Commutative modular group algebras of p-mixed and p-splitting abelian Σ-groups

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Abstract. Let G be a p-mixed abelian group and R is a commutative perfect integral domain of char $R = p > 0$. Then, the first main result is that the group of all normalized invertible elements $V(RG)$ is a Σ-group if and only if G is a Σ-group. In particular, the second central result is that if G is a Σ-group, the R-algebras isomorphism $RA \cong RG$ between the group algebras RA and RG for an arbitrary but fixed group A implies A is a p-mixed abelian Σ-group and even more that the high subgroups of A and G are isomorphic, namely, $\mathcal{H}_A \cong \mathcal{H}_G$. Besides, when G is p-splitting and R is an algebraically closed field of char $R = p \neq 0$, $V(RG)$ is a Σ-group if and only if G_p and G/G_t are both Σ-groups. These statements combined with our recent results published in Math. J. Okayama Univ. (1998) almost exhausted the investigations on this theme concerning the description of the group structure.

Keywords: group algebras, high subgroups, p-mixed and p-splitting groups, Σ-groups

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1. Introduction

Standardly, throughout the text, denote by RG the R-group algebra of an arbitrary abelian group G over a commutative unitary ring R in prime characteristic, for instance, p. As usual, $V(RG)$ will designate the group of normed units (i.e. units of augmentation equal to 1), and $S(RG)$ is its p-primary Sylow component; $I(RG; G_p)$ is the relative augmentation ideal of RG with respect to the p-primary part G_p of G. For G a group, we let G^1 be the group of all infinite heights in G, or in other words $G^1 = \bigcap_p G^{p^\omega}$, where G^{p^ω} equal to the intersection of all G^{p^n} for $n < \omega$, is the first Ulm subgroup of G. A subgroup of G is said to be a high subgroup if it is maximal with respect to the intersection with G^1 equal to 1. All other unexplained notation and terminology will be in agreement with the classical books of L. Fuchs [16] and our articles [1]–[15].

The group $V(RG)$, that is on the focus of our interest, was studied in [1]–[15]. We established there criteria under which this group has some important properties, as more especially, necessary and sufficient conditions were found for $V(RG)$ to belong to certain classes of abelian groups. Our purpose here is to continue these studies for the class of so-called Σ-groups, introduced by Irwin-Walker in 1961 year (see, for example, [19]). The Σ-groups are mixed in general and form a quite large class of abelian groups that contains many other abelian groups as their subclasses; for instance such as all direct sums of countable torsion abelian groups.

By definition (see [19]), an abelian group A is said to be a Σ-group if some of its high subgroups is a direct sum of cyclic groups (consequently all of its high subgroups are direct sums of cyclics — see [21] and [18, 19]). A criterion for p-primary abelian groups to be Σ-groups was obtained by us in [3]. However, for the general mixed case there is no useful necessary and sufficient condition than the definition yet. Using the above mentioned criterion for the p-torsion case, we have obtained in [3] some results about the commutative group algebras of Σ-groups. That is why, it is a real goal to supply these statements with a new treatment of the Σ-structure in the commutative mixed modular group rings. So, our global aim here is to strengthen these facts by proving such similar assertions and removing some not needed conditions. Thus we almost settled a problem posed by us in [10] of finding a suitable criterion for $V(RG)$ to be a Σ-group.

Although there are many high subgroups of an abelian group A that are unisomorphic in general (cf. [19]), with no confusion, we shall consider in the sequel (no fixed) high subgroup defined as $\mathcal{H}_A$. The torsion subgroup and its p-component are denoted as A_t and A_p, respectively.

We continue with

2. High subgroups and the Direct Factor Problem in modular commutative group rings

We begin this section with the following technical matter.

Lemma 1. *Let G be a p-mixed abelian group and let R be an integral domain that is commutative with nonzero characteristic p. Then*

$$V(RG) = GS(RG).$$

Proof: The canonical epimorphism $G \to G/G_p$ can be linearly extended to epimorphism $V(RG) \to V(R(G/G_p))$ with kernel $1 + I(RG; G_p)$. By virtue of a classical result due to G. Higman [17], $V(R(G/G_p)) = G/G_p$. Thus $V(RG) \to G/G_p$ and $S(RG) \to 1$. Consequently $V(RG)/S(RG) \overset{\varphi}{\cong} G/G_p$ and $S(RG) = 1 + I(RG; G_p)$ (see also [2]). On the other hand, it is easy to see that the isomorphism φ^{-1} maps G/G_p onto $GS(RG)/S(RG)$, so $G/G_p \cong GS(RG)/S(RG)$, where this last isomorphism is clearly induced by φ^{-1}. Combining these claims, $V(RG)/S(RG) \overset{\varphi}{\cong} G/G_p \overset{\varphi^{-1}}{\cong} GS(RG)/S(RG)$. Finally, we derive, $V(RG) = GS(RG)$, thus concluding the proof. $\square$

In this aspect we formulate

Lemma 2. *Assume G is a p-mixed abelian group and R is a commutative integral domain of positive characteristic p. Then G is a pure subgroup of $V(RG)$.*

PROOF: Owing to the previous lemma, $V(RG) = GS(RG)$. Take an arbitrary natural number n. If $p \mid n$, $n = p^{k_1}p_2^{k_2}\dots p_t^{k_t}$, where $p_2 < \dots < p_t$ are prime numbers different from p. Thus $V^n(RG) = (GS(RG))^n = G^nS^n(RG) = G^nS^{p^{k_1}}(RG)$, hence $G \cap (G^nS^{p^{k_1}}(RG)) = G^n(G \cap S(R^{p^{k_1}}G^{p^{k_1}})) = G^nG_p^{p^{k_1}} = G^nG_p^n = G^n$, according to the modular law.

If now $p \nmid n$, $S^n(RG) = S(RG)$ and so $G \cap (GS(RG))^n = G \cap [G^nS(RG)] = G^n(G \cap S(RG)) = G^nG_p = G^nG_p^n = G^n$, employing the modular law as well. The proof is finished. □

The next technical affirmation is classical and well-known, but for the sake of completeness, we include its new proof.

Lemma 3. *A pure subgroup of an abelian Σ-group is also an abelian Σ-group.*

PROOF: Presume N is a pure subgroup of an abelian Σ-group C. Therefore $\mathcal{H}_C$ is a direct sum of cyclics. But $\mathcal{H}_N \cap C^1 = \mathcal{H}_N \cap N \cap C^1 = \mathcal{H}_N \cap N^1 = 1$, whence with no harm of generality we may presume that $\mathcal{H}_N \subseteq \mathcal{H}_C$ (actually $\mathcal{H}_N$ may be expanded in some high subgroup of C). A theorem due to L. Kulikov (see for example [16, p. 110, Theorem 18.1]) leads us to the fact that $\mathcal{H}_N$ is a direct sum of cyclics or equivalently N is a Σ-group too, as stated. The proof is verified. □

Lemma 4. *Let M be a subgroup of the abelian p-group C such that $M^{p^n}[p] = C^{p^n}[p]$ for each $n \geq 0$. Then $M = C$.*

PROOF: For $n = 0$, we write $M[p] = C[p]$. Next, by [16], [19] it remains to verify only that M is pure in C. To this end, $M \cap C^{p^n}[p] = M \cap M^{p^n}[p] = M^{p^n}[p]$ whenever $n \geq 1$. So, consulting with [16], [20], it holds true. The lemma is proved. □

Lemma 5. *Suppose C is an abelian p-group and N is its pure subgroup. Then*

$$\mathcal{H}_C \cap N = \mathcal{H}_N.$$

PROOF: Employing Lemma 4, it is enough to show that $(\mathcal{H}_C \cap N)^{p^n}[p] = (\mathcal{H}_N)^{p^n}[p]$ for some high subgroup $\mathcal{H}_N$ of N. Indeed, $(\mathcal{H}_C \cap N)^{p^n}[p] \subseteq (\mathcal{H}_C)^{p^n}[p] \cap N^{p^n}[p]$. In view of [19], we detect that $N^{p^n}[p] = (\mathcal{H}_N)^{p^n}[p] \times N^{p^\omega}[p]$ for some $\mathcal{H}_N$ such that $\mathcal{H}_N \subseteq \mathcal{H}_C$ as in Lemma 3 above argued. Since the modular law in [16] implies $(\mathcal{H}_C \cap N)^{p^n}[p] \subseteq (\mathcal{H}_N)^{p^n}[p]$ and the converse relation is evident, everything is proved. □

As an immediate consequence, we obtain

Corollary 6. *Let G be an abelian group and R a commutative ring with identity of prime characteristic p. Then*

$$\mathcal{H}_{S(RG)} \cap G_p = \mathcal{H}_{G_p}.$$

PROOF: It is a straightforward that G_p is pure in $S(RG)$ (see cf. [2], [4]), whence Lemma 5 is applicable to finish the proof. □

The following technical matter is crucial for the further conclusions.

Lemma 7. *Let G be an abelian group and let R be a commutative ring with unity in prime characteristic p. Then*

$$(\mathcal{H}_{S(RG)}G_p)[p] = \mathcal{H}_{S(RG)}[p]G[p].$$

PROOF: Let z be an arbitrary element from the left hand-side. So, $z = hg$ and $h^p = g^{-p}$, where $h \in \mathcal{H}_{S(RG)}$ and $g \in G_p$. Certainly, $g^p \in \mathcal{H}_{S(RG)} \cap G_p$, hence by using Corollary 6, $g^p \in \mathcal{H}_{G_p}$. But $\mathcal{H}_{G_p}$ is pure in G_p according to [18], therefore it is a routine matter to obtain that $g^p \in \mathcal{H}^p_{G_p}$. That is why, $g \in \mathcal{H}_{G_p}G[p]$. Finally, $z \in \mathcal{H}_{S(RG)}[p]G[p]$, and thus the relation "$\subseteq$" is fulfilled.

The reverse dependence "$\supseteq$" is elementary. The proof is finished. □

Now, we are in a position to attack the significant affirmation that characterizes the high subgroups in $V(RG)$, namely we formulate

Theorem 8. *Let G be a p-mixed abelian group and R a commutative integral domain of positive characteristic p. Then the following explicit formula is valid:*

$$(*) \qquad \mathcal{H}_{V(RG)} = \mathcal{H}_G\mathcal{H}_{S(RG)}.$$

PROOF: Foremost, we will show that $(\mathcal{H}_G\mathcal{H}_{S(RG)}) \cap V^1(RG) = 1$. In fact, first, complying with Lemma 1, $V(RG) = GS(RG)$. Therefore, we need to compute $(GS(RG))^1 = \bigcap_{q \neq p} (GS(RG))^{q^\omega} \cap (GS(RG))^{p^\omega}$, where all q are primes. And so, we observe that $(GS(RG))^{p^\omega} = G^{p^\omega}S(R^{p^\omega}G^{p^\omega})$. Really, choose an arbitrary element x from the left hand-side. Hence $x \in \bigcap_{n<\omega} (GS(RG))^{p^n} = \bigcap_{n<\omega} (G^{p^n}S(R^{p^n}G^{p^n}))$ and $x = g^{p^n}\sum_i r_i^{p^n}g_i^{p^n} = g'^{p^m}\sum_i r_i'^{p^m}g_i'^{p^m} = \dots$, where $g, g' \in G$; $r_i, r_i' \in R$; $g_i, g_i' \in G$; $i \in \mathbb{N}$ and $n < m$ are positive integers. Furthermore, the canonical forms imply $g^{p^n}g_i^{p^n} = g'^{p^m}g_i'^{p^m}$ and $r_i^{p^n} = r_i'^{p^m}$. Since $\sum_i r_i^{p^n}g_i^{p^n} \in S(RG)$, there is a member of this sum that belongs to G_p, say $g_1^{p^n} \in G_p^{p^n}$. By the same reason, we can presume that $g_1'^{p^m} \in G_p^{p^m}$. Finally, $x = (gg_1)^{p^n} \sum_i r_i^{p^n}(g_ig_1^{-1})^{p^n}$. But $(gg_1)^{p^n} \in G_p^{p^m}$ and $\sum_i r_i^{p^n}(g_ig_1^{-1})^{p^n} = \sum_i r_i'^{p^m}(g_i'g_1'^{-1})^{p^m} \in S(R^{p^m}G^{p^m})$. Applying the same inductive procedure for infinitely many element's rations, we can calculate that $x \in G^{p^\omega}S(R^{p^\omega}G^{p^\omega})$, as required.

Next, we see that $(GS(RG))^{q^\omega} = G^{q^\omega}S(RG)$ for every prime $q \neq p$, whence $\bigcap_{q\neq p}(GS(RG))^{q^\omega} = \bigcap_{q\neq p}(G^{q^\omega}S(RG)) = (\bigcap_{q\neq p}G^{q^\omega})S(RG)$.

Indeed, take an arbitrary element x from the left hand-side of $(GS(RG))^{q^\omega}$. Then $x \in \bigcap_{n<\omega}(GS(RG))^{q^n} = \bigcap_{n<\omega}(G^{q^n}S(RG))$, and so $x = g_n^{q^n}v_n = g_m^{q^m}v_m = \dots$, where $g_n, g_m \in G$; $v_n, v_m \in S(RG)$. Therefore, $g_n^{q^n}g_m^{-q^m} \in S(RG) \cap G^{q^n} = (G^{q^n})_p \subseteq G_p = G_p^{q^m}$, whence $g_n^{q^n} \in G^{q^m}$, i.e. $g_n^{q^n} \in \bigcap_{n<\omega}G^{q^n} = G^{q^\omega}$. Thus, the first equality is manifestly satisfied.

Now we take an arbitrary element x from $\bigcap_{q\neq p}(G^{q^\omega}S(RG))$. Hence, $x = g_{q_1}v_1 = g_{q_2}v_2 = \dots$, where $g_{q_1} \in G^{q_1^\omega}$, $v_1 \in S(RG)$; $g_{q_2} \in G^{q_2^\omega}$, $v_2 \in S(RG)$; $q_1 \neq q_2$ are prime numbers different from p. Therefore, $g_{q_1}g_{q_2}^{-1} \in S(RG) \cap G = G_p$. Finally, we get that $g_{q_1} \in G^{q_2^\omega}$ because $G_p = G_p^{q_2}$ whenever $q_2 \neq p$. Thus $(GS(RG))^1 = [(\bigcap_{q\neq p}G^{q^\omega})S(RG)] \cap [G^{p^\omega}S(R^{p^\omega}G^{p^\omega})] = (\bigcap_p G^{p^\omega})S(R^{p^\omega}G^{p^\omega})$. In order to prove this, we see by the modular law in [16] that $[(\bigcap_{q\neq p}G^{q^\omega})S(RG)] \cap (G^{p^\omega}S(R^{p^\omega}G^{p^\omega})) = S(R^{p^\omega}G^{p^\omega})[(\bigcap_{q\neq p}G^{q^\omega})S(RG)] \cap G^{p^\omega}]$. Further, given an arbitrary element $x \in [(\bigcap_{q\neq p}G^{p^\omega})S(RG)] \cap G^{p^\omega}$, we derive $x = av = g$, where $a \in \bigcap_{q\neq p}G^{q^\omega}$, $v \in S(RG)$ and $g \in G^{p^\omega}$. Since $v \in S(RG)$, there exists $c_p \in G_p$ such that $ac_p = g$. Henceforth, it is plain that $x \in (\bigcap_p G^{p^\omega}) = G^1 = \bigcap_n G^n$. Finally, we deduce that $[GS(RG)]^1 = G^1S^1(RG) = G^1S(R^{p^\omega}G^{p^\omega})$. After this, we concentrate on the intersection $(\mathcal{H}_G\mathcal{H}_{S(RG)}) \cap V^1(RG) = [\mathcal{H}_G\mathcal{H}_{S(RG)}] \cap [G^1S(R^{p^\omega}G^{p^\omega})]$. Foremost, we shall show that it possesses only trivial p-elements. In order to argue this, bearing in mind that $(\mathcal{H}_G)_p = \mathcal{H}_{G_p}$ [19, Theorem 13], we observe that $[\mathcal{H}_G\mathcal{H}_{S(RG)}]_p \cap [G^1S(R^{p^\omega}G^{p^\omega})]_p = [\mathcal{H}_{G_p}\mathcal{H}_{S(RG)}] \cap [G_p^1S(R^{p^\omega}G^{p^\omega})] = \mathcal{H}_{S(RG)} \cap S(R^{p^\omega}G^{p^\omega}) = 1$. And so, next choose an arbitrary element x belonging to the intersection that we examine. Hence, there is a natural d with the property $x^{p^d} \in \mathcal{H}_G \cap G^1 = 1$. Thus x is a p-element, and by what we have just shown above, $x = 1$, as desired. Then, $[\mathcal{H}_G\mathcal{H}_{S(RG)}] \cap V^1(RG) = 1$.

Suppose now there exists $v \in V(RG) \setminus \mathcal{H}_G\mathcal{H}_{S(RG)}$ such that $\langle \mathcal{H}_G\mathcal{H}_{S(RG)}, v\rangle \cap (G^1 S(R^{p^\omega} G^{p^\omega})) = 1$. Writing by Lemma 1 that $v = gw$, where $g \in G$ and $w \in S(RG)$, we extract $\langle \mathcal{H}_G\mathcal{H}_{S(RG)}, gw\rangle \cap (G^1 S(R^{p^\omega} G^{p^\omega})) = 1$. From this, it follows that $\langle \mathcal{H}_{S(RG)}, gw\rangle \cap S(R^{p^\omega} G^{p^\omega}) = 1$ and $\langle \mathcal{H}_G, gw\rangle \cap G^1 = 1$. Let us presume that $w^{p^t} = 1$ for some $t \in \mathbb{N}$.

If now $g \in G_p$, then $gw \in S(RG) \setminus \mathcal{H}_{S(RG)}$ and so $\langle \mathcal{H}_{S(RG)}, gw\rangle \cap S(R^{p^\omega} G^{p^\omega}) = 1$ leads us to a contradiction.

Otherwise, when $g \in G \setminus G_p$, we detect $\langle \mathcal{H}_G, gw\rangle^{p^t} = \langle \mathcal{H}_G, g\rangle^{p^t} = \langle (\mathcal{H}_G)^{p^t}, g^{p^t}\rangle = \langle \mathcal{H}_{G^{p^t}}, g^{p^t}\rangle$, where the last equality holds true by virtue of [18]. If $g^{p^t} \notin \mathcal{H}_G$, i.e. $g^{p^t} \notin \mathcal{H}_{G^{p^t}}$, we obtain $\langle \mathcal{H}_{G^{p^t}}, g^{p^t}\rangle \cap G^1 = \langle \mathcal{H}_G, gw\rangle^{p^t} \cap G^1 = 1$, that is false.

In the remaining case when $g^{p^t} \in \mathcal{H}_G$, i.e. by the purity of $\mathcal{H}_G$ in G (see [19]) $g^{p^t} \in \mathcal{H}_G^{p^t}$, we have $g \in a\mathcal{H}_G$ whenever $a \in G$ with $a^{p^t} = 1$. Thus, $\langle \mathcal{H}_G\mathcal{H}_{S(RG)}, gw\rangle = \langle \mathcal{H}_G\mathcal{H}_{S(RG)}, aw\rangle$, where $aw \in S(RG) \setminus \mathcal{H}_{S(RG)}$. That is why $\langle \mathcal{H}_{S(RG)}, aw\rangle \cap S(R^{p^\omega} G^{p^\omega}) \subseteq \langle \mathcal{H}_G\mathcal{H}_{S(RG)}, gw\rangle \cap (G^1 S(R^{p^\omega} G^{p^\omega})) = 1$, which is wrong.

Finally, our above supposition is invalid, so $\mathcal{H}_G\mathcal{H}_{S(RG)}$ is a high subgroup of $V(RG)$, as stated. The proof is finished. □

Remark. The maximal divisible subgroups and p-basic subgroups of the p-mixed group $V(RG)$ were classified in [8], [11]; [8], [12], [13], [14], respectively. Moreover, no every high subgroup of $S(RG)$ is of the above kind.

Now, we can prepare the other key statement which determine the high subgroups in the factor-group $S(RG)/G_p$, namely we state

Theorem 9. *Let G be an abelian group and R be a commutative unitary ring without nilpotent elements in prime characteristic p. Then the following explicit formula is fulfilled:*

$$(**) \qquad \mathcal{H}_{S(RG)/G_p} = \mathcal{H}_{S(RG)}G_p/G_p \cong \mathcal{H}_{S(RG)}/\mathcal{H}_{G_p}.$$

Proof: First, we consider the intersection $[(\mathcal{H}_{S(RG)}G_p/G_p) \cap (S(RG)/G_p)^{p^\omega}][p]$. Since G_p is nice in $S(RG)$ (see, for instance, [3], [5], [11]), this ration may be represented as $(\mathcal{H}_{S(RG)}G_p/G_p)[p] \cap (S^{p^\omega}(RG)G_p/G_p)$. Because, as we have observed, G_p is pure in $S(RG)$, whence in $\mathcal{H}_{S(RG)}G_p$, then adapting Lemma 7, $(\mathcal{H}_{S(RG)}G_p/G_p)[p] = (\mathcal{H}_{S(RG)}G_p)[p]G_p/G_p = \mathcal{H}_{S(RG)}[p]G_p/G_p$.

Furthermore, the examined intersection is equal to $(\mathcal{H}_{S(RG)}[p]G_p/G_p) \cap (S^{p^\omega}(RG)G_p/G_p) = [(\mathcal{H}_{S(RG)}[p]G_p) \cap (S^{p^\omega}(RG)G_p)]/G_p$. The modular law (cf. [16]) does imply that $(\mathcal{H}_{S(RG)}[p]G_p) \cap (S^{p^\omega}(RG)G_p) = G_p[\mathcal{H}_{S(RG)}[p] \cap (G_pS^{p^\omega}(RG))]$. But [2], [3] together with [18], [19] give $S^{p^\omega}(RG) = S(R^{p^\omega} G^{p^\omega})$

and $(G_pS(R^{p^\omega}G^{p^\omega}))[p] = G[p]S(R^{p^\omega}G^{p^\omega})[p] = \mathcal{H}_{G_p}[p]S(R^{p^\omega}G^{p^\omega})[p]$. Thus, again in view of the cited modular law, $\mathcal{H}_{S(RG)}[p] \cap (G_pS^{p^\omega}(RG)) = \mathcal{H}_{G_p}[p]$. Consequently, the studied intersection is precisely 1.

Moreover, by making use of [18], [19], $S(RG)[p] = \mathcal{H}_{S(RG)}[p] \times S^{p^\omega}(RG)[p]$. Therefore, by what we have just demonstrated,

$$S(RG)[p]G_p/G_p = (\mathcal{H}_{S(RG)}[p]G_p/G_p) \times (S^{p^\omega}(RG)[p]G_p/G_p),$$

i.e. in other words $(S(RG)/G_p)[p] = (\mathcal{H}_{S(RG)}G_p/G_p)[p] \times (S^{p^\omega}(RG)G_p/G_p)[p] = (\mathcal{H}_{S(RG)}G_p/G_p)[p] \times (S(RG)/G_p)^{p^\omega}[p]$.

Next, we will argue that $\mathcal{H}_{S(RG)}G_p/G_p$ is pure in $S(RG)/G_p$ which by [16] is equivalent to prove that $\mathcal{H}_{S(RG)}G_p$ is pure in $S(RG)$ because as we have seen above, G_p is pure in $S(RG)$. In order to show this, exploiting [20] or [16], it is sufficient to compute that $(\mathcal{H}_{S(RG)}G_p)[p] \cap S^{p^n}(RG) = (\mathcal{H}_{S(RG)}G_p)^{p^n}[p]$ for each natural number n. Well, owing to Lemma 7, $(\mathcal{H}_{S(RG)}G_p)[p] = \mathcal{H}_{S(RG)}[p]G[p]$. From [18], it follows that $G[p] = \mathcal{H}_{G_p}[p] \times G^{p^\omega}[p]$. Thus, using the modular law in [16] together with the purity of the high subgroups [18], [19], it obviously holds that $(\mathcal{H}_{S(RG)}G_p)[p] \cap S^{p^n}(RG) = G^{p^\omega}[p]\mathcal{H}^{p^n}_{S(RG)}[p] \subseteq (G_p^{p^n}\mathcal{H}^{p^n}_{S(RG)})[p] = (\mathcal{H}_{S(RG)}G_p)^{p^n}[p]$, as required. Our claim on purity is extracted. As a final, combining these two general conclusions, we derive that $\mathcal{H}_{S(RG)}G_p/G_p$ is indeed a high subgroup of $S(RG)/G_p$, as claimed. The isomorphism relation is valid taking into account Corollary 6. We are done. □

Remark. Maximal divisible subgroups and basic subgroups for the quotient group $S(RG)/G_p$ were described in [11]; [8], [12], [13], [14], respectively. Besides, no each high subgroup of $S(RG)/G_p$ is of the present type.

We conclude the major part with

3. Main attainments and their proofs

We now come to the first important attainment, namely

Theorem 10. *Suppose G is a p-mixed abelian group and R is a perfect commutative integral domain in characteristic $p \neq 0$. Then $V(RG)$ is a Σ-group if and only if G is a Σ-group.*

Suppose G is a p-splitting abelian group and R is an algebraically closed field in characteristic $p > 0$. Then $V(RG)$ is a Σ-group if and only if G_p and G/G_t are Σ-groups. In particular, if G is a torsion Σ-group, then $V(RG)$ and $V(RG)/G$ are Σ-groups.

Proof: Foremost, we deal with the first half. For the necessity, we observe that Lemmas 2 and 3 are applicable.

For the sufficiency, by making use of formula $(*)$ from Theorem 8 in the preliminary paragraph, we derive $\mathcal{H}_{V(RG)} = \mathcal{H}_G\mathcal{H}_{S(RG)}$. On the other hand, applying [19] or Lemma 3, G a Σ-group implies that so is G_p, hence according to [3], the same holds for $S(RG)/G_p$. Therefore, owing to Theorem 9, $\mathcal{H}_{S(RG)/G_p} \cong \mathcal{H}_{S(RG)}/\mathcal{H}_{G_p}$ is a direct sum of cyclics. Moreover, a claim of [18], [19] asserts that $\mathcal{H}_{G_p}$ is pure in G_p, whence in $S(RG)$, whence in $\mathcal{H}_{S(RG)}$. Consequently, a result due to L. Kulikov ([16, p. 143, Theorem 28.2]) guarantees that $\mathcal{H}_{G_p}$ is a direct factor of $\mathcal{H}_{S(RG)}$ with a direct sum of cyclic groups complementary factor. Furthermore, it follows from the present equality that $\mathcal{H}_{V(RG)} \cong \mathcal{H}_G \times \mathcal{H}_{S(RG)/G_p} \cong \mathcal{H}_G \times \mathcal{H}_{V(RG)/G}$. That is why, $\mathcal{H}_{V(RG)}$ is a direct sum of cyclics since $\mathcal{H}_G$ is, or equivalently by the definition, $V(RG)$ is a Σ-group, as claimed.

For the second part, complying with [9], we write $G = G_p \times G/G_p$ and $V(RG) \times R^* \cong S(RG) \times \times_{|G|}(R^* \times G/G_t)$. Because of the fact that R^* is divisible whence a Σ-group, the claim is fulfilled by using of [3], [19] and Lemma 3.

Moreover, via [9], we derive $V(RG)/G \cong S(RG)/G_p \times V(R(G/G_p))/(G/G_p)$. But when G is torsion, $V(R(G/G_p))$ is divisible again from [9], hence so is $V(R(G/G_p))/(G/G_p)$. Since G_p is a Σ-group, we can apply [3] and [19] to conclude $V(RG)/G$ is a Σ-group. The proof is completed.

□

We are now ready to proceed by proving the second central assertion, namely we formulate and argue

Theorem 11. *Suppose G is a p-mixed abelian Σ-group and let R be a commutative ring with unity of prime characteristic p. Then $RA \cong RG$ as R-algebras for any group A yields that $\mathcal{H}_A \cong \mathcal{H}_G$, and thus A is a p-mixed abelian Σ-group as well.*

Proof: First, we know that $RA \cong RG$ does imply $PA \cong PG$ as P-algebras for some perfect field P with $\mathrm{char}\, P = p$. It is trivial that this P-isomorphism ensures that A must be also a p-mixed abelian group. Moreover, $V(PA) \cong V(PG)$ and $\mathcal{H}_{V(PA)} \cong \mathcal{H}_{V(PG)}$ because in virtue of Theorem 10 and [19], [21] the high subgroups would be p-basic, so [16] is applicable. By what we have just proved in Theorem 10, A is a Σ-group, too. So, $\mathcal{H}_G$ and $\mathcal{H}_A$ are both direct sums of cyclics, whence they split. Written down, $\mathcal{H}_G = (\mathcal{H}_G)_p \times \mathcal{H}_G/(\mathcal{H}_G)_p$ and by the same reason $\mathcal{H}_A = (\mathcal{H}_A)_p \times \mathcal{H}_A/(\mathcal{H}_A)_p$. But, conforming with [19, Theorem 13], $(\mathcal{H}_G)_p = \mathcal{H}_{G_p}$ and by symmetry $(\mathcal{H}_A)_p = \mathcal{H}_{A_p}$, respectively. On the other hand, $PG \cong PA$ gives $\mathcal{H}_{G_p} \cong \mathcal{H}_{A_p}$ consulting with [3]. Besides, the formula $(*)$ from Theorem 8 along with Corollary 6 mean that $\mathcal{H}_G/\mathcal{H}_{G_p} \cong \mathcal{H}_G\mathcal{H}_{S(PG)}/\mathcal{H}_{S(PG)} = \mathcal{H}_{V(PG)}/\mathcal{H}_{S(PG)} \cong \mathcal{H}_{V(PA)}/\mathcal{H}_{S(PA)} \cong \mathcal{H}_A/\mathcal{H}_{A_p}$. Finally, we deduce $\mathcal{H}_G \cong \mathcal{H}_A$, as desired. The proof of the statement is verified. □

Remark. The last statement improves a result of this type obtained by us in ([3, Theorem - Corollary, p. 83]).

Moreover, it is well to note that $\mathcal{H}_{G/G_p} \ncong \mathcal{H}_G/\mathcal{H}_{G_p}$ in general, so the usage of formula $(*)$ is needed.

We close the manuscript with

4. Concluding discussion and remarks

The question is still left open about obtaining a general criterion for $V(RG)$ to be a Σ-group (see, for example, [10]). As we have observed above, some of our own structural characterizations and descriptions of $V(RG)$ given in [9] along with the results established here and listed above yield such a restricted necessary and sufficient condition for the modular and semisimple case. Thus we almost complete the problem, but the main case is unsolved yet.

Of some majority and interest is also the question whether the isomorphism of commutative group algebras over all fields preserves the property of being a Σ-group (when G/G_p is reduced see [3]), or, in other words, if G is a Σ-group and for every field F the F-algebra isomorphism $FG \cong FA$ holds, is it true that A is a Σ-group?

The other central problem is the Direct Factor Conjecture for Σ-groups, namely is it fulfilled that G is a direct factor of $V(RG)$ whenever G is a p-mixed abelian Σ-group, or more generally for such a group G does it follow that $V(RG)/G$ is a simply presented p-group?

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References

[1] Danchev P.V., *Units in abelian group rings of prime characteristic*, C.R. Acad. Bulgare Sci. **48** (8) (1995), 5–8.

[2] Danchev P.V., *Commutative group algebras of σ-summable abelian groups*, Proc. Amer. Math. Soc. **125** (9) (1997), 2559–2564.

[3] Danchev P.V., *Commutative group algebras of abelian Σ-groups*, Math. J. Okayama Univ. **40** (1998), 77–90.

[4] Danchev P.V., *Isomorphism of modular group algebras of totally projective abelian groups*, Comm. Algebra **28** (5) (2000), 2521–2531.

[5] Danchev P.V., *Modular group algebras of coproducts of countable abelian groups*, Hokkaido Math. J. **29** (2) (2000), 255–262.

[6] Danchev P.V., *Unit groups and Sylow p-subgroups in commutative group rings of prime characteristic p*, C.R. Acad. Bulgare Sci. **54** (1) (2001), 7–10.

[7] Danchev P.V., *Sylow p-subgroups of modular abelian group rings*, C.R. Acad. Bulgare Sci. **54** (2) (2001), 5–8.

[8] Danchev P.V., *Sylow p-subgroups of commutative modular and semisimple group rings*, C.R. Acad. Bulgare Sci. **54** (6) (2001), 5–6.

[9] Danchev P.V., *Normed units in abelian group rings*, Glasgow Math. J. **43** (3) (2001), 365–373.

[10] Danchev P.V., *Criteria for unit groups in commutative group rings*, submitted.
[11] Danchev P.V., *Maximal divisible subgroups in modular group algebras of p-mixed and p-splitting abelian groups*, submitted.
[12] Danchev P.V., *Basic subgroups in abelian group rings*, Czechoslovak Math. J. **52** (2002), no. 1, 129–140.
[13] Danchev P.V., *Basic subgroups in commutative modular group rings*, submitted.
[14] Danchev P.V., *Basic subgroups in group rings of abelian groups*, J. Group Theory, to appear.
[15] Danchev P.V., *Commutative modular group algebras of Warfield Abelian groups*, Trans. Amer. Math. Soc., to appear.
[16] Fuchs L., *Infinite Abelian Groups*, vol. I and II, Mir, Moscow, 1974 and 1977 (in Russian).
[17] Higman G., *The units of group rings*, Proc. London Math. Soc. **46** (1938–39), 231–248.
[18] Irwin J.M., *High subgroups of abelian torsion groups*, Pacific J. Math. **11** (1961), 1375–1384.
[19] Irwin J.M., Walker E.A., *On N-high subgroups of abelian groups*, Pacific J. Math. **11** (1961), 1363–1374.
[20] Irwin J.M., Walker E.A., *On isotype subgroups of abelian groups*, Bull. Soc. Math. France **89** (1961), 451–460.
[21] Irwin J.M., Peercy C., Walker E.A., *Splitting properties of high subgroups*, Bull. Soc. Math. France **90** (1962), 185–192.

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