\omega_1}$ is identified with the corresponding subset of K^{ω_1}. Our method of defining X is a "reminiscence" of the construction of an Aronszain tree given in [2]. In what follows we use the symbol $\leq$ to denote the usual linear order on $\mathbb{Q}$. For every $\alpha < \omega_1$, denote by $\mathrm{In}(\alpha)$ the subset of $\mathbb{Q}^\alpha$ consisting of all strictly increasing functions, that is,

$$\mathrm{In}(\alpha) = \{x \in \mathbb{Q}^\alpha : x(\nu) < x(\mu) \text{ if } \nu < \mu < \alpha\}.$$

Let us construct a family $\{X_\alpha : 0 < \alpha < \omega_1\}$ satisfying the following conditions for all $\alpha, \beta, \gamma \in \omega_1 \setminus \{0\}$:

(1) $X_\alpha \subseteq \mathbb{Q}^\alpha$ and $|X_\alpha| = \omega$;
(2) X_α is an independent subset of K^α under the natural embedding $\mathbb{Q}^\alpha \hookrightarrow K^\alpha$;
(3) $\pi^\beta_\alpha(X_\beta) = X_\alpha$ whenever $\alpha < \beta$, where $\pi^\beta_\alpha : K^\beta \to K^\alpha$ is the projection;
(4) $x(\alpha) \le x(\beta)$ whenever $x \in X_\gamma$ and $\alpha < \beta < \gamma$;
(5) if $x \in X_{\alpha+1}$ and $x(\alpha) = q$, then $(x, q) \in X_{\alpha+2}$;
(6) if $\alpha + 1 < \gamma$, $x \in X_\gamma$ and $x(\alpha) = x(\alpha + 1)$, then $x(\beta) = x(\alpha)$ for each β satisfying $\alpha < \beta < \gamma$;
(7) if $\alpha < \beta$, $x \in X_{\alpha+1} \cap \mathrm{In}(\alpha + 1)$, $q, r \in \mathbb{Q}$ and $x(\alpha) \le q < r$, then there exists $y \in X_{\beta+1} \cap \mathrm{In}(\beta + 1)$ such that $y|_{\alpha+1} = x$ and $q < y(\beta) < r$;
(8) if $\alpha < \gamma$ and γ is limit, then for every $x \in X_{\alpha+1} \cap \mathrm{In}(\alpha + 1)$ and $q \in \mathbb{Q}$ satisfying $x(\alpha) < q$, there exists $y \in X_\gamma \cap \mathrm{In}(\gamma)$ such that $y|_{\alpha+1} = x$ and $y(\beta) < q$ for each $\beta < \gamma$;
(9) if $\gamma > 0$ is limit, then every $x \in X_\gamma$ is bounded, i.e., there exists $q \in \mathbb{Q}$ such that $x(\alpha) \le q$ for each $\alpha < \gamma$.

Note that if $x \in \mathbb{Q}^\alpha$ and $q \in \mathbb{Q}$, then we write $y = (x, q)$ instead of the usual $x^\frown q$ to denote the element $y \in \mathbb{Q}^{\alpha+1}$ defined by $y|_\alpha = x$ and $y(\alpha) = q$.

Put $X_1 = \mathbb{Q}$. Clearly (1)–(9) are fulfilled. Suppose that for some γ with $1 < \gamma < \omega_1$, we have defined a sequence $\{X_\alpha : 0 < \alpha < \gamma\}$ satisfying (1)–(9). Let us consider the following three cases.

I. Suppose that $\gamma = \alpha + 2$ for some $\alpha < \omega_1$. Put $\beta = \alpha + 1$. By (1), the set X_β is countable, so we can find a disjoint family $\{S_x : x \in X_\beta\}$ of subsets of $\mathbb{Q}$ such that each S_x is dense in $\mathbb{Q}$ with respect to the interval topology on $\mathbb{Q}$. Then we put $Z_\beta = X_\beta \cap \mathrm{In}(\beta)$ and

$$X_{\beta+1} = \{(x, x(\alpha)) : x \in X_\beta\} \cup \{(x, q) : x \in Z_\beta,\ q \in S_x,\ x(\alpha) < q\}.$$

It is easy to see that the sequence $\{X_\alpha : \alpha \le \beta + 1\}$ satisfies (1) and (3)–(9), so it remains to verify (2). Assume to the contrary that the set $X_{\beta+1}$ contains elements satisfying a non-trivial linear relation in $K^{\beta+1}$, say,

$$k_1(x_1, q_1) + k_2(x_2, q_2) + \cdots + k_n(x_n, q_n) = 0_{\beta+1}, \tag{2.1}$$

where $k_1, \ldots, k_n \in \mathbb{Z}\setminus\{0\}$ and $(x_1, q_1), \ldots, (x_n, q_n)$ are distinct elements of $X_{\beta+1}$ (so that $x_1, \ldots, x_n \in X_\beta$ and $q_1, \ldots, q_n \in \mathbb{Q}$). We can assume without loss of generality that q_1 is minimal among $q_1, \ldots, q_n$. It is easy to see that (2.1) must contain an expression

$$(y_1, r_1) - (y_2, r_2) + (y_3, r_3) - (y_4, r_4) + \ldots + (y_{2s-1}, r_{2s-1}) - (y_{2s}, r_{2s}) \tag{2.2}$$

with $(y_j, r_j) \in \{(x_1, q_1), \ldots, (x_n, q_n)\}$ for each $j \le 2s$, such that $(y_1, r_1) = (x_1, q_1)$, $(y_j, r_j) \neq (y_{j+1}, r_{j+1})$ for $j = 1, \ldots, 2s-1$, and $r_1 = r_2$, $y_2 = y_3$, $r_3 = r_4$, $\ldots$, $r_{2s-1} = r_{2s}$, $y_{2s} = y_1$. In particular, the sum (2.2) is equal to the neutral element of $K^{\beta+1}$.

Clearly, $(y_i, r_i) \in X_{\beta+1}$, so $y_i(\alpha) \le r_i$ for each $i = 1, \ldots, 2s$. Since $(y_1, r_1) \neq (y_2, r_2)$ and $r_1 = r_2$, we have $y_1 \neq y_2$. From $S_{y_1} \cap S_{y_2} = \emptyset$ and our definition of $X_{\beta+1}$ it follows that either $y_1(\alpha) = r_1$ or $y_2(\alpha) = r_2$. It suffices to consider the case $y_2(\alpha) = r_2$ (one reduces the first case to the second one by changing the signs in (2.2) and the enumeration of summands). Then $y_1(\alpha) \le r_1 = r_2 = y_2(\alpha)$. We claim that

$$r_{2i} = y_{2i}(\alpha) = y_{2i+1}(\alpha) < r_{2i+1} \text{ for } i = 1, \ldots, s-1. \tag{2.3}$$

Indeed, from $y_2 = y_3$ it follows that $r_2 \neq r_3$, and by the minimality of $r_1 = r_2$, $r_2 < r_3$. So, $r_2 = y_2(\alpha) = y_3(\alpha) < r_3$, which implies (2.3) for $i = 1$. Further, from $r_3 = r_4$ it follows that $y_3 \neq y_4$, and since $S_{y_3} \cap S_{y_4} = \emptyset$, we conclude that either $y_3(\alpha) = r_3$ or $y_4(\alpha) = r_4$. The first case is impossible, so we have $y_4(\alpha) = r_4$. Again, from $y_4 = y_5$ it follows that $r_4 \neq r_5$. Combining this with $r_4 = y_4(\alpha) = y_5(\alpha) \le r_5$, we infer that $r_4 < r_5$, that is, (2.3) holds for $i = 2$. Continuing this way, one proves (2.3) for each $i \le s-1$.

Finally, $r_{2s-1} = r_{2s}$ implies that $y_{2s-1} \neq y_{2s}$. Since $S_{y_{2s-1}} \cap S_{y_{2s}} = \emptyset$, we have either $y_{2s-1}(\alpha) = r_{2s-1}$ or $y_{2s}(\alpha) = r_{2s}$. The first case is impossible in view of (2.3) with $i = s-1$, so $y_{2s}(\alpha) = r_{2s}$. Then the equality $y_{2s} = y_1$ implies that $y_{2s}(\alpha) = y_1(\alpha) \le r_1$, and hence $r_{2s} \le r_1$. However, (2.3) and the equalities $r_{2i-1} = r_{2i}$ for $i = 1, \ldots, s$ together imply that

$$r_1 = r_2 < r_3 = r_4 < \cdots < r_{2s-1} = r_{2s} \le r_1,$$

which is a contradiction. This proves that the set $X_{\beta+1}$ is independent.

II. Suppose that $\gamma = \alpha+1$, where α is a limit ordinal. In this case the definition of X_γ is a little bit more complicated. Let $Y_\alpha = X_\alpha \setminus \mathrm{In}(\alpha)$. If $x \in Y_\alpha$, then there exists $\mu < \alpha$ such that $x(\mu) = x(\mu+1)$. Then by (6), $x(\nu) = x(\mu)$ for each ν satisfying $\mu < \nu < \alpha$. Denote this special value $x(\mu)$ of x by $c(x)$.

As in the previous case, there exists a disjoint family $\{S_x : x \in X_\alpha\}$ of dense subsets of the space $\mathbb{Q}$ endowed with the interval topology. We put $Z_\alpha = X_\alpha \cap \mathrm{In}(\alpha)$ and

$$\begin{aligned} X_{\alpha+1} = \{(x, c(x)) : x \in Y_\alpha\} \cup \{(x, q) : x \in Z_\alpha,\ q \in S_x,& \\ x(\nu) < q \text{ for each } \nu < \alpha\}&. \end{aligned}$$

A routine verification shows that the family $\{X_\nu : \nu \le \alpha+1\}$ satisfies (1) and (4)–(9). Since every $x \in X_\alpha$ is bounded by (9), we also have (3) at the step $\alpha+1$.

Therefore, we only have to check (2). If (2) fails to hold at step $\alpha+1$, we can find, as in case I, an expression

$$(2.4)\qquad (y_1,r_1)-(y_2,r_2)+(y_3,r_3)-(y_4,r_4)+\ldots+(y_{2s-1},r_{2s-1})-(y_{2s},r_{2s})$$

with $(y_i,r_i)\in X_{\alpha+1}$ for each $i\le 2s$, such that $(y_i,r_i)\ne(y_{i+1},r_{i+1})$ if $1\le i\le 2s-1$ and $r_1=r_2$, $y_2=y_3$, $r_3=r_4$, … , $r_{2s-1}=r_{2s}$, $y_{2s}=y_1$. In particular, the sum in (2.4) is equal to the neutral element of $K^{\alpha+1}$. Again, we can assume that $r_1\le r_i$ for each $i\le 2s$.

If $i\le 2s$ and $y_i\notin \mathrm{In}(\alpha)$, there exists $\mu_i<\alpha$ such that $y_i(\mu_i)=y_i(\mu_i+1)$, and hence (6) implies that $y_i(\mu_i)=c(y_i)=r_i$. Choose an ordinal $\nu<\alpha$ such that $\mu_i<\nu$ for each $i\le 2s$ with $x_i\notin\mathrm{In}(\alpha)$ and $z_i=\pi^\alpha_{\nu+1}(y_i)\ne\pi^\alpha_{\nu+1}(y_j)=z_j$ whenever $y_i\ne y_j$, $1\le i,j\le 2s$. Our choice of ν implies that the following conditions hold for each $i\le 2s$:

(i) if $z_i\notin\mathrm{In}(\nu+1)$, then $z_i(\nu)=r_i$;
(ii) if $z_i\in\mathrm{In}(\nu+1)$, then $z_i(\nu)<r_i$;
(iii) for every $j\le s$, either $z_{2j-1}(\nu)=r_{2j-1}$ or $z_{2j}(\nu)=r_{2j}$.

Indeed, if $z_i\notin\mathrm{In}(\nu+1)$, then $y_i\notin\mathrm{In}(\alpha)$, and hence $z_i(\nu)=y_i(\nu)=c(y_i)=r_i$ by the choice of ν. This gives (i). Similarly, if $z_i\in\mathrm{In}(\nu+1)$, then $y_i\in X_\alpha\cap\mathrm{In}(\alpha)=Z_\alpha$. Since $(y_i,r_i)\in X_{\alpha+1}$, our definition of $X_{\alpha+1}$ implies that $z_i(\nu)=y_i(\nu)<r_i$. This proves (ii). To verify (iii), assume that $z_{2j-1}(\nu)\ne r_{2j-1}$ and $z_{2j}(\nu)\ne r_{2j}$ for some $j\le s$. Then (i) implies that $z_{2j-1},z_{2j}\in\mathrm{In}(\nu+1)$, which in turn gives $y_{2j-1},y_{2j}\in\mathrm{In}(\alpha)\cap X_\alpha=Z_\alpha$. Since (y_{2j-1},r_{2j-1}) and (y_{2j},r_{2j}) are elements of $X_{\alpha+1}$, our definition of $X_{\alpha+1}$ implies that $r_{2j-1}\in S_{y_{2j-1}}$ and $r_{2j}\in S_{y_{2j}}$. By assumption, $r_{2j-1}=r_{2j}$ and $(y_{2j-1},r_{2j-1})\ne(y_{2j},r_{2j})$, so $y_{2j-1}\ne y_{2j}$. However, $r_j\in S_{y_{2j-1}}\cap S_{y_{2j}}\ne\emptyset$, which is a contradiction. This proves (iii).

Finally, consider the sum

$$(z_1,r_1)-(z_2,r_2)+(z_3,r_3)-(z_4,r_4)+\ldots+(z_{2s-1},r_{2s-1})-(z_{2s},r_{2s})$$

and apply the same argument as in case I along with (i)–(iii) to show that

$$r_1=r_2<r_3=r_4<\cdots<r_{2s-1}=r_{2s}\le r_1,$$

which gives a contradiction and finishes the verification of (2).

III. Suppose that γ is a limit ordinal. Consider the family

$$\mathcal{F}_\gamma=\{(x,q): x\in X_{\alpha+1}\cap\mathrm{In}(\alpha+1),\ q\in\mathbb{Q},\ \alpha<\gamma,\ x(\alpha)<q\}.$$

For every pair $(x,q)\in\mathcal{F}_\gamma$ with $x\in X_{\alpha+1}$, we shall define a function $y\in\mathbb{Q}^\gamma$ satisfying the following conditions for each $\beta<\gamma$:

(iv) $y\in\mathrm{In}(\gamma)$ and $\pi^\gamma_{\alpha+1}(y)=x$;
(v) $\pi^\gamma_\beta(y)\in X_\beta$;
(vi) $y(\beta)<q$.

For a given pair $(x,q) \in \mathcal{F}_\gamma$, choose a strictly increasing sequence $\{\beta_n : n \in \omega\} \subseteq \gamma$ such that $\beta_0 = \alpha$ and $\gamma = \lim_{n\in\omega} \beta_n$. Let also $\{q_n : n \in \omega\}$ be a strictly increasing sequence in $\mathbb{Q}$ such that $q_0 = x(\alpha)$ and $q_n < q$ for each $n \in \omega$. Using (7), we define by induction a sequence $\{y_n : n \in \omega\}$ such that $y_0 = x$, $y_n \in X_{\beta_n+1} \cap \mathrm{In}(\beta_n + 1)$, $q_n < y_n(\beta_n) < q_{n+1}$ and $\pi^{\beta_n+1}_{\beta_m+1}(y_n) = y_m$ whenever $m < n < \omega$. Then there exists a unique element $y \in \mathbb{Q}^\gamma$ such that $\pi^\gamma_{\beta_n+1}(y) = y_n$ for each $n \in \omega$. An easy verification shows that y satisfies (iv)–(vi). Denote this element $y \in \mathbb{Q}^\gamma$ by $y(x,q)$. In addition, if $\alpha < \gamma$ and $x \in X_{\alpha+1}$, denote by $\tilde{x}$ the element of $\mathbb{Q}^\gamma$ defined by $\tilde{x}|_{\alpha+1} = x$ and $\tilde{x}(\nu) = x(\alpha)$ for each ν satisfying $\alpha < \nu < \gamma$. It remains to put

$$X_\gamma = \{y(x,q) : (x,q) \in \mathcal{F}_\gamma\} \cup \{\tilde{x} : x \in X_{\alpha+1} \text{ for some } \alpha < \gamma\}.$$

A direct verification that the family $\{X_\beta : \beta < \gamma\}$ satisfies (1)–(9) is left to the reader. This finishes our recursive construction of the family $\{X_\alpha : \alpha < \omega_1\}$.

Let X be the inverse limit of the system $\{X_\alpha, \pi^\beta_\alpha : \alpha < \beta < \omega_1\}$. In other words, X consists of all $x \in \mathbb{Q}^{\omega_1}$ such that $\pi_\alpha(x) \in X_\alpha$ for each $\alpha < \omega_1$. From (4) it follows that every $x \in X$ is a non-decreasing function from ω_1 to $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, (6) implies that x is eventually constant, that is, there exists an ordinal $\alpha < \omega_1$ such that $x(\beta) = x(\alpha)$ for each β with $\alpha < \beta < \omega_1$. In what follows we identify $\mathbb{Q}^{\omega_1}$ with the corresponding subspace of K^{ω_1}. We claim that X has the following properties:

(d) $\pi_\beta(X) = X_\beta$ for each $\beta < \omega_1$;
(e) X is a linearly independent subset of K^{ω_1};
(f) X is closed in $(K^{\omega_1})_\omega$;
(g) X is not Lindelöf.

First, we check (d). Suppose that $\beta < \omega_1$ and $x \in X_\beta$. By (3), $\pi^{\alpha+1}_\alpha(X_{\alpha+1}) = X_\alpha$ for each $\alpha < \omega_1$, so we can assume that $\beta = \alpha + 1$. Use (5) and (6) to conclude that the function $y \in \mathbb{Q}^{\omega_1}$ defined by $y(\nu) = x(\nu)$ if $\nu \le \alpha$ and $y(\nu) = x(\alpha)$ if $\alpha < \nu < \omega_1$, belongs to X. Clearly, $\pi_\beta(y) = x$, which gives (d).

If $x_1, \dots, x_n$ is a finite subset of pairwise distinct elements of X, one can find $\alpha < \omega_1$ such that the elements $\pi_\alpha(x_1), \dots, \pi_\alpha(x_n)$ of X_α are also pairwise distinct. By (2), the set X_α is independent, so (e) is immediate.

Since $\mathbb{Q}^{\omega_1}$ is closed in $(K^{\omega_1})_\omega$, (f) will follow if we show that X is closed in $(\mathbb{Q}^{\omega_1})_\omega$, where $\mathbb{Q}$ carries the discrete topology. Suppose that $x \in \overline{X}$ for some $x \in (\mathbb{Q}^{\omega_1})_\omega$. Note that the projection $\pi_\alpha\colon (\mathbb{Q}^{\omega_1})_\omega \to (\mathbb{Q}^\alpha)_\omega$ onto the discrete space $(\mathbb{Q}^\alpha)_\omega$ is continuous for each $\alpha < \omega_1$. Therefore, $\pi_\alpha(x) \in \overline{X}_\alpha = X_\alpha$ for all $\alpha < \omega_1$. Since X is the inverse limit of the sets X_α's, we conclude that $x \in X$. This proves (f).

Finally, for every $x \in X$ take the minimal ordinal $\alpha = \alpha(x) < \omega_1$ such that $x(\alpha) = x(\alpha + 1)$, and define

$$U_x = \{y \in \mathbb{Q}^{\omega_1} : y(\alpha) = y(\alpha + 1)\}.$$

Then U_x is an open neighborhood of x in the Tychonoff topology on $\mathbb{Q}^{\omega_1}$ (we recall that $\mathbb{Q}$ is discrete). Clearly, $\mathcal{U} = \{U_x : x \in X\}$ is an open cover of X in $(\mathbb{Q}^{\omega_1})_\omega$, but no countable subfamily of $\mathcal{U}$ covers X. Indeed, let $\mathcal{V} = \{U_x : x \in C\}$ be a subfamily of $\mathcal{U}$, where $C \subseteq X$ is countable. Then there exists a limit ordinal $\gamma < \omega_1$ such that $\alpha(x) < \gamma$ for each $x \in C$. By (8), we can find $y \in X_\gamma \cap \mathrm{In}(\gamma)$, and (d) implies the existence of an element $z \in X$ such that $\pi_\gamma(z) = y$. It is clear that $z \in X \setminus \bigcup \mathcal{V}$, so (g) holds.

Our final step is to show that the subgroup H of K^{ω_1} generated by X is closed in $(K^{\omega_1})_\omega$. Suppose that $y \in \overline{H}$ for some $y \in K^{\omega_1} \setminus \{0\}$. Since the projection $\pi_\alpha \colon (K^{\omega_1})_\omega \to (K^\alpha)_\omega$ onto the discrete group $(K^\alpha)_\omega$ is continuous, we have $\pi_\alpha(y) \in \overline{\pi_\alpha(H)} = \pi_\alpha(H) = \langle X_\alpha \rangle$ for each $\alpha < \omega_1$. Therefore, if $\alpha < \omega_1$ and $\pi_\alpha(y) \neq 0_\alpha$, we can find non-zero integers $k_{\alpha,1}, \dots, k_{\alpha,n_\alpha}$ and pairwise distinct elements $x_{\alpha,1}, \dots, x_{\alpha,n_\alpha} \in X_\alpha$ such that

$$\pi_\alpha(y) = k_{\alpha,1}x_{\alpha,1} + \cdots + k_{\alpha,n_\alpha}x_{\alpha,n_\alpha}. \tag{2.5}$$

Suppose that $\alpha < \beta < \omega_1$ and let

$$\pi_\beta(y) = k_{\beta,1}x_{\beta,1} + \cdots + k_{\beta,n_\beta}x_{\beta,n_\beta} \tag{2.6}$$

be the representation of $\pi_\beta(y)$ corresponding to the ordinal β. Then

$$k_{\alpha,1}x_{\alpha,1} + \cdots + k_{\alpha,n_\alpha}x_{\alpha,n_\alpha} = k_{\beta,1}\pi^\beta_\alpha(x_{\beta,1}) + \cdots + k_{\beta,n_\beta}\pi^\beta_\alpha(x_{\beta,n_\beta}). \tag{2.7}$$

Since $x_{\alpha,i}$ and $\pi^\beta_\alpha(x_{\beta,j})$ with $i \le n_\alpha$ and $j \le n_\beta$ are elements of the independent set X_α, we conclude that some of $\pi^\beta_\alpha(x_{\beta,j})$ are equal to 0_α while the non-trivial part on the right-hand part of (2.7) coincides (after possible cancellations) with its left-hand part up to a permutation. This implies, in particular, that $k_\alpha = \sum_{i=1}^{n_\alpha} |k_{\alpha,i}| \le \sum_{j=1}^{n_\beta} |k_{\beta,j}| = k_\beta$. Therefore, the sequence $\{k_\alpha : \alpha < \omega_1\}$ stabilizes, i.e., there exist an ordinal $\nu < \omega_1$ and a positive integer k such that $k_\alpha = k$ for each α with $\nu \le \alpha < \omega_1$.

Suppose now that the ordinals α and β in (2.7) satisfy $\nu \le \alpha < \beta < \omega_1$. Then $k_\alpha = k = k_\beta$. Since X_α is independent, we have $n_\alpha = n_\beta = n$ and, in addition, there exists a permutation $\sigma = \sigma_{\beta,\alpha}$ of the set $\{1, \dots, n\}$ such that $\pi^\beta_\alpha(x_{\beta,i}) = x_{\alpha,\sigma(i)}$ and $k_{\beta,i} = k_{\alpha,\sigma(i)}$ for each $i = 1, \dots, n$. Since the representation (2.5) is unique for each $\alpha < \omega_1$, we must have

$$\sigma_{\beta,\alpha} \circ \sigma_{\gamma,\beta} = \sigma_{\gamma,\alpha} \ \text{ whenever } \ \nu \le \alpha < \beta < \gamma < \omega_1. \tag{2.8}$$

For every $\alpha > \nu$, use $\sigma_{\alpha,\nu}$ to renumerate the elements $x_{\alpha,1}, \dots, x_{\alpha,n}$ in order to have $x_{\nu,i} = \pi^\alpha_\nu(x_{\alpha,i})$ and $k_{\nu,i} = k_{\alpha,i}$ for each $i = 1, \dots, n$. Then (2.8) implies that $x_{\alpha,i} = \pi^{\beta}_{\alpha}(x_{\beta,i})$ and $k_{\alpha,i} = k_{\beta,i} = k_i$ whenever $\nu \le \alpha < \beta < \omega_1$ and $1 \le i \le n$. Since X is the inverse limit of X_α's, for every $i = 1, \dots, n$ there exists $x_i \in X$ such that $\pi_\alpha(x_i) = x_{\alpha,i}$ for all $\alpha < \omega_1$. We conclude, therefore, that $y = k_1x_1 + \cdots + k_nx_n \in H$. This proves that H is closed in $(K^{\omega_1})_\omega$. In other words, H satisfies (a) announced in the beginning of the proof.

From (f) it follows that X is closed in H and by (g), the group H is not Lindelöf. This implies (b). In addition, by (d) and (1), the group

$$\pi_\alpha(H) = \pi_\alpha(\langle X\rangle) = \langle \pi_\alpha(X)\rangle = \langle X_\alpha\rangle$$

is countable for each $\alpha < \omega_1$, which gives (c). So, H satisfies (a)–(c), and hence it is a complete $\aleph_0$-bounded P-group which fails to be $\mathbb{R}$-factorizable. $\square$

Note that the complete group H in Theorem 2.4 cannot be embedded as a subgroup into a Lindelöf topological group — otherwise H would be closed in such a group, hence Lindelöf. We conjecture that $\mathbb{R}$-factorizability has a stronger impact on topological groups:

Problem 1. Is every $\mathbb{R}$-factorizable P-group G topologically isomorphic to a subgroup of a Lindelöf group?

The combination of $\mathbb{R}$-factorizability and completeness looks even more promising:

Problem 2. Must every complete $\mathbb{R}$-factorizable P-group be Lindelöf?

We do not know the answer to the following question related to Lemma 2.3:

Problem 3. Let G be a complete topological group. Is then the group $(G)_\omega$ necessarily complete?

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