On a parabolic problem with nonlinear Newton boundary conditions

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Abstract. The paper is concerned with the study of a parabolic initial-boundary value problem with nonlinear Newton boundary condition considered in a two-dimensional domain. The goal is to prove the existence and uniqueness of a weak solution to the problem in the case when the nonlinearity in the Newton boundary condition does not satisfy any monotonicity condition and to analyze the finite element approximation.

Keywords: parabolic convection-diffusion equation, nonlinear Newton boundary condition, Galerkin method, compactness method, finite element approximation, error estimates

Classification: 35K60, 65N30, 65N15

Introduction

A number of problems of technology and science are described by partial differential equations equipped with nonlinear Newton boundary conditions. Let us mention, e.g. radiation and heat transfer problems ([2], [22], [27]), modeling of electrolysis of aluminium with turbulent flow at the boundary ([11], [29]) and some problems of elasticity ([18]). Our paper was inspired by some nonstandard applications in biology, where the nutrition of kernels of plants can be described by a parabolic partial differential equation equipped by mixed Dirichlet - nonlinear Newton boundary conditions (see, e.g., [1], [7]).

Up to now, elliptic problems equipped with Newton nonlinear boundary conditions have been analyzed analytically as well as numerically. In the analysis of these problems one meets a number of obstacles, particularly in the very topical case when the nonlinearity is unbounded and has a polynomial behaviour. The first results for a problem of this type were obtained in [11], where the existence and uniqueness of the solution of the continuous problem was proved with the aid of the monotone operator theory and the convergence of the approximate solutions to the exact one was established under the assumption that all integrals appearing in the discrete problem were evaluated exactly. In [12], the convergence of the finite element method was proved in the case that both volume and boundary integrals were calculated with the aid of quadrature formulae. In the analysis of the boundary terms it was not possible to apply the well-known Ciarlet–Raviart theory ([5], [6]) of the finite element numerical integration because of the nonlinearity on the boundary. The convergence analysis was obtained with the aid of a suitable modification of results from [33]. Furthermore, the work [13] is concerned with the derivation of error estimates. They were obtained thanks to the uniform monotonicity of the problem in [13]. However, in contrast to standard nonlinear situations treated, e.g., in [4], [16], [17], [34], where strong monotonicity was used, an optimal $O(h)$ error estimate for linear finite elements was not achieved. The order of convergence is reduced due to the fact that only uniform monotonicity with growth of degree $t^{2+\alpha}$, $\alpha > 0$, holds now, and due to the nonlinearity in the boundary integrals. Moreover, also the application of numerical integration in the nonlinear boundary integral can lead to a further reduction of the rate of convergence. The theoretically established decrease of the order of convergence caused by the nonlinearity in the Newton boundary condition was confirmed with the aid of numerical experiments in [15]. Finally, in [14], the effect of the approximation of a curved boundary is analyzed with the aid of Zlámal's concepts of ideal triangulation and ideal interpolation ([35]). Let us also mention that another approach was used in [19] and [20], where the problem for the Laplace equation with nonlinear Newton boundary condition was transformed to a nonlinear boundary integral equation.

Practical applications often require the solution of nonstationary transient problems with Newton boundary conditions. In this paper we shall be concerned with the analysis of nonstationary convection-diffusion problem equipped with mixed Dirichlet - nonlinear Newton boundary conditions. In Section 1, the continuous problem is formulated. The concept of a weak solution is introduced and some auxiliary results are established. We assume that the nonlinearity in the Newton boundary condition has a linear growth and is Lipschitz-continuous. Section 2 is devoted to the proof of the existence and uniqueness of the weak solution. In Section 3, under the assumption that the space domain is polygonal, the finite element solution is analyzed and error estimates for the semidiscretization in space are obtained.

1. Formulation of the problem

1.1 Function spaces and classical formulation.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$ consisting of three parts Γ_1, Γ_2, Γ_3, see Figure 1. By $\overline{\Omega}$ we denote the closure of Ω. For $T > 0$ let us denote by Q_T the space-time cylinder $\Omega \times (0, T)$. Let $\mathbf{n}$ denote the unit outer normal to Γ. By $\mathbb{N}$ we denote the set of all positive integers.

We introduce the following notation of function spaces:

$C(\overline{\Omega})$ — space of functions continuous in $\overline{\Omega}$;
$C^k(\overline{\Omega})$, $k \in \mathbb{N}$ — space of functions having continuous derivatives of order k in $\overline{\Omega}$;

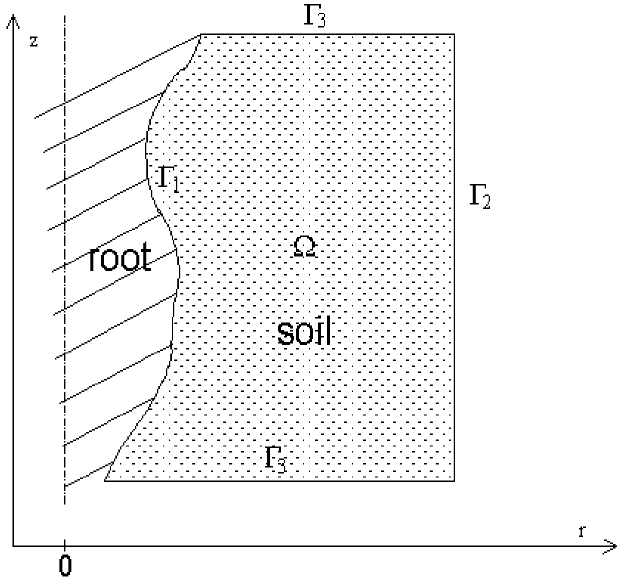

Figure 1: Computational domain

$C_0^\infty(0,T)$ — space of infinitely differentiable functions with compact support in $(0,T)$;
$L^p(\Omega)$, $1 \le p < \infty$ — space of measurable functions whose pth power is Lebesgue integrable over Ω, equipped with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{1/p}; \tag{1.1}$$

$L^2_\alpha(\Omega)$, where $\alpha \in C(\overline{\Omega})$, $\alpha_1 \ge \alpha \ge \alpha_0 > 0$, $\alpha_0, \alpha_1 = \text{const}$, is the α-weighted L^2-space which is a Hilbert space with the scalar product

$$(u,v)_\alpha = \int_\Omega \alpha(x) u(x) v(x) \, dx; \tag{1.2}$$

$W^{k,p}(\Omega)$, $1 \le p < \infty$ — Sobolev space of functions from $L^p(\Omega)$ whose distribution derivatives of order $\le k$ are elements of $L^p(\Omega)$, equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}, \tag{1.3}$$

where $D^\alpha u = \frac{\partial^{\alpha_1+\alpha_2} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$, and $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$. We set $H^k(\Omega) = W^{k,2}(\Omega)$. In $H^1(\Omega)$ we shall also work with the seminorm

$$|u|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2}; \tag{1.4}$$

$H^\mu(\Omega), \mu \in (\frac{1}{2}, 1)$ — space of functions $u \in L^2(\Omega)$, for which

$$\begin{aligned} I(u) = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{\|x-y\|^{2(1+\mu)}} \, dx dy \right)^{1/2} < \infty, \text{ with the norm} \\ \|u\|_{H^\mu(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + I^2(u) \right)^{1/2}. \end{aligned} \tag{1.5}$$

Further, we shall introduce the Bochner spaces. Let X be a Banach space. Then we define: $C([0,T];X)$ — space of functions $u : [0,T] \to X$, continuous, for which

$$\|u\|_{C([0,T];X)} = \sup_{t \in [0,T]} \|u(t)\|_X < \infty; \tag{1.6}$$

$C^1([0,T];X)$ — space of functions $u : [0,T] \to X$ continuously differentiable in $[0,T]$;
$L^p(0,T;X), 1 \le p < \infty$ — space of functions $u : (0,T) \to X$, strongly measurable such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u\|^p_X \, dt \right)^{1/p} < \infty. \tag{1.7}$$

For $p = 2$, $X = L^2(\Omega)$ we have $L^2(0,T;L^2(\Omega)) \equiv L^2(Q_T)$;
$L^\infty(0,T;L^2(\Omega))$ — space of functions $u : (0,T) \to L^2(\Omega)$ such that

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} = \operatorname*{ess\,sup}_{t \in (0,T)} \|u(t)\|_{L^2(\Omega)} < \infty. \tag{1.8}$$

It is known that $L^p(\Omega)$, $1 < p < \infty$, $H^\mu(\Omega)$, $\mu \in (\frac{1}{2}, 1]$, $L^p(0,T;X)$, $1 < p < \infty$, are reflexive Banach spaces.

In virtue of the Sobolev imbedding theorems,

$$H^s(\Omega) \hookrightarrow\hookrightarrow H^r(\Omega), \qquad r, s \in \mathbb{N}, \quad 0 \le r < s, \tag{1.9}$$

(compact imbedding)

$$H^1(\Omega) \hookrightarrow L^q(\Omega), \qquad q \in [1, \infty). \tag{1.10}$$

(continuous imbedding)

By [26] (in the case of domains with infinitely smooth boundary) and [8] (for Lipschitz-continuous boundary),

$$H^\mu(\Omega) \hookrightarrow\hookrightarrow H^{\mu-\varepsilon}(\Omega), \qquad \text{if } \mu \geq \varepsilon > 0. \tag{1.11}$$

Hence, for $\mu \in (0,1)$ we have

$$H^1(\Omega) \hookrightarrow\hookrightarrow H^\mu(\Omega) \hookrightarrow\hookrightarrow H^0(\Omega) = L^2(\Omega). \tag{1.12}$$

Due to the theorem on traces (see [26] for domains with infinitely smooth boundaries and [23] for Lipschitz-continuous boundaries), for every $\mu \in (\frac{1}{2}, 1]$ the trace mapping $\theta : H^\mu(\Omega) \to L^2(\partial\Omega)$, is continuous. This means that there exists $C_{\mathrm{Tr}}(\mu) > 0$ such that

$$\|u\|_{L^2(\partial\Omega)} \leq C_{\mathrm{Tr}}(\mu)\|u\|_{H^\mu(\Omega)}, \quad u \in H^\mu(\Omega). \tag{1.13}$$

We set $C_{\mathrm{Tr}} = C_{\mathrm{Tr}}(1)$.

Now we introduce the following *initial-boundary value problem*: Find a function $u = u(x,t)$ defined in Q_T such that

$$\alpha(x)\frac{\partial u(x,t)}{\partial t} = \operatorname{div}\left(\beta(x)\nabla u(x,t) + \mathbf{v}(x)u(x,t)\right) + q(x) \quad \text{in} \quad Q_T, \tag{1.14}$$

$$\beta(x)\frac{\partial u(x,t)}{\partial n} + \gamma(x)u(x,t) = G(x, u(x,t)) \qquad \text{on} \quad \Gamma_1, \tag{1.15}$$

$$u(x,t) = 0 \qquad \text{on} \quad \Gamma_2, \tag{1.16}$$

$$\beta(x)\frac{\partial u(x,t)}{\partial n} = 0 \qquad \text{on} \quad \Gamma_3, \tag{1.17}$$

$$u(x,0) = u^0(x), \qquad x \in \Omega. \tag{1.18}$$

Let us assume that the functions from (1.14)–(1.18) have the following properties:

$$\alpha \in C(\overline{\Omega}), \qquad \alpha_1 \geq \alpha \geq \alpha_0 > 0, \quad \alpha_0, \alpha_1 = \text{const}, \tag{1.19}$$

$$\beta \in C^1(\overline{\Omega}), \qquad \beta_1 \geq \beta \geq \beta_0 > 0, \quad \beta_0, \beta_1 = \text{const}, \tag{1.20}$$

$$\mathbf{v} \in \left[C^1(\overline{\Omega})\right]^2, \tag{1.21}$$

$$\gamma \in C(\overline{\Gamma}_1), \qquad |\gamma| \leq \gamma_1 = \text{const}, \tag{1.22}$$

$$q \in L^2(\Omega), \tag{1.23}$$

$$u_0 \in L^2(\Omega). \tag{1.24}$$

Moreover, $G : \Gamma_1 \times \mathbb{R} \to \mathbb{R}$, $G = G(x,u)$, is continuous, has a linear growth and is Lipschitz-continuous with respect to u. This means that there exist $g \geq 0$, $g \in L^2(\Gamma_1)$, $K \geq 0$, L_{G}, such that

$$|G(x,u)| \leq g(x) + K|u|, \quad \forall\, x \in \Gamma_1, \quad \forall\, u \in \mathbb{R}, \tag{1.25}$$

$$|G(x,u) - G(x,u^*)| \leq L_{\mathrm{G}}|u - u^*|, \quad \forall\, x \in \Gamma_1, \quad \forall\, u, u^* \in \mathbb{R}. \tag{1.26}$$

(Let us note that (1.25) follows from (1.26).)

Classical solution of the above initial-boundary value problem is a function $u \in C^2(\overline{Q}_T)$ satisfying equation (1.14), boundary conditions (1.15)–(1.17) and initial condition (1.18) pointwise.

In the analysis of this problem it will be necessary to work with a number of various constants. Constants with fixed meaning in the whole paper will be denoted by symbols K, L_{G}, γ_1, α_0, α_1, β_0, C_{Tr}, C_{F}, C_0, C_1, $\dots$. On the other hand, by C we shall denote a generic constant having, in general, different values in different places.

1.2 Weak solution.

In order to define the concept of a weak solution, we introduce the space of test functions

$$V = \{\varphi \in H^1(\Omega); \quad \varphi\,|_{\Gamma_2} = 0\}. \tag{1.27}$$

Let us remind that the well-known Friedrichs inequality holds in this space: there exists a constant $C_{\mathrm{F}} > 0$ such that

$$\|\varphi\|_{H^1(\Omega)} \leq C_{\mathrm{F}} |\varphi|_{H^1(\Omega)} \quad \forall\, \varphi \in V. \tag{1.28}$$

This implies that $|\cdot|_{H^1(\Omega)}$ is a norm on V equivalent with the norm $\|\cdot\|_{H^1(\Omega)}$. The norm $|\cdot|_{H^1(\Omega)}$ on V is induced by the scalar product $((\cdot,\cdot))_V$ defined by

$$((u,v))_V = \int_\Omega \nabla u \cdot \nabla v\, dx. \tag{1.29}$$

The weak formulation is derived in a standard way: We assume that u is a classical solution, multiply equation (1.14) by an arbitrary test function $\varphi \in V$, integrate over Ω and use Green's theorem and conditions (1.15)–(1.17). We obtain the relation

$$\begin{aligned}
\int_\Omega \alpha \frac{\partial u}{\partial t} \varphi\, dx &= \int_\Omega \operatorname{div}\,[\beta \nabla u + \mathbf{v} u]\, \varphi\, dx + \int_\Omega q\varphi\, dx \\
&= \int_{\partial\Omega} [\beta \nabla u + \mathbf{v} u] \cdot \mathbf{n} \varphi\, dS - \int_\Omega [\beta \nabla u + \mathbf{v} u] \cdot \nabla \varphi\, dx + \int_\Omega q\varphi\, dx \\
&= \int_{\Gamma_1} [G(x,u) - \gamma u + \mathbf{v} \cdot \mathbf{n} u]\, \varphi\, dS + \int_{\Gamma_3} \mathbf{v} \cdot \mathbf{n} u \varphi\, dS \\
&\quad - \int_\Omega \beta \nabla u \cdot \nabla \varphi\, dx - \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} u \varphi\, dS + \int_\Omega \operatorname{div}(\mathbf{v} u) \varphi\, dx + \int_\Omega q\varphi\, dx.
\end{aligned}$$

Hence,

$$
\begin{aligned}
(1.30)\qquad &\int_\Omega \alpha \frac{\partial u}{\partial t}\varphi\, dx + \int_\Omega [\beta \nabla u\cdot\nabla\varphi - \operatorname{div}(\mathbf{v}u)\varphi]\, dx \\
&\quad = \int_{\Gamma_1} [G(x,u) - \gamma u]\,\varphi\, dS + \int_\Omega q\varphi\, dx.
\end{aligned}
$$

Let us introduce the following notation:

$$(1.31)\qquad (u,\varphi)_\alpha = \int_\Omega \alpha u\varphi\, dx,$$

$$(1.32)\qquad a_0(u,\varphi) = \int_\Omega \beta\nabla u\cdot\nabla\varphi\, dx,$$

$$(1.33)\qquad a_1(u,\varphi) = -\int_\Omega \operatorname{div}(\mathbf{v}u)\varphi\, dx,$$

$$(1.34)\qquad a(u,\varphi) = a_0(u,\varphi) + a_1(u,\varphi),$$

$$(1.35)\qquad d_0(u,\varphi) = \int_{\Gamma_1} G(x,u)\varphi\, dS,$$

$$(1.36)\qquad d_1(u,\varphi) = -\int_{\Gamma_1} \gamma u\varphi\, dS,$$

$$(1.37)\qquad d(u,\varphi) = d_0(u,\varphi) + d_1(u,\varphi).$$

Then (1.30) can be written in the form

$$(1.38)\qquad \frac{d}{dt}(u(t),\varphi)_\alpha + a(u(t),\varphi) = d(u(t),\varphi) + (q,\varphi) \qquad \forall\,\varphi\in V.$$

In what follows we prove several important properties of these forms.

Lemma 1. *The forms a, d are defined for $u,\varphi\in V$. For these functions we have*

$$(1.39)\qquad |a_0(u,\varphi)| \le C_1|u|_{H^1(\Omega)}|\varphi|_{H^1(\Omega)},$$

$$(1.40)\qquad |a_1(u,\varphi)| \le C_2|u|_{H^1(\Omega)}|\varphi|_{H^1(\Omega)},$$

$$(1.41)\qquad |d_0(u,\varphi)| \le C_3(1+|u|_{H^1(\Omega)})|\varphi|_{H^1(\Omega)},$$

$$(1.42)\qquad |d_1(u,\varphi)| \le C_4|u|_{H^1(\Omega)}|\varphi|_{H^1(\Omega)},$$

$$(1.43)\qquad a(u,u) \ge \frac{\beta_0}{2}|u|^2_{H^1(\Omega)} - C_0\|u\|^2_{L^2(\Omega)} \quad \text{for } \ u\in V,$$

with constants $C_0,\dots,C_4>0$ independent of u, φ and $\beta_0>0$ from (1.20). *If, moreover,*

$$(1.44)\qquad \mathbf{v}\cdot\mathbf{n}\le 0 \qquad \text{on } \ \Gamma_1\cup\Gamma_3,$$

$$(1.45)\qquad \operatorname{div}\,\mathbf{v}\le 0 \qquad \text{in } \ \Omega,$$

then

$$a(u,u) \geq \beta_0 |u|^2_{H^1(\Omega)}, \quad u \in V. \tag{1.46}$$

Proof: Using the Cauchy inequality, the theorem on traces, the Friedrichs inequality and assumptions (1.20)–(1.25), we find that

$$\begin{aligned}
|a_0(u,\varphi)| &\leq \int_\Omega |\beta \nabla u \cdot \nabla \varphi| \, dx \leq \beta_1 \int_\Omega |\nabla u \cdot \nabla \varphi| \, dx \\
&\leq C_1 |u|_{H^1(\Omega)} |\varphi|_{H^1(\Omega)}, \\
|a_1(u,\varphi)| &\leq \int_\Omega |\operatorname{div}(\mathbf{v}u)\varphi| \, dx \leq \int_\Omega |\operatorname{div}\, \mathbf{v}\, u\varphi| \, dx + \int_\Omega |\mathbf{v} \cdot \nabla u \, \varphi| \, dx \\
&\leq C_{\mathbf{v}} (\|u\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} + |u|_{H^1(\Omega)} \|\varphi\|_{L^2(\Omega)}) \\
&\leq C_{\mathrm{F}} C_{\mathbf{v}} (C_{\mathrm{F}} + 1) |u|_{H^1(\Omega)} |\varphi|_{H^1(\Omega)} = C_2 |u|_{H^1(\Omega)} |\varphi|_{H^1(\Omega)}, \\
|d_0(u,\varphi)| &\leq \int_{\Gamma_1} |G(u,x)\varphi| \, dS \leq \int_{\Gamma_1} g|\varphi| \, dS + K \int_{\Gamma_1} |u\varphi| \, dS \\
&\leq \|g\|_{L^2(\Gamma_1)} \|\varphi\|_{L^2(\partial\Omega)} + K \|u\|_{L^2(\partial\Omega)} \|\varphi\|_{L^2(\partial\Omega)} \\
&\leq C_{\mathrm{Tr}} C_{\mathrm{F}} \|g\|_{L^2(\Gamma_1)} |\varphi|_{H^1(\Omega)} + K C_{\mathrm{Tr}}^2 C_{\mathrm{F}}^2 |\varphi|_{H^1(\Omega)} |u|_{H^1(\Omega)} \\
&= C_3 (1 + |u|_{H^1(\Omega)}) |\varphi|_{H^1(\Omega)}, \\
|d_1(u,\varphi)| &\leq \int_{\Gamma_1} |\gamma u \varphi| \, dS \leq \gamma_1 \int_{\Gamma_1} |u\varphi| \, dS \leq \gamma_1 \|u\|_{L^2(\partial\Omega)} \|\varphi\|_{L^2(\partial\Omega)} \\
&\leq C_4 |u|_{H^1(\Omega)} |\varphi|_{H^1(\Omega)},
\end{aligned}$$

where $C_{\mathbf{v}} = \|\mathbf{v}\|_{C^1(\overline{\Omega})}$, C_{Tr} is the constant from the theorem on traces (1.13) and C_{F} is the constant from the Friedrichs inequality (1.28) and $C_1 = \beta_1$, $C_2 = C_{\mathrm{F}} C_{\mathbf{v}} (C_{\mathrm{F}} + 1)$, $C_3 = C_{\mathrm{Tr}} C_{\mathrm{F}} \max\{\|g\|_{L^2(\Gamma_1)}, K C_{\mathrm{Tr}} C_{\mathrm{F}}\}$, $C_4 = \gamma_1 C_{\mathrm{F}}^2 C_{\mathrm{Tr}}^2$. Further, we have

$$a(u,u) \geq \beta_0 |u|^2_{H^1(\Omega)} - \int_\Omega \operatorname{div}(\mathbf{v}u) u \, dx \geq \frac{\beta_0}{2} |u|^2_{H^1(\Omega)} - C_0 \|u\|^2_{L^2(\Omega)}, \tag{1.47}$$

with $C_0 = C_{\mathbf{v}}(1 + 1/(2\beta_0))$, since

$$\begin{aligned}
\int_\Omega \operatorname{div}(\mathbf{v}u) u \, dx &\leq \Big| \int_\Omega u^2 \operatorname{div}\, \mathbf{v} \, dx \Big| + \Big| \int_\Omega (\mathbf{v} \cdot \nabla u) u \, dx \Big| \\
&\leq C_{\mathbf{v}} (\|u\|^2_{L^2(\Omega)} + \int_\Omega |\nabla u| \, |u| \, dx) \\
&\leq C_{\mathbf{v}} (\|u\|^2_{L^2(\Omega)} + |u|_{H^1(\Omega)} \|u\|_{L^2(\Omega)}) \\
&\leq \frac{\beta_0}{2} |u|^2_{H^1(\Omega)} + C_0 \|u\|^2_{L^2(\Omega)},
\end{aligned}$$

where we have used Young's inequality:

$$ab \le \frac{1}{2}\varepsilon a^2 + \frac{1}{2\varepsilon}b^2, \qquad a, b \ge 0,\ \varepsilon > 0 \tag{1.48}$$

with $\varepsilon = \beta_0$.

Under assumptions (1.44), (1.45), we obtain

$$a(u,u) \ge \beta_0 |u|^2_{H^1(\Omega)} - \int_\Omega \operatorname{div}(\mathbf{v}u)u\,dx \ge \beta_0 |u|^2_{H^1(\Omega)}, \tag{1.49}$$

since

$$\begin{aligned}\int_\Omega \operatorname{div}(\mathbf{v}u)u\,dx &= \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}u^2\,dS - \int_\Omega \mathbf{v}\cdot\nabla u\,u\,dx \\ &= \int_{\Gamma_1\cup\Gamma_3} \mathbf{v}\cdot\mathbf{n}u^2\,dS - \frac{1}{2}\left[\int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}u^2\,dS - \int_\Omega u^2 \operatorname{div}\mathbf{v}\,dx\right] \\ &= \frac{1}{2}\int_{\Gamma_1\cup\Gamma_3} \mathbf{v}\cdot\mathbf{n}u^2\,dS + \frac{1}{2}\int_\Omega u^2 \operatorname{div}\mathbf{v}\,dx \le 0.\end{aligned}$$

□

Definition of a weak solution. We say that u is a weak solution of problem (1.14)–(1.18), if
(a) $u \in L^2(0,T;V) \cap L^\infty\big(0,T;L^2(\Omega)\big)$,
(b) u satisfies identity (1.38) for all $\varphi \in V$ in the sense of distributions on $(0,T)$, i.e.,

$$\begin{aligned}-\int_0^T (u(t),\varphi)_\alpha \vartheta'(t)\,dt + \int_0^T a(u(t),\varphi)\vartheta(t)\,dt &= \int_0^T d(u(t),\varphi)\vartheta(t)\,dt \\ &+ \int_0^T (q,\varphi)\vartheta(t)\,dt \qquad \forall\,\vartheta \in C_0^\infty(0,T),\ \forall\,\varphi \in V,\end{aligned} \tag{1.50}$$

and,
(c) u satisfies the initial condition

$$u(0) = u^0. \tag{1.51}$$

2. Existence of a weak solution

The goal of the next two paragraphs will be the proof of the existence and uniqueness of a weak solution of the problem (1.14)–(1.18):

Theorem 1. *There exists a unique solution of problem* (1.50), (1.51).

2.1 Proof of existence.

The existence of a weak solution will be proved with the aid of the Galerkin method. The space V is separable and, hence, there exists its basis $\{w_k\}_{k=1}^{\infty}$ such that

$$V = \overline{\bigcup_{k=1}^{\infty} X_k}^{H^1(\Omega)}, \quad \text{where} \quad X_k = \operatorname{span}\{w_1, \dots, w_k\}. \tag{2.1}$$

Let us define the Galerkin approximation $u_k \in C^1([0,T], X_k)$ which satisfies the conditions

$$\frac{d}{dt}(u_k(t), w_i)_\alpha + a(u_k(t), w_i) = d(u_k(t), w_i) + (q, w_i), \qquad i = 1, \dots, k, \quad t \in (0,T), \tag{2.2}$$

$$u_k(0) = u_k^0 = P_k u^0. \tag{2.3}$$

Here, the mapping P_k is the L^2-projection on X_k, i.e., for $u \in L^2(\Omega)$, we define $P_k u \in X_k$ so that

$$(P_k u, \varphi) = (u, \varphi) \quad \forall \varphi \in X_k. \tag{2.4}$$

In the sequel, we shall employ the following inequality from [3]:

$$\|w\|^2_{L^2(\partial\Omega)} \le C_5 \|w\|_{L^2(\Omega)} |w|_{H^1(\Omega)} \qquad \forall w \in V, \tag{2.5}$$

called the multiplicative trace inequality.

Now we derive a priori estimates of approximate solutions u_k.

Lemma 2. *There exists a constant $C > 0$ such that each solution u_k of problem* (2.2), (2.3) *satisfies the estimates*

$$\|u_k\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{2.6}$$

$$\|u_k\|_{L^2(0,T;V)} \le C \qquad \forall k = 1, 2, \dots . \tag{2.7}$$

PROOF: Conditions (2.2) can be written as

$$(u_k'(t), \varphi)_\alpha + a(u_k(t), \varphi) = d(u_k(t), \varphi) + (q, \varphi) \quad \forall \varphi \in X_k, \quad t \in (0,T), \tag{2.8}$$

where we simply write u_k' instead of $\frac{\partial u_k}{\partial t}$. Substituting $u_k(t)$ for φ in (2.8), we obtain

$$\int_\Omega \alpha u_k'(t) u_k(t)\, dx + a(u_k(t), u_k(t)) = d(u_k(t), u_k(t)) + \int_\Omega q u_k(t)\, dx. \tag{2.9}$$

In virtue of Lemma 1, (2.5), the linear growth of the function G, the Cauchy inequality and Young's inequality, we find that

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \alpha(u_k^2)'(t)\,dx + \frac{\beta_0}{2}|u_k(t)|^2_{H^1(\Omega)} - C_0\|u_k(t)\|^2_{L^2(\Omega)}\\
&\le \Big|\int_{\Gamma_1} G(x,u_k(t))u_k(t)\,dS - \int_{\Gamma_1}\gamma u_k^2(t)\,dS + \int_\Omega q u_k(t)\,dx\Big|\\
&\le \Big|\int_{\Gamma_1} g u_k(t)\,dS\Big| + K\int_{\Gamma_1} u_k^2(t)\,dS + \|q\|_{L^2(\Omega)}\|u_k(t)\|_{L^2(\Omega)} + \int_{\Gamma_1}|\gamma| u_k^2(t)\,dS\\
&\le \frac{1}{2}\|g\|^2_{L^2(\Gamma_1)} + \left(\frac{1}{2}+K\right)\|u_k(t)\|^2_{L^2(\Gamma_1)} + \gamma_1\|u_k(t)\|^2_{L^2(\Gamma_1)}\\
&\quad + \|q\|_{L^2(\Omega)}\|u_k(t)\|_{L^2(\Omega)}\\
&\le \left(\frac{1}{2}+K+\gamma_1\right)\|u_k(t)\|^2_{L^2(\Gamma_1)} + \frac{1}{2}\|u_k(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|q\|^2_{L^2(\Omega)} + \frac{1}{2}\|g\|^2_{L^2(\Gamma_1)}\\
&\le C_6\|u_k(t)\|_{L^2(\Omega)}|u_k(t)|_{H^1(\Omega)} + \frac{1}{2}\|u_k(t)\|^2_{L^2(\Omega)} + C_7\\
&\le \left(\frac{1}{2}+\frac{C_6^2}{\beta_0}\right)\|u_k(t)\|^2_{L^2(\Omega)} + \frac{\beta_0}{4}|u_k(t)|^2_{H^1(\Omega)} + C_7,
\end{aligned}
$$

where $C_6 = C_5(\frac{1}{2}+K+\gamma_1)$, $C_7 = \frac{1}{2}(\|q\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Gamma_1)})$. Hence,

$$
(2.10)\quad \frac{d}{dt}\|\sqrt{\alpha}u_k(t)\|^2_{L^2(\Omega)} + \frac{\beta_0}{2}|u_k(t)|^2_{H^1(\Omega)} \le 2\Big(\frac{1}{2}+C_0+\frac{C_6^2}{\beta_0}\Big)\|u_k(t)\|^2_{L^2(\Omega)} + 2C_7.
$$

The integration with respect to time yields

$$
\begin{aligned}
\alpha_0\|u_k(t)\|^2_{L^2(\Omega)} + \frac{\beta_0}{2}\int_0^t |u_k(\xi)|^2_{H^1(\Omega)}\,d\xi &\le 2\Big(\frac{1}{2}+C_0+\frac{C_6^2}{\beta_0}\Big)\int_0^t \|u_k(\xi)\|^2_{L^2(\Omega)}\,d\xi\\
&\quad + 2C_7T + \alpha_1\|u_k(0)\|^2_{L^2(\Omega)},
\end{aligned}
$$

and, thus,

$$
(2.11)\quad \|u_k(t)\|^2_{L^2(\Omega)} + C_{10}\int_0^t |u_k(\xi)|^2_{H^1(\Omega)}\,d\xi \le C_8\int_0^t \|u_k(\xi)\|^2_{L^2(\Omega)}\,d\xi + C_9,
$$

where $C_8 = 2(\frac{1}{2}+C_0+\frac{C_6^2}{\beta_0})/\alpha_0$, $C_9 = (2C_7T + \alpha_1\|u_k(0)\|^2_{L^2(\Omega)})/\alpha_0$ and $C_{10} = \beta_0/(2\alpha_0)$. Now we use Gronwall's lemma in the form from [9]:

Let $y, w, z, r \in C([0,T])$, $w, r \ge 0$ and $y(t)+w(t) \le z(t) + \int_0^t r(s)y(s)\,ds$. Then

$$
(2.12)\quad y(t)+w(t) \le z(t) + \int_0^t r(\vartheta)z(\vartheta)\exp\Big(\int_\vartheta^t r(s)ds\Big)\,d\vartheta.
$$

Here we set $z(t) = C_9$, $r(s) = C_8$, $y(t) = \|u_k(t)\|^2_{L^2(\Omega)}$,
$w(t) = C_{10} \int_0^t |u_k(\xi)|^2_{H^1(\Omega)}\, d\xi$. Then (2.11) implies that

$$\begin{aligned} &\|u_k(t)\|^2_{L^2(\Omega)} + C_{10} \int_0^t |u_k(\xi)|^2_{H^1(\Omega)}\, d\xi \\ &\le C_9 + C_8 C_9 \int_0^t \exp\big(\int_\vartheta^t C_8\, ds\big)\, d\vartheta = C_9 \exp(C_8 t) \le C_9 \exp(C_8 T). \end{aligned} \tag{2.13}$$

From this we get

$$\max_{t\in[0,T]} \|u_k(t)\|^2_{L^2(\Omega)} + C_{10} \int_0^T |u_k(\xi)|^2_{H^1(\Omega)}\, d\xi \le 2C_9 \exp(C_8 T) = \mathrm{const}, \tag{2.14}$$

which immediately yields (2.6) and (2.7). □

Let us continue in the proof of the existence of the weak solution. Since $\{w_1, \dots, w_k\}$ is a basis in X_k, there exist functions $\zeta_1(t), \dots, \zeta_k(t)$ such that

$$u_k(t) = \sum_{i=1}^k \zeta_i(t) w_i. \tag{2.15}$$

Conditions (2.8) represent a system of ordinary differential equations for unknown functions $\zeta_i(t)$, $i = 1, \dots, k$. Its right-hand side satisfies the Carathéodory conditions and is Lipschitz-continuous with respect to ζ_i, $i = 1, \dots, k$, which implies the existence of a unique generalized (i.e. absolutely continuous) solution in some time interval $[0, T^*]$. From the uniform boundedness (2.6) and (2.7), it follows that there exists a unique approximate solution u_k in the whole time interval $[0, T]$ (see e.g. [24]).

With the aid of a modification of Theorem 4.11 in [28, p. 290], there exists such a basis $\{w_i\}_{i=1}^\infty$ in V that

$$((w_i, \varphi))_V = \lambda_i (w_i, \varphi)_\alpha \qquad \forall\, \varphi \in V, \tag{2.16}$$

$$(w_i, w_j)_\alpha = \delta_{ij} \qquad \forall\, i, j \in \mathbb{N}, \tag{2.17}$$

$$\Big(\Big(\frac{w_i}{\sqrt{\lambda_i}}, \frac{w_j}{\sqrt{\lambda_j}}\Big)\Big)_V = \delta_{ij} \qquad \forall\, i, j \in \mathbb{N}, \tag{2.18}$$

$$0 < C \le \lambda_1 \le \lambda_2 \le \dots \quad \text{and} \quad \lambda_r \to \infty \quad \text{as} \quad r \to \infty, \tag{2.19}$$

$$\{w_i/\sqrt{\lambda_i}\}_{i=1}^\infty \quad \text{forms an orthonormal basis in} \quad V. \tag{2.20}$$

(The scalar product $((\cdot, \cdot))_V$ is defined in (1.29).)

Moreover, for $X_k = \operatorname{span}\{w_1, \dots, w_k\}$ and

$$P_k^\alpha v = \sum_{i=1}^{k} (v, w_i)_\alpha w_i : V \to X_k \subset V, \tag{2.21}$$

we obtain

$$|P_k^\alpha v|_{H^1(\Omega)} \le |v|_{H^1(\Omega)} \qquad \forall\, v \in V. \tag{2.22}$$

Actually, in virtue of (2.16), (2.18)–(2.21), for $v \in V$ we have

$$\begin{aligned} |P_k^\alpha v|^2_{H^1(\Omega)} &= \sum_{i=1}^{k} (v, w_i)^2_\alpha ((w_i, w_i))_V \\ &= \sum_{i=1}^{k} \frac{1}{\lambda_i} ((v, w_i))^2_V = \sum_{i=1}^{k} ((v, \frac{w_i}{\sqrt{\lambda_i}}))^2_V \le |v|^2_{H^1(\Omega)}. \end{aligned} \tag{2.23}$$

Further, for every $\varphi \in X_k$ we have

$$(P_k^\alpha v, \varphi)_\alpha = \sum_{i=1}^{k} (v, w_i)_\alpha (w_i, \varphi)_\alpha = (v, \varphi)_\alpha. \tag{2.24}$$

Now let us return to the definition (2.2) of the approximate solution u_k, rewritten in the form (2.8). Since $X_k \subset V \subset L^2_\alpha(\Omega) \equiv L^2_\alpha(\Omega)^* \subset V^*$, the derivative $u_k' = \partial u_k / \partial t$ can be considered as an element of V^*. If we denote by $\langle \cdot, \cdot \rangle$ the duality between V^* and V in such a way that

$$\langle \vartheta, \varphi \rangle = (\vartheta, \varphi)_\alpha \qquad \forall\, \varphi \in V,\ \forall\, \vartheta \in L^2_\alpha(\Omega), \tag{2.25}$$

then we have

$$\langle u_k', \varphi \rangle = (u_k', \varphi)_\alpha \qquad \forall\, \varphi \in V. \tag{2.26}$$

Let $v \in V$. Then, according to (2.24), (2.26) and (2.8), since $u_k' \in X_k$, we find that

$$\begin{aligned} \langle u_k'(t), v \rangle &= (u_k'(t), v)_\alpha = (u_k'(t), P_k^\alpha v)_\alpha \\ &= -a(u_k(t), P_k^\alpha v) + d(u_k(t), P_k^\alpha v) + (q, P_k^\alpha v)_{L^2(\Omega)}. \end{aligned}$$

This, Lemma 1 and (2.22) imply that

$$|\langle u_k'(t), v \rangle| \le C(|u_k(t)|_{H^1(\Omega)} + 1)|v|_{H^1(\Omega)} \quad \forall\, v \in V,\ t \in (0, T), \tag{2.27}$$

where the constant C depends on $C_1, \dots, C_4$. Hence,

$$\begin{aligned} \|u_k'(t)\|^2_{L^2(0,T;V^*)} &= \int_0^T \|u_k'(t)\|^2_{V^*}\, dt \le 2C^2 \int_0^T \left(|u_k(t)|^2_{H^1(\Omega)} + 1 \right) dt \\ &= 2C^2 T + 2C^2 \|u_k\|^2_{L^2(0,T;V)},\ k = 1, 2, \dots, \end{aligned} \tag{2.28}$$

which is bounded by a constant independent of k, as follows from (2.7).

The obtained results can be summarized in the following way:

Theorem 2. *The sequence* $\{u_k\}_{k=1}^{\infty}$ *is bounded in* $L^\infty(0,T;L^2(\Omega))$ *and in* $L^2(0,T;V)$. *The sequence* $\{u_k'\}_{k=1}^{\infty}$ *is bounded in* $L^2(0,T;V^*)$.

In what follows, we shall apply the well-known Aubin–Lions lemma (see, e.g., [25] or [10]):

Theorem 3. *Let* X_0, X, X_1 *be Banach spaces with the following properties:*
(a) $X_0 \hookrightarrow X \hookrightarrow X_1$ (*continuous imbedding*),
(b) X_0, X_1 *are reflexive,*
(c) $X_0 \hookrightarrow\hookrightarrow X$ (*compact imbedding*).
Let us put

$$W = \left\{ v \in L^2(0,T;X_0); \frac{\partial v}{\partial t} \in L^2(0,T;X_1) \right\}. \tag{2.29}$$

Then

$$W \hookrightarrow\hookrightarrow L^2(0,T;X). \tag{2.30}$$

Now we prove the following results:

Theorem 4. *The sequence of approximate solutions* $\{u_k\}_{k=1}^{\infty}$ *is compact in* $L^2(0,T;H^\mu(\Omega))$ *for* $\mu \in (\frac{1}{2},1)$. *The sequence of traces* $\{u_k\,|_{\partial\Omega\times(0,T)}\}_{k=1}^{\infty}$ *is compact in* $L^2(0,T;L^2(\partial\Omega))$.

Proof: It is necessary to show that there exists a function u and a subsequence $\{u_k\}_{k=1}^{\infty}$ (for simplicity we use the same notation) such that

$$u_k \to u \text{ in } L^2(0,T;H^\mu(\Omega)) \text{ for } k \to \infty, \tag{2.31}$$

$$u_k\,|_{\partial\Omega\times(0,T)} \to u\,|_{\partial\Omega\times(0,T)} \text{ in } L^2(0,T;L^2(\partial\Omega)) \text{ for } k \to \infty. \tag{2.32}$$

Let us set $X_0 = V$, $X_1 = V^*$, $X = H^\mu(\Omega)$ for $\mu \in (\frac{1}{2},1)$. Then, in view of the results from Paragraph 1.1, the conditions (a), (b), (c) from the previous theorem are satisfied, because $V \subset H^1(\Omega) \hookrightarrow\hookrightarrow H^\mu(\Omega) \hookrightarrow L^2(\Omega) \equiv L^2(\Omega)^* \hookrightarrow V^*$. Moreover, the trace operator θ is defined on $H^\mu(\Omega)$ and $\theta : H^\mu(\Omega) \to L^2(\partial\Omega)$ is a continuous mapping. By the Aubin–Lions lemma, there exists a subsequence of approximate solutions $\{u_k\}_{k=1}^{\infty}$ such that

$$u_k \to u \quad \text{in} \;\; L^2(0,T;H^\mu(\Omega)) \quad \text{as} \;\; k \to \infty. \tag{2.33}$$

This means that

$$\|u_k - u\|^2_{L^2(0,T;H^\mu(\Omega))} = \int_0^T \|u_k(t) - u(t)\|^2_{H^\mu(\Omega)}\,dt \to 0 \quad \text{as} \;\; k \to \infty. \tag{2.34}$$

Further, from the property (1.13) of the trace operator $\theta : H^\mu(\Omega) \to L^2(\partial\Omega)$ it follows that

$$\begin{aligned} &\|u_k(t)\,|\,\partial\Omega - u(t)\,|\,\partial\Omega\|_{L^2(\partial\Omega)} \\ &= \|\theta u_k(t) - \theta u(t)\|_{L^2(\partial\Omega)} \le C_{\mathrm{Tr}}(\mu)\|u_k(t) - u(t)\|_{H^\mu(\Omega)}. \end{aligned} \tag{2.35}$$

Hence,

$$\begin{aligned} \|u_k - u\|^2_{L^2(0,T;L^2(\partial\Omega))} &= \int_0^T \left(\int_{\partial\Omega} |u_k - u|^2\,dS\right) dt \\ &= \int_0^T \|u_k(t)\,|\,\partial\Omega - u(t)\,|\,\partial\Omega\|^2_{L^2(\partial\Omega)}\,dt \\ &\le C^2_{\mathrm{Tr}}(\mu) \int_0^T \|u_k(t) - u(t)\|^2_{H^\mu(\Omega)}\,dt \to 0 \quad \text{as } k \to \infty, \end{aligned}$$

what we wanted to prove. □

Since $L^2(0,T;L^2(\partial\Omega)) = L^2(\partial\Omega \times (0,T))$, we obtain:

Corollary. *It is possible to choose a subsequence* $\{u_k\}_{k=1}^\infty$ *of approximate solutions satisfying* (2.31), (2.32) *and*

$$u_k \to u \quad \text{a.e. in } \partial\Omega \times (0,T). \tag{2.36}$$

Remark. The convergence of the traces of $u_k(t)$ for $k \to \infty$ can also be proved by putting $X_0 = V$, $X = L^2(\Omega)$, $X_1 = V^*$ in the Aubin–Lions lemma. This yields the strong convergence of a subsequence (since $W \hookrightarrow\hookrightarrow L^2(0,T;L^2(\Omega))$):

$$u_k \to u \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{2.37}$$

Further we use the multiplicative trace inequality (2.5)

$$\|w\|^2_{L^2(\partial\Omega)} \le C_5\|w\|_{L^2(\Omega)}|w|_{H^1(\Omega)} \qquad \forall\, w \in V.$$

From this we have

$$\begin{aligned} \|w\|^2_{L^2(0,T;L^2(\partial\Omega))} &= \int_0^T \|w(t)\|^2_{L^2(\partial\Omega)}\,dt \le C_5 \int_0^T \|w(t)\|_{L^2(\Omega)}|w(t)|_{H^1(\Omega)}\,dt \\ &\le C_5 \left(\int_0^T \|w(t)\|^2_{L^2(\Omega)}\,dt\right)^{1/2} \left(\int_0^T |w(t)|^2_{H^1(\Omega)}\,dt\right)^{1/2} \\ &= C_5\|w\|_{L^2(0,T;L^2(\Omega))}\|w\|_{L^2(0,T;V)}. \end{aligned} \tag{2.38}$$

This, the boundedness (2.7) of the sequence $\{u_k\}_{k=1}^{\infty}$ in $L^2(0,T;V)$ and (2.37) imply that

$$\begin{aligned}\|u_k-u\|^2_{L^2(0,T;L^2(\partial\Omega))} &\le C_5\|u_k-u\|_{L^2(0,T;L^2(\Omega))}\|u_k-u\|_{L^2(0,T;V)}\\ &\le C_5(C+\|u\|_{L^2(0,T;V)})\|u_k-u\|_{L^2(0,T;L^2(\Omega))}\to 0,\\ &\qquad\text{as}\quad k\to\infty,\end{aligned}$$

what we wanted to prove.

Now it is possible to pass to the limit in equation (2.8), rewritten in the form

$$\begin{aligned}&-\int_0^T (u_k(t),w_i)_\alpha\vartheta'(t)\,dt+\int_0^T a(u_k(t),w_i)\vartheta(t)\,dt\\ &=\int_0^T d(u_k(t),w_i)\vartheta(t)\,dt+\int_0^T (q,w_i)\vartheta(t)\,dt\quad\forall\,\vartheta\in C_0^\infty(0,T),\\ &\qquad i=1,\dots,k.\end{aligned}\tag{2.39}$$

The sequence u_k satisfies

$$u_k\rightharpoonup u \qquad \text{*-weakly in } L^\infty(0,T;L^2(\Omega)),\tag{2.40}$$

$$u_k\rightharpoonup u \qquad \text{weakly in } L^2(0,T;V),\tag{2.41}$$

$$u_k\to u \qquad \text{strongly in } L^2(0,T;H^\mu(\Omega)),\ \mu\in(\tfrac12,1),\tag{2.42}$$

$$u_k\to u \qquad \text{strongly in } L^2(Q_T),\tag{2.43}$$

$$u_k|_{\partial\Omega\times(0,T)}\to u|_{\partial\Omega\times(0,T)} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)),\tag{2.44}$$

$$u_k\to u \qquad \text{a.e. in } \partial\Omega\times(0,T).\tag{2.45}$$

Let $\vartheta\in C_0^\infty(0,T)$. It is obvious that the mappings

$$\phi\in L^2(0,T;V)\mapsto\int_0^T(\phi(t),w_i)_\alpha\vartheta'(t)\,dt\ \in\mathbb{R},\tag{2.46}$$

$$\phi\in L^2(0,T;V)\mapsto\int_0^T a(\phi(t),w_i)\vartheta(t)\,dt\ \in\mathbb{R},\tag{2.47}$$

are continuous linear functionals on $L^2(0,T;V)$. On the basis of the definition of the weak convergence in $L^2(0,T;V)$ we can immediately pass to the limit in the first two terms in (2.39). Further, we split the third term in two parts:

$$\int_0^T d(u_k(t),w_i)\vartheta(t)\,dt=\int_0^T\{d_0(u_k(t),w_i)+d_1(u_k(t),w_i)\}\,\vartheta(t)\,dt.\tag{2.48}$$

The part with d_1 is again linear in $u_k(t)$ and we proceed as above. Concerning the part with d_0 we have to prove that

$$\Big|\int_0^T \{d_0(u_k(t),w_i)-d_0(u(t),w_i)\}\,\vartheta(t)\,dt\Big|\to 0 \quad \text{as } k\to\infty. \tag{2.49}$$

This is a consequence of the Lipschitz continuity of the function G, the Cauchy inequality and (2.44):

$$\begin{aligned}
&\Big|\int_0^T \{d_0(u_k(t),w_i)-d_0(u(t),w_i)\}\,\vartheta(t)\,dt\Big|\\
&=\Big|\int_0^T \Big\{\int_{\Gamma_1}(G(x,u_k(t))-G(x,u(t)))\,w_i\vartheta(t)\,dS\Big\}\,dt\Big|\\
&\le \int_0^T \Big\{\int_{\Gamma_1} L_{\mathrm{G}}|u_k(t)-u(t)|\,|w_i|\,dS\Big\}|\vartheta(t)|\,dt\\
&\le C\Big(\int_{\Gamma_1}|w_i|^2\,dS\Big)^{1/2}\Big(\int_0^T\int_{\Gamma_1}|u_k(t)-u(t)|^2\,dS\,dt\Big)^{1/2}\to 0,
\end{aligned} \tag{2.50}$$

where $C=L_G\|\vartheta\|_{L^2(0,T)}$.

Summarizing the above results, we see that the limit function u satisfies conditions $u\in L^2(0,T;V)\cap L^\infty(0,T;L^2(\Omega))$ and (1.50) with $\varphi:=w_i$, $i=1,2,\dots$. This and (2.1) imply that (1.50) holds for all $\varphi\in V$. It remains to verify condition (1.51). For each $v\in L^2(\Omega)$,

$$(P_k v,\varphi)\to(v,\varphi),\qquad \forall\,\varphi\in V \ \text{ as } \ k\to\infty. \tag{2.51}$$

Actually, for $\varphi\in V$ there exists $\{\varphi_k\}_{k=1}^\infty$, $\varphi_k\in X_k$ such that $\varphi_k\to\varphi$ in V (cf. (2.1)) and

$$\begin{aligned}
|(P_kv-v,\varphi)|&=|(P_kv-v,\varphi-\varphi_k)|\\
&\le C_{\mathrm{F}}(\|P_kv\|_{L^2(\Omega)}+\|v\|_{L^2(\Omega)})\|\varphi-\varphi_k\|_V\\
&\le 2C_{\mathrm{F}}\|v\|_{L^2(\Omega)}\|\varphi-\varphi_k\|_V\to 0 \ \text{ as } \ k\to\infty.
\end{aligned} \tag{2.52}$$

Further, in view of (2.28), we can assume that the sequence u_k is chosen in such a way that

$$u_k'=\frac{\partial u_k}{\partial t}\rightharpoonup\widetilde{u} \quad \text{in} \quad L^2(0,T;V^*). \tag{2.53}$$

Then for all $\varphi\in V$ and all $\vartheta\in C_0^\infty(0,T)$

$$\begin{aligned}
\int_0^T(\varphi,u(t))\vartheta'(t)\,dt&=\lim_{k\to\infty}\int_0^T(\varphi,u_k(t))\vartheta'(t)\,dt\\
&=-\lim_{k\to\infty}\int_0^T(u_k'(t),\varphi)\vartheta(t)\,dt=-\int_0^T(\widetilde{u}(t),\varphi)\vartheta(t)\,dt,
\end{aligned} \tag{2.54}$$

which means that $u' = \partial u/\partial t = \widetilde{u}$, and hence,

$$u_k' \rightharpoonup u' \quad \text{in} \;\; L^2(0,T;V^*). \tag{2.55}$$

Obviously for all $\varphi \in V$ and $\vartheta \in C_0^\infty[0,T)$ with $\vartheta(T) = 0$, $\vartheta(0) \neq 0$, we have $\varphi\vartheta(t), \varphi\vartheta'(t) \in L^2(0,T;V)$, and, hence,

$$\begin{aligned} &\int_0^T \left(u_k'(t) - u'(t), \varphi\right) \vartheta(t)\, dt \\ &\qquad = -(u_k(0) - u(0), \varphi)\vartheta(0) - \int_0^T (u_k(t) - u(t), \varphi)\vartheta'(t)\, dt. \end{aligned} \tag{2.56}$$

In virtue of (2.55) and (2.41), the integrals in the first expression as well as the first integral in the last expression have zero limit as $k \to \infty$. Hence, using $u_k(0) = P_k u^0$ and (2.51), we find that $(u^0 - u(0), \varphi) = 0$ for all $\varphi \in V$, which means that

$$u(0) = u^0. \tag{2.57}$$

Thus, we have proven that u is a weak solution of problem (1.14)–(1.18).

2.2 Proof of uniqueness.
Let us assume that there exist two weak solutions u_1, u_2 of problem (1.14)–(1.18). This means that the following equations are satisfied:

$$\frac{d}{dt}(u_i(t), \varphi)_\alpha + a(u_i(t), \varphi) = d(u_i(t), \varphi) + (q, \varphi) \quad \forall\, \varphi \in V, \quad i = 1, 2, \tag{2.58}$$

in the sense of distribution on $(0,T)$.
On the basis of results from [32, Chapter III, Lemma 1.2], or [21],

$$\frac{d}{dt}(u_i(t), \varphi)_\alpha = \langle \frac{\partial u_i}{\partial t}(t), \varphi \rangle, \quad \varphi \in V, \quad i = 1, 2, \quad \text{for a.e.} \;\; t \in (0,T). \tag{2.59}$$

(See (2.25).) From (2.58) and (2.59), writing $w = u_1 - u_2$, we obtain

$$\begin{aligned} &\langle \frac{\partial w}{\partial t}(t), \varphi \rangle + a(w(t), \varphi) = d(u_1(t), \varphi) - d(u_2(t), \varphi), \\ &\qquad\qquad\qquad\qquad \forall\, \varphi \in V, \; \text{for a.e.} \;\; t \in (0,T). \end{aligned} \tag{2.60}$$

Now, we substitute $\varphi := w(t)$ and find from (2.60) that

$$\begin{aligned} &\langle \frac{\partial w}{\partial t}(t), w(t) \rangle + a(w(t), w(t)) = d(u_1(t), w(t)) - d(u_2(t), w(t)), \\ &\qquad\qquad\qquad\qquad \text{for a.e.} \;\; t \in (0,T). \end{aligned} \tag{2.61}$$

From the above references it follows that $w \in C([0,T]; L^2_\alpha(\Omega))$ and

$$\begin{aligned}\frac{d}{dt}\int_\Omega \alpha|w(t)|^2\,dx &= \frac{d}{dt}(w(t),w(t))_\alpha \\ &= 2\langle\frac{\partial w}{\partial t}(t), w(t)\rangle \quad \text{for a.e. } t\in(0,T).\end{aligned} \tag{2.62}$$

This and (2.61) imply that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \alpha|w(t)|^2\,dx + a(w(t),w(t))\,dx \tag{2.63}$$
$$= \int_{\Gamma_1}[G(x,u_2(t)) - G(x,u_1(t))]\,w(t)\,dS - \int_{\Gamma_1}\gamma|w(t)|^2\,dS \quad \text{for a.e. } t\in(0,T).$$

The individual terms will be estimated with the aid of Young's inequality (1.48), inequality (2.5), Lemma 1 and assumptions (1.19)–(1.23) and (1.26). Thus, for a.e. $t \in (0,T)$,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \alpha|w(t)|^2\,dx + \beta_0|w(t)|^2_{H^1(\Omega)} \\ &\le 2(\gamma_1 + L_{\mathrm{G}})C_5\|w(t)\|_{L^2(\Omega)}|w(t)|_{H^1(\Omega)} + 2C_0\|w(t)\|^2_{L^2(\Omega)} \\ &\le \frac{\beta_0}{2}|w(t)|^2_{H^1(\Omega)} + C_{11}\|w(t)\|^2_{L^2(\Omega)},\end{aligned} \tag{2.64}$$

where $C_{11} = 2C_0 + 2(L_G+\gamma_1)^2C_5^2/\beta_0$. Thus

$$\frac{d}{dt}\int_\Omega \alpha|w(t)|^2\,dx + \frac{\beta_0}{2}|w(t)|^2_{H^1(\Omega)} \le C_{11}\int_\Omega |w(t)|^2\,dx.$$

The integration with respect to time is possible and yields

$$\begin{aligned}\alpha_0\int_\Omega |w(t)|^2\,dx + \frac{\beta_0}{2}\int_0^t |w(\xi)|^2_{H^1(\Omega)}\,d\xi &\le C_{11}\int_0^t\int_\Omega |w(\xi)|^2\,dx\,d\xi, \\ \int_\Omega |w(t)|^2 dx &\le \frac{C_{11}}{\alpha_0}\int_0^t\int_\Omega |w(\xi)|^2\,dx\,d\xi,\end{aligned} \tag{2.65}$$

because $|w(0)|^2 = 0$.
From the last inequality, using Gronwall's lemma (2.12), we find that

$$\int_\Omega |w(t)|^2\,dx \le 0, \qquad t \in (0,T). \tag{2.66}$$

This already implies that $w \equiv 0$ and, hence, $u_1 = u_2$, which proves the uniqueness of the weak solution.

3. Finite element approximation

Let us assume that the domain Ω is polygonal. By $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$, we denote a system of triangulations of Ω with standard properties from the finite element theory (see, e.g., [5]): $\mathcal{T}_h$ is formed by a finite number of closed triangles K and

$$(3.1)\qquad \begin{aligned} &\text{(a)}\ \overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K,\\ &\text{(b)}\ \text{if } K_1,\, K_2 \in \mathcal{T}_h,\ K_1 \neq K_2, \text{ then either } K_1 \cap K_2 = \emptyset\\ &\qquad \text{or } K_1 \cap K_2 \text{ is a common vertex or } K_1 \cap K_2 \text{ is a common side}\\ &\qquad \text{of } K_1 \text{ and } K_2. \end{aligned}$$

Let the end points of $\overline{\Gamma}_1$, $\overline{\Gamma}_2$, $\overline{\Gamma}_3$ be vertices of the triangulations $\mathcal{T}_h$.

By h_K and ϑ_K we denote the length of the maximal side and the magnitude of the minimal angle of $K \in \mathcal{T}_h$, respectively, and set

$$(3.2)\qquad h = \max_{K\in\mathcal{T}_h} h_K, \qquad \vartheta_h = \min_{K\in\mathcal{T}_h} \vartheta_K.$$

Let us assume that the system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ is *regular*. This means that there exists a constant $\vartheta_0 > 0$ such that

$$(3.3)\qquad \vartheta_h \geq \vartheta_0 \quad \forall\, h \in (0, h_0).$$

We define the following finite dimensional spaces:

$$(3.4)\qquad \begin{aligned} X_h &= \{v_h \in C(\overline{\Omega}); v_h\,|_K \in P_1(K) \ \ \forall\, K \in \mathcal{T}_h\},\\ V_h &= X_h \cap V = \{v_h \in X_h; v_h|_{\Gamma_2} = 0\}, \end{aligned}$$

where $P_1(K)$ is the space of all linear polynomials on K.

The *approximate solution* is defined as a function u_h with the following properties:

$$(3.5)\qquad \begin{aligned} &\text{(a)}\ u_h \in C^1([0,T]; V_h),\\ &\text{(b)}\ \frac{d}{dt}(u_h(t), \varphi_h)_\alpha + a(u_h(t), \varphi_h) = d(u_h(t), \varphi_h) + (q, \varphi_h) \quad \forall\, \varphi_h \in V_h,\\ &\text{(c)}\ u_h(0) = u_h^0 = \pi_h u^0, \end{aligned}$$

where π_h is a suitable interpolation operator from V into V_h.

Similarly as in the case of the Galerkin approximation we can prove the existence of a unique solution of the *discrete problem* (3.5).

If we denote by $\{v_1, v_2, \dots, v_N\}$ a basis of the space V_h, then there exist functions $\xi_j(t)$, $j = 1, \dots, N$, such that

$$u_h(t) = \sum_{j=1}^{N} \xi_j(t) v_j \tag{3.6}$$

and condition (3.5), (b) can be rewritten in the form

$$\frac{d}{dt}(\sum_{j=1}^{N} \xi_j(t) v_j, v_i)_\alpha + a(\sum_{j=1}^{N} \xi_j(t) v_j, v_i) = d(\sum_{j=1}^{N} \xi_j(t) v_j, v_i) + (q, v_i), \quad i = 1, \dots, N, \tag{3.7}$$

or

$$\sum_{j=1}^{N} (v_j, v_i)_\alpha \frac{d\xi_j(t)}{dt} = -\sum_{j=1}^{N} a(v_j, v_i) \xi_j(t) + d(\sum_{j=1}^{N} \xi_j(t) v_j, v_i) + (q, v_i), \quad i = 1, \dots, N. \tag{3.8}$$

This is a system of nonlinear ordinary differential equations which can be solved by a suitable discrete method for the solution of ODE's. Let us mention several simple numerical schemes. To this end, we construct a partition $\{t_k\}_{k=0}^{M}$ of the time interval $[0, T]$, where $t_k = k\tau$ and $\tau = T/M$.

We have several possibilities of the time discretization:

(1) We use the approximation $\xi_j^k \approx \xi_j(t_k)$ and

$$\frac{d\xi_j(t_k)}{dt} \approx \frac{\xi_j^{k+1} - \xi_j^k}{\tau} \tag{3.9}$$

and all other terms with ξ_j are considered on the time level t_k. In this way we obtain a simple explicit forward Euler scheme whose stability is conditioned by a rather restrictive limitation of the time step τ.

(2) The use of the backward time difference

$$\frac{d\xi_j(t_{k+1})}{dt} \approx \frac{\xi_j^{k+1} - \xi_j^k}{\tau} \tag{3.10}$$

on the time level t_{k+1} leads to fully implicit unconditionally stable scheme. This requires to solve a nonlinear algebraic system on each time level t_{k+1} for unknowns $\xi_1^{k+1}, \dots, \xi_N^{k+1}$.

(3) If we use the approximation

$$\frac{d\xi_j(t_{k+1})}{dt} \approx \frac{\xi_j^{k+1} - \xi_j^k}{\tau}, \tag{3.11}$$

and consider the linear terms $\sum_{j=1}^N a(v_j, v_i)\xi_j(t)$ on the $(k+1)$-st time level, whereas the nonlinear terms $d(\sum_{j=1}^N \xi_j(t)v_j, v_i)$ are linearized with respect to $\xi_1^{k+1}, \dots, \xi_N^{k+1}$, as e.g.,

$$\begin{aligned} d(\sum_{j=1}^N \xi_j(t_{k+1})v_j, v_i) &\approx -\int_{\Gamma_1} \gamma \sum_{j=1}^N \xi_j^{k+1} v_j v_i \, dS + \int_{\Gamma_1} G(x, \sum_{j=1}^N \xi_j^k v_j) v_i \, dS \\ &= \sum_{j=1}^N d_1(v_j, v_i)\xi_j^{k+1} + \int_{\Gamma_1} G(x, \sum_{j=1}^N \xi_j^k v_j) v_i \, dS \end{aligned} \tag{3.12}$$

we obtain a semiimplicite conditionally stable scheme, requiring the solution of a linear system with respect to unknowns $\xi_1^{k+1}, \dots, \xi_N^{k+1}$ on each time level.

In what follows we shall be concerned with the investigation of the convergence and error estimates for the space semidiscretization (3.5). We shall assume that the continuous problem (1.14)–(1.18) possesses a unique *strong solution* u with the following regularity properties:

$$u \in L^2(0, T; H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0, T; H^1(\Omega)). \tag{3.13}$$

The main result of this section can be formulated in the following way:

Theorem 6. *Let the exact solution satisfy the regularity conditions* (3.13) *and let $u_h \in C^1([0, T]; V_h)$ be the approximate solution obtained with the aid of the method defined in* (3.5), (a)–(c). *Then there exists a constant C independent of h such that*

$$\max_{t \in [0,T]} \|u(t) - u_h(t)\|_{L^2(\Omega)} \le Ch, \tag{3.14}$$

$$\|u - u_h\|_{L^2(0,T;H^1(\Omega))} \le Ch. \tag{3.15}$$

Proof: The exact solution and the approximate solution satisfy the relations

$$\left(\frac{\partial u}{\partial t}, \varphi\right)_\alpha + a(u, \varphi) = d(u, \varphi) + (q, \varphi) \qquad \forall \varphi \in V, \tag{3.16}$$

and

$$\left(\frac{\partial u_h}{\partial t}, \varphi_h\right)_\alpha + a(u_h, \varphi_h) = d(u_h, \varphi_h) + (q, \varphi_h) \qquad \forall \varphi_h \in V_h \subset V, \tag{3.17}$$

respectively. Subtracting these equations from each other and putting $\varphi = \varphi_h := \xi = \pi_h u - u_h \in V_h$, where $\pi_h : V \to V_h$ is Clément's interpolation (see [5, Paragraph 3.2.3]), we obtain

$$\left(\frac{\partial(u-u_h)}{\partial t}, \xi\right)_\alpha + a(u-u_h,\xi) = d(u,\xi) - d(u_h,\xi), \tag{3.18}$$

$$\begin{aligned}\left(\frac{\partial\xi}{\partial t}, \xi\right)_\alpha + \left(\frac{\partial\eta}{\partial t}, \xi\right)_\alpha + a(\xi,\xi) + a(\eta,\xi) &= d(u,\xi) - d(u_h,\xi)\\ &= d(u,\xi) - d(\pi_h u,\xi) + d(\pi_h u,\xi) - d(u_h,\xi),\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\left(\frac{\partial\xi}{\partial t}, \xi\right)_\alpha + a(\xi,\xi)\\ &= d(u,\xi) - d(\pi_h u,\xi) + d(\pi_h u,\xi) - d(u_h,\xi) - a(\eta,\xi) - \left(\frac{\partial\eta}{\partial t}, \xi\right)_\alpha,\end{aligned} \tag{3.20}$$

where we denote $\eta = u - \pi_h u$ and thus $u - u_h = \eta + \xi$.

We estimate individual terms in (3.20) with the use of Lemma 1, Young's inequality (1.48), the Friedrichs inequality (1.28) (with constant C_{F}), inequality (2.5) (with constant C_5), theorem on traces (1.13) (with constant C_{Tr}) and assumptions (1.19)–(1.26). If we set $C_{12} = C_1 + C_2$, $C_{13} = 2((L_G+\gamma_1)C_{\mathrm{Tr}}^2 C_{\mathrm{F}}^2 + C_{12})^2/\beta_0$, $C_{14} = 2(L_G+\gamma_1)^2 C_5^2 C_{\mathrm{F}}^2/\beta_0$, then

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\alpha}\xi(t)\|^2_{L^2(\Omega)} + \frac{\beta_0}{2}|\xi|^2_{H^1(\Omega)} - C_0\|\xi\|^2_{L^2(\Omega)}\\
&\le (L_{\mathrm{G}}+\gamma_1)\|\eta\|_{L^2(\partial\Omega)}\|\xi\|_{L^2(\partial\Omega)}\\
&\quad + (L_{\mathrm{G}}+\gamma_1)\|\xi\|^2_{L^2(\partial\Omega)} + C_{12}|\eta|_{H^1(\Omega)}|\xi|_{H^1(\Omega)} + \alpha_1\|\frac{\partial\eta}{\partial t}\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)}\\
&\le (L_{\mathrm{G}}+\gamma_1)C_{\mathrm{Tr}}^2 C_{\mathrm{F}}^2|\eta|_{H^1(\Omega)}|\xi|_{H^1(\Omega)} + (L_{\mathrm{G}}+\gamma_1)C_5 C_{\mathrm{F}}\|\xi\|_{L^2(\Omega)}|\xi|_{H^1(\Omega)}\\
&\quad + C_{12}|\eta|_{H^1(\Omega)}|\xi|_{H^1(\Omega)} + \alpha_1\|\frac{\partial\eta}{\partial t}\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)}\\
&= |\xi|_{H^1(\Omega)}|\eta|_{H^1(\Omega)}((L_{\mathrm{G}}+\gamma_1)C_{\mathrm{Tr}}^2 C_{\mathrm{F}}^2 + C_{12})\\
&\quad + (L_{\mathrm{G}}+\gamma_1)C_5 C_{\mathrm{F}}\|\xi\|_{L^2(\Omega)}|\xi|_{H^1(\Omega)} + \alpha_1\|\frac{\partial\eta}{\partial t}\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)}\\
&\le \frac{\beta_0}{4}|\xi|^2_{H^1(\Omega)} + C_{13}|\eta|^2_{H^1(\Omega)} + C_{14}\|\xi\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2}\|\frac{\partial\eta}{\partial t}\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2}\|\xi\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\alpha}\xi(t)\|^2_{L^2(\Omega)} + \frac{\beta_0}{4}|\xi|^2_{H^1(\Omega)}\\
&\qquad\le C_{13}|\eta|^2_{H^1(\Omega)} + (C_{14} + C_0 + \frac{\alpha_1}{2})\|\xi\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2}\|\frac{\partial\eta}{\partial t}\|^2_{L^2(\Omega)}.\end{aligned}$$

Under the assumptions that the system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ is regular, Clément's interpolation has the following interpolation properties:

$$\|v-\pi_h v\|_{L^2(\Omega)} \le C_{\mathrm{C}} h |v|_{H^1(\Omega)}, \quad v\in H^1(\Omega), \tag{3.21}$$
$$|v-\pi_h v|_{H^1(\Omega)} \le C_{\mathrm{C}} h |v|_{H^2(\Omega)}, \quad v\in H^2(\Omega),$$
$$\|v-\pi_h v\|_{L^2(\Omega)} \le C_{\mathrm{C}} h^2 |v|_{H^2(\Omega)}, \quad v\in H^2(\Omega). \tag{3.22}$$

(See [5, Paragraph 3.2.3].) This and (3.13) imply that

$$\|\frac{\partial \eta}{\partial t}\|_{L^2(\Omega)} \le C_{\mathrm{C}} h |\frac{\partial u}{\partial t}|_{H^1(\Omega)}, \tag{3.23}$$
$$|\eta|_{H^1(\Omega)} \le C_{\mathrm{C}} h |u|_{H^2(\Omega)}, \tag{3.24}$$
$$\|\eta\|_{L^2(\Omega)} \le C_{\mathrm{C}} h^2 |u|_{H^2(\Omega)}. \tag{3.25}$$

Let us set $C_{15} = \frac{1}{\alpha_0}\max(C_{13}C_{\mathrm{C}}^2, \frac{\alpha_1}{2}C_{\mathrm{C}}^2, \frac{\alpha_1}{2}+C_{14}+C_0)$. Then we have

$$\frac{d}{dt}\|\sqrt{\alpha}\xi(t)\|^2_{L^2(\Omega)} + \frac{\beta_0}{4}|\xi(t)|^2_{H^1(\Omega)} \le \alpha_0 C_{15}\|\xi(t)\|^2_{L^2(\Omega)} + \alpha_0 C_{15} h^2 \left(|\frac{\partial u(t)}{\partial t}|^2_{H^1(\Omega)} + |u(t)|^2_{H^2(\Omega)}\right),$$

$$\|\sqrt{\alpha}\xi(t)\|^2_{L^2(\Omega)} - \|\sqrt{\alpha}\xi(0)\|^2_{L^2(\Omega)} + \frac{\beta_0}{4}\int_0^t |\xi(\vartheta)|^2_{H^1(\Omega)}\,d\vartheta$$
$$\le \alpha_0 C_{15}\int_0^t \|\xi(\vartheta)\|^2_{L^2(\Omega)}\,d\vartheta + \alpha_0 C_{15}h^2\int_0^t \left(|\frac{\partial u(\vartheta)}{\partial t}|^2_{H^1(\Omega)} + |u(\vartheta)|^2_{H^2(\Omega)}\right)d\vartheta,$$

$$\|\xi(t)\|^2_{L^2(\Omega)} + \frac{\beta_0}{4\alpha_0}\int_0^t |\xi(\vartheta)|^2_{H^1(\Omega)}\,d\vartheta \le C_{15}\int_0^t \|\xi(\vartheta)\|^2_{L^2(\Omega)}\,d\vartheta + C_{15}h^2 C_u + \frac{\alpha_1}{\alpha_0}\|\xi(0)\|^2_{L^2(\Omega)}, \tag{3.26}$$

where

$$C_u = \|\frac{\partial u}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} + \|u\|^2_{L^2(0,T;H^2(\Omega))}. \tag{3.27}$$

The last term in (3.26) vanishes in virtue of the relation $u_h^0 = \pi_h u^0 = \pi_h u(0)$.

Now Gronwall's lemma (2.12), where we set

$$y(t) = \|\xi(t)\|^2_{L^2(\Omega)},\ w(t) = \frac{\beta_0}{4\alpha_0}\int_0^t |\xi(\vartheta)|^2_{H^1(\Omega)}\,d\vartheta,\ z(t) = C_{15}C_u h^2,\ r(t) = C_{15},$$

yields the estimate

$$
\begin{aligned}
&\|\xi(t)\|^2_{L^2(\Omega)} + \frac{\beta_0}{4\alpha_0}\int_0^t |\xi(\vartheta)|^2_{H^1(\Omega)}\,d\vartheta \\
&\le C_{15}C_u h^2 + C_{15}^2 C_u h^2 \int_0^t \exp\left(C_{15}(t-\vartheta)\right) d\vartheta \\
&= C_{15}C_u h^2 \exp\left(C_{15}t\right).
\end{aligned}
\tag{3.28}
$$

From this it follows that

$$\max_{t\in[0,T]} \|u_h(t) - \pi_h u(t)\|_{L^2(\Omega)} \le \overline{C}_u h, \tag{3.29}$$

$$\|u_h - \pi_h u\|_{L^2(0,T;H^1(\Omega))} \le \overline{C}_u h. \tag{3.30}$$

Finally, with the aid of (3.23)–(3.25) and triangular inequality we arrive at the error estimates

$$\max_{t\in[0,T]} \|u(t) - u_h(t)\|_{L^2(\Omega)} \le \widehat{C}_u h, \tag{3.31}$$

$$\|u - u_h\|_{L^2(0,T;H^1(\Omega))} \le \widehat{C}_u h, \tag{3.32}$$

where the constants $\overline{C}_u$, $\widehat{C}_u$ depend on u, but are independent of h. This concludes the proof. □

4. Conclusion

In this paper, a nonstationary convection-diffusion problem equipped with mixed Dirichlet - nonlinear Newton boundary conditions have been analyzed. The main result is the proof of the existence and uniqueness of a weak solution and the finite element analysis, carried out in the case of the Lipschitz-continuous boundary nonlinearity. There are several further questions and open problems of the practical importance:

- analysis of the problem with a boundary nonlinearity of more general behaviour (polynomial growth, local Lipschitz-continuity),
- finite element solution using numerical integration for the evaluation of integrals,
- investigation of the effect of the approximation of a curved boundary of the domain Ω,
- extension of the results to higher degree elements,
- analysis of the full space-time discretization.

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References

[1] Barber S.A., *Soil Nutrient Bioavailability: A Mechanistic Approach*, John Wiley & Sons, Inc., New York, 1995.

[2] Bialecki R., Nowak A.J., *Boundary value problems in heat conduction with nonlinear material and nonlinear boundary conditions*, Appl. Math. Model. **5** (1981), 417–421.

[3] Brenner S.C., Scott L.R., *The Mathematical Theory of Finite Element Methods*, Springer, New York, 1994.

[4] Chow S.S., *Finite element error estimates for nonlinear elliptic equations of monotone type*, Numer. Math. **54** (1989), 373–393.

[5] Ciarlet P.G., *The Finite Element Method for Elliptic Problems*, North Holland, Amsterdam, 1978.

[6] Ciarlet P.G., Raviart P.A., *The combined effect of curved boundaries and numerical integration in isoparametric finite element method*, in: The Mathematical Foundations of the Finite Element Method with Application to Partial Differential Equations (A. K. Aziz, Ed.), Academic Press, New York, 1972.

[7] Claassen N., Barber S.A., *Simulation model for nutrient uptake from soil by a growing plant root system*, Agronomy Journal **68** (1976), 961–964.

[8] Cwikel M., *Real and complex interpolation and extrapolation of compact operators*, Duke Math. J. **65.2** (1992), 333–343.

[9] Dolejší V., Feistauer M., Schwab C., *A finite volume discontinuous Galerkin scheme for nonlinear convection-diffusion problems*, preprint, Forschungsinstitut für Mathematik, ETH Zürich, January 2001 (to appear in Calcolo).

[10] Feistauer M., *Mathematical Methods in Fluid Mechanics*, The Pitman Monographs and Surveys in Pure and Applied Mathematics 67, Longman Scientific and Technical Series, Harlow, 1993.

[11] Feistauer M., Kalis H., Rokyta M., *Mathematical modelling of an electrolysis process*, Comment Math. Univ. Carolinae **30** (1989), 465–477.

[12] Feistauer M., Najzar K., *Finite element approximation of a problem with a nonlinear Newton boundary condition*, Numer. Math. **78** (1998), 403–425.

[13] Feistauer M., Najzar K., Sobotíková V., *Error estimates for the finite element solution of elliptic problems with nonlinear Newton boundary conditions*, Numer. Funct. Anal. Optim. **20** (1999), 835–851.

[14] Feistauer M., Najzar K., Sobotíková V., *On the finite element analysis of problems with nonlinear Newton boundary conditions in nonpolygonal domains*, Appl. Math. **46** (2001), 353–382.

[15] Feistauer M., Najzar K., Sobotíková V., Sváček P., *Numerical analysis of problems with nonlinear Newton boundary conditions*, in: Proc. of the 3rd European Conference Numerical Mathematics and Advanced Applications (P. Neittaanmäki, T. Tiihonen, P. Tarvainen, Editors), World Scientific, Singapore, 2000, pp. 486–493.

[16] Feistauer M., Sobotíková V., *Finite element approximation of nonlinear elliptic problems with discontinuous coefficients*, M^2AN **24** (1990), 457–500.

[17] Feistauer M., Ženíšek A., *Finite element solution of nonlinear elliptic problems*, Numer. Math. **50** (1987), 451–475.

[18] Ganesh M., Graham I.G., Sivaloganathan J., *A pseudospectral three-dimensional boundary integral method applied to a nonlinear model problem from finite elasticity*, SIAM J. Numer. Anal. **31** (1994), 1378–1414.

[19] Ganesh M., Steinbach O., *Boundary element methods for potential problems with nonlinear boundary conditions*, Applied Mathematics Report AMR98/17, School of Mathematics, The University of New South Wales, Sydney, 1998.

[20] Ganesh M., Steinbach O., *Nonlinear boundary integral equations for harmonic problems*, Applied Mathematics Report AMR98/20, School of Mathematics, The University of New South Wales, Sydney, 1998.
[21] Girault V., Raviart P.A., *Finite Element Approximation of the Navier-Stokes Equations*, Lecture Notes in Mathematics 749, Springer-Verlag, Berlin–Heidelberg–New York, 1979.
[22] Křížek M., Liu L., Neittaanmäki P., *Finite element analysis of a nonlinear elliptic problem with a pure radiation condition*, in: Applied Nonlinear Analysis, Kluwer, Amsterdam, 1999, pp. 271–280.
[23] Kufner A., John O., Fučík S., *Function Spaces*, Academia, Prague, 1977.
[24] Kurzweil J., *Ordinary Differential Equations*, Elsevier, Amsterdam–Oxford–New York–Tokyo, 1986.
[25] Lions J.L., *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod, Paris, 1969.
[26] Lions J.L., Magenes E., *Problémes aux limites non homogénes et applications*, Dunod, Paris, 1968.
[27] Liu L., Křížek M., *Finite element analysis of a radiation heat transfer problem*, J. Comput. Math. **16** (1998), 327–336.
[28] Málek J., Nečas J., Rokyta M., Růžička M., *Weak and Measure-Valued Solutions to Evolutionary PDEs*, Chapman & Hall, London, 1996.
[29] Moreau R., Ewans J.W., *An analysis of the hydrodynamics of aluminium reduction cells*, J. Electrochem. Soc. **31** (1984), 2251–2259.
[30] Nečas J., *Les méthodes directes en théories des équations elliptiques*, Academia, Prague, 1967.
[31] Sváček P., *Higher order finite element method for a problem with nonlinear boundary condition*, in: Proc. of the 13th Summer School "Software and Algorithms of Numerical Mathematics", West Bohemian University Pilsen, 1999, pp. 301–308.
[32] Temam R., *Navier-Stokes Equations*, North-Holland, Amsterdam–New York–Oxford, 1977.
[33] Ženíšek A., *Nonhomogeneous boundary conditions and curved triangular finite elements*, Appl. Math. **26** (1981), 121–141.
[34] Ženíšek A., *The finite element method for nonlinear elliptic equations with discontinuous coefficients*, Numer. Math. **58** (1990), 51–77.
[35] Zlámal M., *Curved elements in the finite element method*, I. SIAM J. Numer. Anal. **10** (1973), 229–240.

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