Čech-completeness and ultracompleteness in “nice” spaces

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Abstract. We prove that if X^n is a union of n subspaces of pointwise countable type then the space X is of pointwise countable type. If X^ω is a countable union of ultracomplete spaces, the space X^ω is ultracomplete. We give, under CH, an example of a Čech-complete, countably compact and non-ultracomplete space, giving thus a partial answer to a question asked in [BY2].

Keywords: ultracompleteness, Čech-completeness, countable type, pointwise countable type

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0. Introduction

Given a space X and an $n \in \mathbb{N}$, we say that a property $\mathcal{P}$ is n-additive in power n, if $X \in \mathcal{P}$ in case $X^n = X_1 \cup X_2 \cup \ldots \cup X_n$, where $X_i \in \mathcal{P}$ for all $i \leq n$. We will say that $\mathcal{P}$ is additive in finite powers if it is n-additive in power n for every $n \in \mathbb{N}$. In [Tk] it was proved that many non-additive properties are additive in finite powers. In particular, it was shown that weight, character, pseudocharacter and tightness are additive in finite powers. The paper [BGT] provides some examples where metrizability is not 2-additive in power 2. In [Lo] it was established that there are models of ZFC where the Čech-completeness is not additive in power 2. Thus it is a natural question to ask which completeness-like properties are additive in finite powers. In this paper we prove additivity in finite powers for pointwise countable type.

In 1987, V.I. Ponomarev and V.V. Tkachuk introduced in [PT] the concept of strongly complete spaces as those X which have countable character in βX. In [BY1] the same property was defined internally and was called ultracompleteness. Clearly, ultracompleteness is a stronger property than Čech-completeness and it is a consequence of local compactness. Hence many categorical properties of ultracompleteness are similar to the properties of Čech-completeness. In particular, ultracompleteness is hereditary with respect to closed subsets and it is not additive. In this paper we prove that ultracompleteness is countably additive in X^ω, that is, if $X^\omega = \bigcup_{i\in\omega} X_i$ and each X_i is ultracomplete, then the

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space X^ω is ultracomplete. We also establish that any ultracomplete topological group is locally compact. Under CH we answer positively a question posed by Buhagiar and Yoshioka [BY2], proving the existence of a Čech-complete countably compact non-ultracomplete space. We also show that in metrizable spaces the Čech-completeness is additive and ultracompleteness is not.

1. Notation and terminology

Throughout this paper all spaces are assumed to be Tychonoff. Given a space X, we denote its topology by $\mathcal{T}(X)$ and $\mathcal{T}^*(X) = \{U \in \mathcal{T}(X) : U \neq \emptyset\}$. The family of clopen non-empty subsets of X will be denoted by $\mathcal{T}_c^*(X)$. If $A \subset X$ then $\mathcal{T}(A, X) = \{U \in \mathcal{T}(X) : A \subset U\}$ and $\mathcal{T}(x, X) = \mathcal{T}(\{x\}, X)$. The space X is ultracomplete if $\chi(X, cX) \leq \omega$ for every compactification cX of the space X. Following a Russian practice we say that a family is centered if it has the finite intersection property. The sequence $\gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ of open covers of X is called complete if, given a centered family $\mathcal{H}$ of subsets of X such that for every $n \in \mathbb{N}$, there exist $H \in \mathcal{H}$ and $U \in \gamma_n$ with $H \subset U$, we have $\bigcap \overline{\mathcal{H}} = \bigcap\{\overline{H} : H \in \mathcal{H}\} \neq \emptyset$. The space X is Čech-complete if it has a complete sequence of open covers and X is of (pointwise) countable type if for any compact set $F \subset X$ (for any $x \in X$) there exists a compact $K \subset X$ such that $F \subset K$ ($x \in K$) and $\chi(K, X) \leq \omega$. Given an $A \subset X$, we denote its closure in βX by $\overline{A}^{\beta X}$. A collection $\mathcal{P} \subset \mathcal{T}^*(X)$ is a π-base in X if given a $U \in \mathcal{T}^*(X)$, there exists a $V \in \mathcal{P}$ such that $V \subset U$. A subset $A \subset X$ is bounded in X if any continuous real-valued function on X is bounded on A. If κ is a cardinal number, the Kowalsky hedgehog with κ spines is formed from the union of κ copies of the unit interval $[0, 1]$ by identifying the zero points of each interval. Its metric is defined by $d(x, y) = |x - y|$ if x and y belong to the same interval (also called spine), and $d(x, y) = x + y$ otherwise. The point obtained from identifying the zeros of the spines is called the vertex of the hedgehog. A normal space X is an F-space if any two disjoint open F_σ subsets of X have disjoint closures in X. If $x \in X$ then $x^n = (x_1, x_2, \ldots, x_n) \in X^n$ where $x_i = x$ for all $i \leq n$.

2. Additivity in powers

We will prove, among other things, that pointwise countable type is additive in finite powers. It turns out that, in metrizable spaces, the Čech-completeness is additive. We also prove that any locally ultracomplete topological group is locally compact.

The next result is well-known (see for example [Ar]).

2.1 Theorem. *Let X be a topological space.*

(i) *If X is of pointwise countable type then X is a union of G_δ-subsets in every compactification cX of the space X.*

(ii) *Let K be a compact space which contains X. If X is a union of G_δ-subsets in K then X is of pointwise countable type.*

2.2 Theorem. *Let $X^n = X_1 \cup X_2 \cup \ldots \cup X_n$ where X_i is of pointwise countable type for all $i \leq n$. Then the space X is of pointwise countable type.*

Proof: Before taking to the proof, we will establish the following result.

2.3 Lemma. *If $X = A_1 \cup A_2 \cup \ldots \cup A_n$ where A_i is of pointwise countable type for all $i \in \{1, 2, \ldots, n\}$ then $\bigcap_{i=1}^n \overline{A_i}$ is of pointwise countable type.*

Proof: By Theorem 2.1(ii), it is sufficient to prove that $\bigcap_{i=1}^n \overline{A_i}$ is a union of G_δ-subsets of the compact set $P = \bigcap_{i=1}^n \overline{A_i}^{\beta X}$. Since each A_i is of pointwise countable type, we have $A_i = \bigcup_{\alpha \in \mathcal{I}_i} G_\alpha^i$ where G_α^i is a G_δ-subspace of $\overline{A_i}^{\beta X}$ for all $\alpha \in \mathcal{I}_i$ and $i = 1, 2, \ldots, n$. Let $A_i' = A_i \cap P$ for $i \leq n$. We have $A_i' = \bigcup_{\alpha \in \mathcal{I}_i} (G_\alpha^i \cap P)$, and hence the subspace A_i' is a union of G_δ-subsets of the compact space P for every $i \leq n$. Since $\bigcup_{i=1}^n A_i' = \bigcup_{i=1}^n (A_i \cap P) = P \cap \left(\bigcup_{i=1}^n A_i\right) = X \cap P = X \cap \left(\bigcap_{i=1}^n \overline{A_i}^{\beta X}\right) = \bigcap_{i=1}^n \overline{A_i}$, the set $\bigcap_{i=1}^n \overline{A_i}$ is a union of G_δ-subsets of P and hence it is of pointwise countable type.

We will prove Theorem 2.2 by induction on n. The case $n = 1$ is clear. Suppose that its conclusion is valid for every $m < n$.

Given an $x_0 \in X$, consider the subspace of pointwise countable type $F = \bigcap_{i=1}^n \overline{X_i}$ and the closed subset $Y = \{(x_0^{n-1}, y) : y \in X\}$. If $Y \subset F$ then Y is of pointwise countable type. Hence the space X is of pointwise countable type because X and Y are homeomorphic.

If $Y - F \neq \emptyset$, we pick $(x_0^{n-1}, y) \in Y - F$. There are $U \in \mathcal{T}(x_0, X)$, $V \in \mathcal{T}(y, X)$ and $i_0 \in \{1, 2, \ldots, n\}$ such that $(U^{n-1} \times V) \cap X_{i_0} = \emptyset$. Without loss of generality, we may assume that $i_0 = 1$.

Since $U^{n-1} \times V \subset X_2 \cup X_3 \cup \ldots \cup X_n$, we obtain $U^{n-1} \times V = X_2' \cup X_3' \cup \ldots \cup X_n'$ where $X_j' = X_j \cap (U^{n-1} \times V)$ is of pointwise countable type for $2 \leq j \leq n$. As $U^{n-1} \times \{y\}$ is a closed subspace of $U^{n-1} \times V$, we have $U^{n-1} \times \{y\} = X_2'' \cup X_3'' \cup \ldots \cup X_n''$ where the space $X_j'' = X_j' \cap (U^{n-1} \times \{y\})$ is of pointwise countable type for all $j = 2, 3, \ldots, n$. Since $U^{n-1} \times \{y\}$ and U^{n-1} are homeomorphic we conclude that $U^{n-1} = X_2''' \cup X_3''' \cup \ldots \cup X_n'''$ where all X_j''' are of pointwise countable type. By the inductive hypothesis the open subspace $U \ni x_0$ is of pointwise countable type. Being $x_0 \in X$ an arbitrary point, the space X is of locally pointwise countable type, so it is of pointwise countable type. □

2.4 Corollary. *If $X^n = X_1 \cup X_2 \cup \ldots \cup X_n$ where X_i is locally Čech-complete for $i \leq n$, then X is of pointwise countable type.*

2.5 Theorem. *Let $X = A_1 \cup A_2 \cup \ldots \cup A_n$ where A_i is an ultracomplete subspace of X for all $i = 1, 2, \ldots, n$. If $X = \overline{A_i}$ for all $i \in \{1, 2, \ldots, n\}$, then X is ultracomplete.*

Proof: Since $X = \overline{A_i}$, we have $\overline{A_i}^{\beta X} = \beta X$. For each $i \in \{1, 2, \ldots, n\}$, we choose a countable base $\mathcal{U}_i \subset \mathcal{T}(\beta X)$ for A_i. Consider the family

$$\mathcal{W} = \{U_1 \cup U_2 \cup \ldots \cup U_n : U_i \in \mathcal{U}_i \text{ for every } i = 1, 2, \ldots, n\}.$$

It is clear that $\mathcal{W} \subset \mathcal{T}(\beta X)$ and $\mathcal{W}$ is countable. We shall prove that $\mathcal{W}$ is a base for X in βX. If $V \in \mathcal{T}(X, \beta X)$ then $A_i \subset V$ for all $i \le n$. Hence for each $i \in \{1, 2, \ldots, n\}$ there exists $U_{j(i)} \in \mathcal{U}_i$ for which $A_i \subset U_{j(i)} \subset V$. This implies that $\bigcup_{i=1}^n U_{j(i)} \in \mathcal{W}$ and

$$X = \bigcup_{i=1}^{n} A_i \subset \bigcup_{i=1}^{n} U_{j(i)} \subset V.$$

□

If not all subspaces A_i are dense in X then X is not necessarily ultracomplete. The following corollary shows what we can have in general case.

2.6 Corollary. *Suppose that $X = A_1 \cup A_2 \cup \ldots \cup A_n$, where A_i is ultracomplete for all $i = 1, 2, \ldots, n$. Then there exists $G \in \mathcal{T}^*(X)$ such that $\overline{G}$ is ultracomplete.*

Proof: We use induction on n. The case $n = 1$ is obvious. Suppose that $n \ge 2$ and our assertion is true for every $k < n$. If $X = \overline{A_i}$ for all $i \le n$, Theorem 2.5 shows that X is ultracomplete and hence, we can take $G = X$, so the assertion is proved. Assume that there is an $i_0 \in \{1, 2, \ldots, n\}$ such that $X \ne \overline{A_{i_0}}$. Without loss of generality, we may suppose that $i_0 = 1$. Since $X - \overline{A_1} \subset A_2 \cup A_3 \cup \ldots \cup A_n$ there exists $W \in \mathcal{T}^*(X)$ for which $W \subset \overline{W} \subset X - \overline{A_1} \subset A_2 \cup A_3 \cup \ldots \cup A_n$. Therefore, $\overline{W} = A_2' \cup A_3' \cup \ldots \cup A_n'$ where $A_j' = A_j \cap \overline{W}$ is ultracomplete for $2 \le j \le n$. By the inductive hypothesis, there exists $H = \overline{W} \cap U \ne \emptyset$ for some $U \in \mathcal{T}^*(X)$ such that $\overline{H}$ is ultracomplete. There exists $G \in \mathcal{T}^*(X)$ for which $G \subset \overline{G} \subset W \cap U \subset \overline{H}$, because $W \cap U \in \mathcal{T}^*(X)$. Since $\overline{H}$ is ultracomplete, the subspace $\overline{G}$ is ultracomplete as well. □

2.7 Theorem. *Let X be a metrizable space. Given an $n \in \mathbb{N}$, suppose that $X^n = X_1 \cup X_2 \cup \ldots \cup X_n$ where each X_i is ultracomplete. Then X is ultracomplete.*

Proof: By Corollary 11 of [PT] there is a compact $K_i \subset X_i$ such that $X_i - K_i$ is locally compact. Denote by $p_j : X^n \to X$ the projection of X^n onto its j-th factor and let $K = \bigcup_{i,j=1}^n p_j(K_i)$. The set $K \subset X$ is compact and $Y^n \cap (K_1 \cup \ldots \cup K_n) = \emptyset$, where $Y = X - K$. If $Y_i = X_i \cap Y^n$ then Y_i is an open subset of the locally compact space $X_i - K_i$. Thus, Y^n is a union of $\leq n$ locally compact spaces. Apply Corollary 1.4 of [Tk] to conclude that Y is locally compact. As a consequence, X is a metrizable space whose points of non-local compactness lie inside a compact set K. Hence X is ultracomplete by Corollary 11 from [PT]. $\square$

2.8 Theorem. *If X^ω is a countable union of ultracomplete subspaces, then the space X^ω is ultracomplete.*

Proof: Call a subset $Z \subset X^\omega$ *strongly dense* in X^ω if $p_A(Z) = X^A$ for any finite $A \subset \omega$; here $p_A : X^\omega \to X^A$ is the natural projection onto the face X^A. This concept belongs to Tkachenko [Tk1] as well as the following

Lemma. *If Z is a strongly dense in X^ω then the map $p_A : Z \to X^A$ is open for any finite $A \subset \omega$.*

Tkachenko also proved in [Tk1, Lemma 2] that if Y is any space such that $Y^\omega = \bigcup\{Z_i : i \in \omega\}$ then either some Z_i is strongly dense in Y^ω or there exists a finite $B \subset \omega$ and a point $z \in Y^B$ such that $Z_i \cap p_B^{-1}(z)$ is strongly dense in $p_B^{-1}(z)$ which is identified with Y^ω in an obvious way. An easy consequence is that there is always an $i \in \omega$ and a closed subspace $F \subset Z_i$ such that F maps openly and continuously onto Y.

Now, assume that $X^\omega = \bigcup\{X_i : i \in \omega\}$ where X_i is ultracomplete for all $i \in \omega$. Letting $Y = X^\omega$ we obtain $Y^\omega = (X^\omega)^\omega = X^\omega = \bigcup_{i\in\omega} X_i$. Applying the mentioned results of Tkachenko we can find $i \in \omega$ and a closed $F \subset X_i$ such that there is an open continuous map of F onto $Y = X^\omega$. The space F is ultracomplete being a closed subspace of an ultracomplete space X_i. Since an open continuous image of an ultracomplete space is ultracomplete, the space $Y = X^\omega$ is also ultracomplete. $\square$

Our next proposition shows that Čech-completeness is finitely additive in metrizable spaces. The original proof given by the authors was somewhat technical. With the permission of O. Okunev, we present here the proof he suggested after being informed about this result.

2.9 Proposition. *Let X be a metrizable space. If $X = A_1 \cup A_2 \cup \ldots \cup A_n$ where A_i is Čech-complete for all $i \leq n$, then X is Čech-complete.*

Proof: Let us consider the completion $\widetilde{X}$ of the space X. Since $\widetilde{X}$ is a metric space, each subspace A_i with $i \leq n$ is a G_δ-subset of $\widetilde{X}$. A finite union of G_δ-sets is a G_δ-set, so the space X is a G_δ-subset of $\widetilde{X}$. Therefore X is Čech-complete. $\square$

2.10 Example. *There exists a non-ultracomplete separable metric space which is a union of two ultracomplete subspaces.*

Proof: Let H be the Kowalsky hedgehog with countably many spines. Then H is locally compact at all points except the vertex h. By [PT, Corollary 11], the space H is ultracomplete. Now let H_i be a homeomorphic copy of H for each $i \in \omega$ with $h_i \in H_i$ being the respective copy of h.

The space $X = \bigoplus_{i\in\omega} H_i$ is separable and metrizable and the set $\{h_i : i \in \omega\}$ of its points of non-local compactness is not compact. Apply again Corollary 11 from [PT] to conclude that X is not ultracomplete. However, $X = A \cup B$, where $A = \{h_i : i \in \omega\}$ and $B = X - A$ and both subspaces A and B are locally compact and hence ultracomplete. □

2.11 Theorem. *Let G be topological group. If there is an open non-empty $U \subset G$ such that $\overline{U}$ is ultracomplete then G is locally compact.*

Proof: The set of points of local compactness is open in any space. Thus, if the set of points of local compactness in $\overline{U}$ is non-empty, then it meets U and therefore there is a point x of local compactness in U. It is clear that x will be point of local compactness in G. By homogeneity of G, all points of G will be points of local compactness, i.e., G is locally compact.

If there are no points of local compactness in $\overline{U}$ then $\overline{U}$ is bounded in $\overline{U}$ [PT], i.e., $\overline{U}$ is pseudocompact. Observe that U is Čech-complete and hence there exists a compact $K \subset G$ with $\chi(K, G) \leq \omega$. It was proved in [Pa] that such groups are paracompact and hence any pseudocompact closed subspace of G has to be compact. As a consequence $\overline{U}$ is compact which is a contradiction. □

2.12 Corollary. *Any locally ultracomplete topological group is locally compact.*

2.13 Corollary. *If a topological group G is a finite union of its ultracomplete subspaces then G is locally compact.*

Proof: By Corollary 2.6 there is an open non-empty $U \subset G$ such that $\overline{U}$ is ultracomplete. By Theorem 2.11 G is locally compact. □

3. An example of a countably compact, Čech-complete but not ultracomplete space

Under CH, we give an example of a countably compact subspace of $\beta\omega$, which is Čech-complete but not ultracomplete, thus answering a question posed in [BY2].

Let $\beta\omega$ be the Stone-Čech compactification of ω. We will deal with the remainder $\omega^* = \beta\omega - \omega$. A point x in a topological space X is a ***P***-point of X if every G_δ-set containing x is a neighborhood of x. It is known that, under CH, there are ***P***-points in ω^*.

3.1 Lemma. *Assuming CH, take any **P**-point $p \in \omega^*$. Then there exist families $\{U_\beta : \beta < \omega_1\}$, $\{V_\beta : \beta < \omega_1\} \subset \mathcal{T}_c^*(\omega^*)$ such that*

(1) $U_\beta \subset U_{\beta'}$ $V_\beta \subset V_{\beta'}$ *if* $\beta < \beta' < \omega_1$;

(2) $\left(\bigcup_{\beta<\omega_1} U_\beta\right) \cap \left(\bigcup_{\beta<\omega_1} V_\beta\right) = \emptyset$;

(3) $\overline{\left(\bigcup_{\beta<\omega_1} U_\beta\right) \cup \left(\bigcup_{\beta<\omega_1} V_\beta\right)} = \omega^*$;

(4) $p \in \overline{\left(\bigcup_{\beta<\omega_1} U_\beta\right)} \cap \overline{\left(\bigcup_{\beta<\omega_1} V_\beta\right)} - \left(\left(\bigcup_{\beta<\omega_1} U_\beta\right) \cup \left(\bigcup_{\beta<\omega_1} V_\beta\right)\right)$.

Proof: Let $\mathcal{O} = \{O_\alpha : \alpha < \omega_1\} \subset \mathcal{T}_c^*(\omega^*)$ be a π-base in ω^* with $p \notin O_\alpha$ for every $\alpha < \omega_1$. Take a local base $\{W_\alpha : \alpha < \omega_1\} \subset \mathcal{T}_c^*(\omega^*)$ at p for which, if $\alpha < \alpha'$ then $W_{\alpha'} \subset W_\alpha$. Take any $x, y \in \omega^* - \{p\}$ with $x \neq y$. Consider $W_0', W_0'' \in \mathcal{T}_c^*(\omega^*)$ such that $W_0' \subset W_0$, $W_0'' \subset W_0$, $W_0' \cap W_0'' = \emptyset$ and $\left(W_0' \cup W_0''\right) \cap \{x, y, p\} = \emptyset$. There are $U_0', V_0' \in \mathcal{T}_c^*(\omega^*)$ for which $x \in U_0'$, $y \in V_0'$, $p \notin \left(U_0' \cup V_0'\right)$, $U_0' \cap V_0' = \emptyset$ and $\left(U_0' \cup V_0'\right) \cap \left(W_0' \cup W_0''\right) = \emptyset$. Let $U_0'' = U_0' \cup W_0'$, $V_0'' = V_0' \cup W_0''$. In case $\left(U_0'' \cup V_0''\right) \cap O_0 \neq \emptyset$, let $U_0 = U_0''$, $V_0 = V_0''$. Otherwise, take $U_0 = U_0'' \cup O_0$ and $V_0 = V_0''$.

Assume that, for some $\alpha < \omega_1$, we constructed families $\{U_\beta : \beta < \alpha\} \subset \mathcal{T}_c^*(\omega^*)$, $\{V_\beta : \beta < \alpha\} \subset \mathcal{T}_c^*(\omega^*)$ such that

(1_α) $U_\beta \subset U_{\beta'}$ $V_\beta \subset V_{\beta'}$ if $\beta < \beta' < \alpha$;

(2_α) $\left(\bigcup_{\beta<\alpha} U_\beta\right) \cap \left(\bigcup_{\beta<\alpha} V_\beta\right) = \emptyset$ and $p \notin \left(\bigcup_{\beta<\alpha} U_\beta\right) \cup \left(\bigcup_{\beta<\alpha} V_\beta\right)$;

(3_α) $\left(U_\beta \cup V_\beta\right) \cap O_\beta \neq \emptyset$ for all $\beta < \alpha$;

(4_α) $U_\beta \cap W_\beta \neq \emptyset$ and $V_\beta \cap W_\beta \neq \emptyset$ for all $\beta < \alpha$.

Let $F_\alpha = \overline{\bigcup_{\beta<\alpha} U_\beta}$ and $G_\alpha = \overline{\bigcup_{\beta<\alpha} V_\beta}$. Being ω^* an F-space, we have $F_\alpha \cap G_\alpha = \emptyset$. Since $p \notin \left(\bigcup_{\beta<\alpha} U_\beta\right) \cup \left(\bigcup_{\beta<\alpha} V_\beta\right)$ and p is a $\boldsymbol{P}$-point, there exists $W \in \mathcal{T}(p, \omega^*)$ with $W \subset \left(\omega^* - \bigcup_{\beta<\alpha} U_\beta\right) \cap \left(\omega^* - \bigcup_{\beta<\alpha} V_\beta\right)$. As a consequence,

$$W \cap \left(\left(\bigcup_{\beta<\alpha} U_\beta\right) \cup \left(\bigcup_{\beta<\alpha} V_\beta\right)\right) = \emptyset.$$

Hence, $p \notin F_\alpha \cup G_\alpha$ and there exists $\beta_\alpha \geq \alpha$ such that $W_{\beta_\alpha} \subset W$.

By normality of ω^*, we can choose clopen, disjoint sets U_α', $V_\alpha' \subset \omega^*$ for which $F_\alpha \subset U_\alpha'$, $G_\alpha \subset V_\alpha'$, and $p \notin \left(U_\alpha' \cup V_\alpha'\right)$. Let W', $W'' \subset W_{\beta_\alpha}$ be disjoint, clopen sets for which $p \notin W' \cup W''$ and $\left(W' \cup W''\right) \cap \left(U_\alpha' \cup V_\alpha'\right) = \emptyset$. Define $U_\alpha'' = U_\alpha' \cup W'$, $V_\alpha'' = V_\alpha' \cup W''$. To end our construction, in case $\left(U_\alpha'' \cup V_\alpha''\right) \cap O_\alpha \neq \emptyset$, we take $U_\alpha = U_\alpha''$ and $V_\alpha = V_\alpha''$. Otherwise, we choose $U_\alpha = U_\alpha'' \cup O_\alpha$ and $V_\alpha = V_\alpha''$. It is easy to see that the properties ($1_{\alpha+1}$)-($4_{\alpha+1}$) are fulfilled and hence the inductive construction goes on until ω_1.

As a result, we obtain families $\{U_\beta : \beta < \omega_1\}$ and $\{V_\beta : \beta < \omega_1\}$ for which the properties (1_α)-(4_α) are fulfilled for all $\alpha < \omega_1$. We conclude by showing that the families $\{U_\alpha : \alpha < \omega_1\}$ and $\{V_\alpha : \alpha < \omega_1\}$ so obtained satisfy the conditions of our Lemma. If $\beta < \beta' < \omega_1$ then the condition ($1_{\beta'+1}$) implies $U_\beta \subset U_{\beta'}$ and

$V_\beta \subset V_{\beta'}$, and hence (1) is fulfilled. The condition (2) is an immediate consequence of the fact that (2_α) holds for all $\alpha < \omega_1$. Now, (3) holds because (3_α) is true for each $\alpha < \omega_1$. Finally, if $U \in \mathcal{T}(p, \omega^*)$ then $W_\alpha \subset U$ for some $\alpha < \omega_1$. By $(4_{\alpha+1})$ we have $U_\alpha \cap W_\alpha \neq \emptyset$ and therefore $\big(\bigcup\{U_\beta : \beta < \omega_1\}\big) \cap U \neq \emptyset$ whence $p \in \overline{\bigcup\{U_\beta : \beta < \omega_1\}}$. Analogously, $p \in \overline{\bigcup\{V_\beta : \beta < \omega_1\}}$, that is, the condition (4) is fulfilled. □

3.2 Example. *Under CH there exists a Čech-complete, countably compact, non-ultracomplete space.*

Proof: Let $A \subset \omega^*$ be a discrete and countable set. Then $\overline{A} - A$ is homeomorphic to ω^*, so we can apply Lemma 3.1 to find a $\boldsymbol{P}$-point p in the space $\overline{A} - A$ and families $\{U_\alpha : \alpha < \omega_1\}$, $\{V_\alpha : \alpha < \omega_1\}$ of clopen subsets of $\overline{A} - A$ which satisfy the conditions (1), (2), (3) and (4) of Lemma 3.1.

Take $X = \omega^* - (F \cup A)$ where $F = \big(\overline{A} - A\big) - \bigcup_{\alpha<\omega_1} V_\alpha$. It is clear that X is Čech-complete. Let us prove that X is a countably compact, non-ultracomplete space.

Choose a local base $\{W_\alpha : \alpha < \omega_1\} \subset \mathcal{T}_c^*(\overline{A} - A)$ at p for which, if $\alpha < \alpha'$ then $W_{\alpha'} \subset W_\alpha$. Suppose that X is ultracomplete. Since ω^* is a compactification of X and $\omega^* - X = F \cup A$, there exists a sequence $\{K_n : n \in \omega\}$ of compact subspaces of $F \cup A$ with $K_n \subset K_{n+1}$ for all $n \in \omega$, which witness the ultracompleteness of X, i.e., for any compact $K \subset F \cup A$ there exists an $n \in \omega$ such that $K \subset K_n$. Let $A_n = A \cap K_n$. Since $\overline{A_n} \subset A \cup F$, we have $A_n^* = \overline{A_n} - A_n \subset F$. Now, $p \notin A_n^*$ for all $n \in \omega$. In fact, as A_n^* is open in $\overline{A} - A$, if $p \in A_n^*$ by condition (4) of Lemma 3.1 we would have $A_n^* \cap \big(\bigcup_{\alpha<\omega_1} V_\alpha\big) \neq \emptyset$, which contradicts $A_n^* \subset F$. From the above it follows that given an $n \in \omega$, there exists $\alpha_n < \omega_1$ such that $W_{\alpha_n} \cap A_n^* = \emptyset$. Hence, if $\alpha > \alpha_n$ for every $n \in \omega$ then $W_\alpha \cap \big(\bigcup_{n<\omega} A_n^*\big) = \emptyset$. We choose a clopen set $O \subset W_\alpha \cap \big(\bigcup_{\alpha<\omega_1} U_\alpha\big)$. As $O = \overline{B} - B = B^*$ for some $B \subset A$, we obtain that $B \cup B^* \subset F \cup A$ is compact. There exists an $n \in \omega$ for which $B \cup B^* \subset K_n$ and therefore $B \subset K_n \cap A = A_n$. Thus $B^* \subset A_n^*$, which contradicts the choice of O.

To prove that X is countably compact, suppose that $D \subset X$ is a countably infinite closed discrete subspace of X. Then $\overline{D} - D \subset \omega^* - X = F \cup A$. Assume first that $(\overline{D} - D) \cap A \neq \emptyset$. If $a \in \overline{D} \cap A$ then take a clopen $U \subset \omega^*$ such that $U \cap A = \{a\}$ and hence $U \cap \overline{A} = \{a\}$. Since $a \in \overline{D}$ and $U \in \mathcal{T}(a, \omega^*)$ the set $\overline{D} \cap U$ has to be infinite. Therefore the closed set $\overline{D} \cap U$ has cardinality $2^{\mathfrak{c}}$ which implies $(\overline{D} \cap U) - (D \cup \{a\}) \neq \emptyset$. If $x \in (\overline{D} \cap U) - (D \cup \{a\})$ then $x \in (\overline{D} - D) - (F \cup A)$ which is a contradiction.

The reasoning above shows that $(\overline{D} - D) \cap A = \emptyset$ and hence $\overline{D} - D \subset F$. Assume first that $D' = D - \overline{A}$ is infinite. Then $D' \cap \overline{A} = \emptyset$ and $A \cap \overline{D'} = \emptyset$ because $\overline{D'} - D' \subset \overline{D} - D \subset F \subset \omega^* - A$. Since ω^* is an F-space, we have $\overline{D'} \cap \overline{A} = \emptyset$ which is a contradiction with the fact that the non-empty set $\overline{D'} - D'$ is a subset of $F \subset \overline{A}$. Thus $D - \overline{A}$ is finite and $E = D \cap \overline{A} = D \cap (\overline{A} - A) \subset X \cap (\overline{A} - A) = \bigcup_{\alpha<\omega_1} V_\alpha$.

The set E is infinite because $E = D - (D - \overline{A})$. The family $\{V_\alpha : \alpha < \omega_1\}$ is increasing and E is countable, so $E \subset V_\alpha$ for some $\alpha < \omega_1$. As a consequence, $\overline{E} - E \subset \overline{E} \subset V_\alpha$ which contradicts the fact that $\overline{E} - E \subset \overline{D} - D \subset F$ and F is disjoint from V_α. This last contradiction proves that X is countably compact.

□

4. Open problems

The following questions suggest a further development of the research undertaken in this paper. They may be difficult or easy, but some new methods are needed to tackle them.

4.1 Problem. *Suppose that $X^2 = X_1 \cup X_2$ where X_i is of countable type for $i = 1, 2$. Is the space X of countable type?*

4.2 Problem. *Suppose that $X^2 = X_1 \cup X_2$ where X_i is locally Čech-complete for $i = 1, 2$. Must X be locally Čech-complete?*

4.3 Problem. *Suppose that $X = X_1 \cup X_2$ where X is a stratifiable space and X_i is Čech-complete for $i = 1, 2$. Must X be Čech-complete?*

4.4 Problem. *Suppose that $X^2 = X_1 \cup X_2$ where X_i is ultracomplete for $i = 1, 2$. Must X be ultracomplete?*

4.5 Problem. *Suppose that $X^2 = X_1 \cup X_2$ where X_i is ultracomplete for $i = 1, 2$. Must X be Čech-complete?*

4.6 Problem. *Suppose that $X^2 = X_1 \cup X_2$ where X_i is locally ultracomplete for $i = 1, 2$. Must X be locally ultracomplete?*

4.7 Problem. *Let X be a pseudocompact ultracomplete space. Must X have points of local compactness?*

4.8 Problem. *Let X be a countably compact ultracomplete space. Must X have points of local compactness?*

4.9 Problem. *Let X be a countably compact space with $\beta X - X$ countable. Must X be ultracomplete?*

4.10 Problem. *Let X be a homogeneous ultracomplete space. Must X be locally compact?*

4.11 Problem. *Let X be a homogeneous countably compact Čech-complete space. Must X be ultracomplete?*

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