More on strongly sequential spaces

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Abstract. Strongly sequential spaces were introduced and studied to solve a problem of Tanaka concerning the product of sequential topologies. In this paper, further properties of strongly sequential spaces are investigated.

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Strongly sequential spaces were introduced in [10] in order to solve a problem of Tanaka [12] of characterizing topologies whose product with every metrizable topology is sequential. In this paper, we identify a sequence $(x_n)_\omega$ with the corresponding filter (generated by $\{x_n : n \geq k\}_{k\in\omega}$) and a decreasing sequence of subsets with the filter it generates. In this way, the definition of the adherence of a filter ([1])

$$\operatorname{adh}\mathcal{H} = \bigcup_{\mathcal{F}\#\mathcal{H}} \lim \mathcal{F},$$

applies to sequences and decreasing sequences of subsets. Let $\mathrm{cl}_{\mathrm{Seq}}$ denote the (idempotent) sequential closure ([2]) and let $\operatorname{adh}_{\mathrm{Seq}}\mathcal{H}$ be the union of limits of sequences $(x_n)_\omega$ that meshes with the filter $\mathcal{H}$.

A topology (more generally a convergence) is *strongly sequential* if

$$\operatorname{adh}\mathcal{H} \subset \mathrm{cl}_{\mathrm{Seq}}(\operatorname{adh}_{\mathrm{Seq}}\mathcal{H}),$$

I am deeply indebted to professor S. Dolecki whose observations and suggestions are not only at the origin of this note but also have importantly improved its content. I would also like to thank professor Y. Tanaka for many valuable comments (in [13] and [14]) about preliminary versions of the present paper and about [10].

[1] Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets *mesh*, in symbol $\mathcal{A}\#\mathcal{B}$, if $A\cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

[2] Let $\operatorname{adh}^0_{\mathrm{Seq}} A = A$ and let $\operatorname{adh}^1_{\mathrm{Seq}} A = \operatorname{adh}_{\mathrm{Seq}} A$ be the union of limits of sequences of A. If α is an ordinal number, let $\operatorname{adh}^\alpha_{\mathrm{Seq}} A = \operatorname{adh}_{\mathrm{Seq}}(\bigcup_{\beta<\alpha} \operatorname{adh}^\beta_{\mathrm{Seq}} A)$. For each subset A of X, there exists the least ordinal α for which $\operatorname{adh}^{\alpha+1}_{\mathrm{Seq}} A = \operatorname{adh}^\alpha_{\mathrm{Seq}} A$. This set is the *sequential closure* $\mathrm{cl}_{\mathrm{Seq}} A$ of A. The supremum of the above α's for every subset A is the *sequential order* of the topology (or convergence). The topology is *sequential* if the closure cl coincide with the sequential closure.

for every countably based filter $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_{\mathrm{ad}_T}$. The notation $\mathcal{H} = \mathcal{H}_{\mathrm{ad}_T}$ means that $\mathcal{H}$ has a filter-base consisting of sets that are unions of the closure of their points. Of course, this condition is always fulfilled in a T_1 (i.e., points are closed) convergence. In other words, a T_1 topology (convergence) is strongly sequential if whenever a decreasing sequence of subsets $(A_n)_\omega$ accumulates at x, the point x belongs to the sequential closure of the set of limit points of convergent sequences $(x_n)_n$ such that $x_n \in A_n$. One can also say that a T_1 topology (convergence) is strongly sequential if it is sequential and satisfies the following: if $(A_n)_{n\in\omega}$ is a decreasing sequence accumulating at x then $x \in \mathrm{cl}\{y : y_n \to y; y_n \in A_n\}$. In the sequel, *regular* means regular and T_1 (contrary to [10]), so that the above characterization applies.

Strongly sequential spaces play a role with respect to sequential spaces similar to that played by strongly Fréchet spaces with respect to Fréchet spaces (see [10]) ([3]).

A topology (convergence) in which $\mathrm{adh}\,\mathcal{H} \neq \emptyset$ implies that $\mathrm{adh}_{\mathrm{Seq}}\,\mathcal{H} \neq \emptyset$ for every countably based filter $\mathcal{H}$ is called *Tanaka space* ([4]). Obviously, every strongly sequential space is a Tanaka space. Proposition 1 below shows that the converse is true among regular sequential spaces. On the other hand, Y. Tanaka [13] asked if a regular sequential inner-one A space is strongly sequential. Recall that a topology is *inner-one A* [9] if $\mathrm{adh}(A_n)_\omega \neq \emptyset$ implies that there exists $x_n \in A_n$ such that $\{x_n : n \in \omega\}$ is not closed.

Proposition 1. *Let X be a regular sequential topology. The following are equivalent.*

1. *X is strongly sequential;*
2. *X is a Tanaka topology;*
3. *X is inner-one A.*

Y. Tanaka pointed out to me in [14] that he proved the equivalence between 2 and 3 in 1986.

PROOF: $1 \Longrightarrow 2 \Longrightarrow 3$ follows immediately from the definitions.

$3 \Longrightarrow 1$. Let $(H_n)_\omega$ fulfill $x \in \bigcap_n \mathrm{cl}\,H_n$. Then $x \in \bigcap_n \mathrm{cl}(H_n \cap W)$ for every closed neighborhood W of x. As X is inner-one A, there exist sequences $(x_n^W)_\omega$ such that $x_n^W \in H_n \cap W$ and $\{x_n^W : n \in \omega\}$ is not closed, hence not sequentially closed, because of sequentiality. Modulo a rearrangement of the terms, $(x_n^W)_\omega$ admits a subsequence $(x_{n_k}^W)_k$ that converges to a point $x_W \in W \cap \mathrm{adh}_{\mathrm{Seq}}(H_n)$. By regularity, $x \in \mathrm{cl}\{x_W : W = \mathrm{cl}\,W \in \mathcal{N}(x)\}$. By sequentiality, this closure is equal to the sequential closure, so that $x \in \mathrm{cl}_{\mathrm{Seq}}\,\mathrm{adh}_{\mathrm{Seq}}(H_n)_\omega$. □

[3] A topology is *Fréchet* if $\mathrm{cl} = \mathrm{adh}_{\mathrm{Seq}}$ and *strongly Fréchet* if whenever a decreasing sequence $(A_n)_{n\in\omega}$ accumulates at x, there exists a sequence $x_n \in A_n$ that converges to x.

[4] This property is called "property (C)" in [12].

In [12], Y. Tanaka proved, in the context of regular (T_1) topologies, that a topology whose product with every first-countable space is sequential is necessarily a Tanaka space. Under supplementary assumptions on X, he gave a characterization for the product $X \times Y$ of X with a first-countable space Y to be sequential. In view of [10, Theorem 5.1] (that gives a similar characterization in terms of strong sequentiality without these assumptions on X) and of Proposition 1, he could have dropped the supplementary assumptions on X in [12, Theorem 1.1]. By the way, these assumptions essentially reduce to the fact that X is Fréchet.

Proposition 2. *A regular Tanaka topology in which each point is G_δ, is strongly Fréchet.*

Notice that, although not stated independently, this result is shown along the lines of the proof of the main theorems of [12].

PROOF: Suppose that $x \in \operatorname{cl} A$. Let $(B_n)_\omega$ be a sequence of open sets such that $\bigcap_n B_n = \{x\}$. By regularity, there is a sequence $(F_n)_\omega$ of closed neighborhoods of x such that $F_n \subset B_n$ for each n. It follows that $x \in \operatorname{cl}(F_n \cap A)$, and, as X is a Tanaka topology, there exists a convergent sequence $x_n \in F_n \cap A$. On the other hand, $\lim(x_n)_n \subset \bigcap_n F_n \subset \bigcap_n B_n = \{x\}$, so that the topology is Fréchet. Moreover, a T_1 Tanaka Fréchet topology is strongly Fréchet (see [12] or [10]). □

Now, if we drop the assumption of regularity, Tanaka and strongly sequential topologies no longer coincide (among sequential spaces).

Example 3. [*A sequential Tanaka topology which is not strongly sequential.*]
Consider the free bisequence

$$x_{n,k} \underset{k}{\longrightarrow} x_n \underset{n}{\longrightarrow} x_\infty,$$

with its usual topology ([5]). Denote $Y_n = \{x_{n,k} : k \in \omega\}$ and consider a family $\mathcal{A}$ of subsets of $\{x_{n,k} : n, k \in \omega\}$ such that $\mathcal{A} \cup \{Y_n : n \in \omega\}$ is maximal almost disjoint (MAD) (see for example [6]). To the already convergent sequences of $Y = \mathcal{A} \cup \{x_{n,k} : n, k \in \omega\} \cup \{x_n : n \in \omega\} \cup \{x_\infty\}$, we add those generated by each $A \in \mathcal{A}$, each of which converges to the respective A seen as an element of Y. Endow Y with the finest topology for which the sequences above converge. This is obviously a sequential topology. It is moreover a Tanaka topology: since all the points but x_∞ are of countable character, it is enough to consider a decreasing sequence (H_p) that fulfills $x_\infty \in \bigcap_p \operatorname{cl} H_p$ and such that every H_p is included in $\{x_{n,k} : n, k \in \omega\}$. If $w_p \in H_p$, then by maximality of $\mathcal{A} \bigcup \{Y_n : n \in \omega\}$, there

[5] More precisely, $(x_n)_{n\in\omega}$ is a free sequence converging to x_∞ and for every n, $(x_{n,k})_{k\in\omega}$ is a free sequence converging to x_n. Sequences of the type $(x_{n,k})_{k\in\omega}$ are disjoint. All points $x_{n,k}$ are isolated, while a neighborhood basis of x_n is given by $\{\{x_n\} \cup \{x_{n,k} : k \geq p\} : p \in \omega\}$. Finally, a neighborhood basis for x_∞ is given by $\{\{x_\infty\} \cup \{x_n\} \cup \{x_{n,k} : k \geq m_n\} : p \in \omega,\ n \geq p, m_n \in \omega\}$.

is a subsequence of $(w_p)_\omega$ that converges to some $A \in \mathcal{A}$. On the other hand, Y is not strongly sequential because the filter $\mathcal{H}$ generated by $(\bigcup_{n \geq m} Y_n)_{m \in \omega}$ verifies $x_\infty \in \operatorname{adh} \mathcal{H}$ and $\operatorname{adh}_{\mathrm{Seq}} \mathcal{H} \subset \mathcal{A}$ which consists of isolated points, so that $x_\infty \notin \operatorname{cl}_{\mathrm{Seq}}(\operatorname{adh}_{\mathrm{Seq}} \mathcal{H})$.

The free bisequence

$$x_{n,k} \underset{k}{\longrightarrow} x_n \underset{n}{\longrightarrow} x_\infty,$$

with its usual topology is not a Tanaka space, hence not strongly sequential, contrary to my claim in [10, p. 150]. Indeed, the filter $\mathcal{H}$ generated by $\{x_{n,k} : n \geq m\}_{m \in \omega}$ fulfills $x_\infty \in \operatorname{adh} \mathcal{H}$ but $\operatorname{adh}_{\mathrm{Seq}} \mathcal{H} = \emptyset$ because no sequence of the type $(x_{n_m,k_m})_m$ with $n_m \geq m$ converges. Other examples of non Fréchet strongly sequential topologies can however be provided. Indeed, in view of [10, Theorem 3.1] (that states that a convergence is strongly sequential if and only if its product with every metrizable topology is sequential) and of the classical theorem [8, Theorem 4.2] of Michael that states that a regular sequential topology is locally countably compact if and only if its product with every sequential topology is sequential, we get

Proposition 4. *A regular sequential, locally countably compact topology* (*convergence*) *is strongly sequential.*

In particular, each MAD compact topology ([6]) is a regular sequential locally countably compact, hence strongly sequential, topology of sequential order 2 [4, Theorem 3.5], hence not a Fréchet space.

On the other hand, a locally relatively countably compact ([7]) sequential topology need not be strongly sequential, as shows Example 3. Indeed, we only need to find a relatively countably compact neighborhood for x_∞ and $Y \setminus \mathcal{A}$ is such.

Proposition 4 can actually be strengthened. Recall that a topology is q if every point has a sequence (Q_n) of neighborhoods such that $x_n \in Q_n$ implies $\operatorname{adh}(x_n) \neq \emptyset$. A slightly more general class of topologies is that of *bi-quasi-k* spaces. In such spaces, every adherent filter meshes with a countable family (Q_n) such that $x_n \in Q_n$ implies $\operatorname{adh}(x_n) \neq \emptyset$. If this property holds only for countably based filters, the space is called *countably bi-quasi-k*.

Proposition 5. *Every regular sequential countably bi-quasi-k topology* (*in particular a regular sequential q-topology or a regular sequential bi-k topology*) *is strongly sequential.*

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[6] that is, the Alexandroff compactification of $N \cup \mathcal{A}$ where $\mathcal{A}$ is a MAD family on a countable set N and where $N \cup \mathcal{A}$ is endowed with the topology in which the neighborhood filter of $A \in \mathcal{A}$ is generated by $\{W : \{A\} \in W,\ A \setminus W \text{ is finite.}\}$. This space has been called Alexandroff compactification of a Mrówka space, or a Franklin space, or an Isbell space or a ψ-space.

[7] i.e., each point has a relatively countably compact neighborhood, that is, a neighborhood on which each countably based filter has non-empty adherence (in the whole set).

Proof: Let $\mathcal{H}$ be a countably based filter (of decreasing base $(H_n)_\omega$) such that $x \in \operatorname{adh}\mathcal{H}$. As X is countably bi-quasi-k, there exists a sequence of sets (Q_n) such that $(Q_n)\#\mathcal{H}$ such that every sequence $x_n \in Q_n$ has non empty adherence. For every n choose $x_n \in H_n \cap Q_n$. The sequence (x_n) has non empty adherence, so that $\{x_n : n \in \omega\}$ is not closed, hence not sequentially closed, by sequentiality. Thus $(x_n)_\omega$ has a convergent subsequence so that $\operatorname{adh}_{\mathrm{Seq}}\mathcal{H} \neq \emptyset$. In view of Proposition 1, X is strongly sequential. $\square$

I thank the referee for having pointed out to me that Proposition 5 can also be deduced from Proposition 1 and [7, Lemma 9.1]. Proposition 5 answers positively a question of Y. Tanaka [13]: Are regular sequential countably bi-k spaces strongly sequential?

In view of Proposition 5, a non Fréchet strongly sequential topology need not be locally countably compact. Indeed, there exists a regular non Fréchet sequential q-topology which is not locally countably compact. For example, the product of a MAD-compact space and of a regular non locally countably compact first-countable space is a regular q-topology as a product of regular q-topologies which is sequential because the MAD-compact space is locally countably compact. Hence it is strongly sequential (of sequential order at least 2) and not locally countably compact (see also [3, Proposition 13]).

Strongly sequential spaces can be characterized in terms of their product properties: They are exactly the topologies whose product with every metrizable (or bisequential) topology is sequential (equivalently strongly sequential) [10, Theorem 3.1]. On the other hand, strong sequentiality appears in other results on product of sequential spaces, like [2, Theorem 12.1] and [11, Corollary 6.13]. This last example can be combined with Proposition 5 to the effect that

Theorem 6. *The product of a sequential regular bi-quasi-k topology with a strongly Fréchet topology is sequential.*

References

[1] Dolecki S., *Convergence-theoretic approach to quotient quest*, Topology Appl. **73** (1996), 1–21.

[2] Dolecki S., Mynard F., *Convergence-theoretic mechanism behind product theorems*, Topology Appl. **104** (2000), 67–99.

[3] Dolecki S., Nogura T., *Two-fold theorem on Fréchetness of products*, Czech. Math. J. **49** (124) (1999), 421–429.

[4] Dolecki S., Nogura T., *Countably infinite products of sequential topologies*, to appear.

[5] Engelking R., *Topology*, PWN, 1977.

[6] Handbook of Set-theoretic Topology, edited by K. Kunen and J.E. Vaughan, North-Holland Publ., 1984.

[7] Michael E., *A quintuple quotient quest*, Gen. Topology Appl. **2** (1972), 91–138.

[8] Michael E., *Local compactness and cartesian product of quotient maps and k-spaces*, Ann. Inst. Fourier (Grenoble) **19** (1968), 281–286.

[9] Michael E., Olson R.C., Siwiec D., *A-spaces and countably biquotient maps*, Dissertationes Math. **133** (1976).
[10] Mynard F., *Strongly sequential spaces*, Comment. Math. Univ. Carolinae **41.1** (2000), 143–153.
[11] Mynard F., *Coreflectively modified continuous duality applied to classical product theorems*, Applied General Topology **2** (2) (2002), 119–154.
[12] Tanaka Y., *Products of sequential spaces*, Proc. Amer. Math. Soc. **54** (1976), 371–375.
[13] Tanaka Y., *Private communication*, November 2000.
[14] Tanaka Y., *Private communication*, June 2001.

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