Remarks on the sobriety of Scott topology and weak topology on posets

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Abstract. We give some necessary and sufficient conditions for the Scott topology on a complete lattice to be sober, and a sufficient condition for the weak topology on a poset to be sober. These generalize the corresponding results in [1], [2] and [4].

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1. Preliminaries

Let X be a T_0 space. Then there is an induced partial order defined by setting $x \leq y$ if and only if $x \in \mathrm{cl}\{y\}$. Conversely, any partial order on X can be deduced in this way. In fact, if $(L, \leq)$ is a partially ordered set (briefly poset), we define the Alexandroff topology $A(L)$ to be the collection of all upper sets in L (i.e. sets U satisfying $x \in U$ and $x \leq y$ implies $y \in U$), and the weak topology $W(L)$ to be the smallest topology for which all sets of the form $\downarrow x$ are closed. A topology on L is said to be compatible if it induces the given partial order. It is well known that a topology Ω on L is compatible if and only if

$$W(L) \subset \Omega \subset A(L).$$

Let L, M be two posets and $f : L \to M$ an isotone map. Then $f : (L, A(L)) \to (M, A(M))$ is continuous. If we do not distinguish $(L, A(L))$ and $A(L)$, then A is a functor from the category $POSET$ of posets and isotone maps to the category $T_0\,TOP$ of T_0 topological spaces and continuous maps.

Lemma 1. *The assignment $P : X \mapsto (X, \leq)$ defines a functor from the category $T_0\,TOP$ to the category $POSET$ (where $\leq$ is the induced partial order) which is a right adjoint to the functor A.*

Proof: It suffices to show that any continuous map $f : A(L) \to X$ factors uniquely through $i : A(P(X)) \to X$ by an isotone map $\bar{f} : L \to X$ for a T_0

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topological space X and a poset L. But this is clear since f preserves order and $\bar{f} = f$.

We call a T_0 space X an Alexandroff space if its topology coincides with the Alexandroff topology for the induced partial order. It is easy to show that X is a Alexandroff space if and only if its topology is closed under arbitrary meets if and only if each point of X has a smallest open neighborhood. $\square$

Proposition 1. *The category ATOP of Alexandroff topological spaces and continuous maps is isomorphic to the category POSET.*

2. Main results

Let L be a poset. It is well known that if Ω is a sober topology on L inducing the given order then $W(L) \subset \Omega \subset \sigma(L)$, where $\sigma(L)$ is the Scott topology on L. In [3], J. Isbell showed that there is a complete lattice for which the Scott topology on it is not sober. In [4], it was shown that if L is a complete lattice such that $\sigma(L)$ is a continuous lattice then the Scott topology on L is sober. In [1], J. Isbell showed that a T_0 topological complete lattice is sober. We give some necessary and sufficient conditions for the Scott topology on a complete lattice to be sober.

Let X be a T_0 space. We call X a weakly Scott topological space if its topology is contained in the Scott topology and X is a complete lattice for the induced partial order. Every complete lattice endowed with the weak topology is a weakly Scott topological space. If $x \in X$, a class of open sets Ψ of X is said to be a prime open neighborhood basis of x if for any prime open neighborhood P of x there is a $Q \in \Psi$ such that $x \in Q \subset P$. A map $f : X \to Y$ is said to be primal continuous if for any prime open set P of Y, $f^{-1}(P)$ is an open subset of X.

Proposition 2. *Let X be a weakly Scott topological space. The following conditions are equivalent:*

(a) *X is sober;*
(b) *for each $x, y \in X$ and $z = x \vee y$, the set $\Psi_z = \{P \cap Q \mid P$ is a prime open neighborhood of x, Q is a prime open neighborhood of $y\}$ is a prime open neighborhood basis of z;*
(c) *for every set I, the I-indexed supremum map $\sup : X^I \to X$ is primal continuous;*
(d) *the supremum map $\sup : X \times X \to X$ is primal continuous.*

Proof: (a) $\Rightarrow$ (b): If X is sober, then any prime open set has the form $X \setminus \{t\}^- = X \setminus (\downarrow t)$, so if $z = x \vee y \in U$ for some prime open set U, we may assume $x \neq \perp$, $y \neq \perp$, where $\perp$ is the least element. Then we have either $x \in U$ or $y \in U$. Assuming $x \in U$, then U is a prime open neighborhood of x and $X \setminus \{\perp\}$ is a prime open neighborhood of y, $U \cap (X \setminus \{\perp\}) \subset U$.

(b) $\Rightarrow$ (c): Let P be a prime open set of X. If $\bigvee_{i \in I} x_i \in P$, there exist finitely many members $x_{i_1}, \dots, x_{i_n}$, such that $x_{i_1} \vee \cdots \vee x_{i_n} \in P$ since P is open in the Scott topology. By (b), we have prime open sets $P_1, \dots, P_n$ with $x_{i_k} \in P_k$, $k = 1, \dots, n$, such that $P_1 \cap \dots \cap P_n \subset P$, i.e. $P_1 \vee \dots \vee P_n \subset P$, so $\prod_{i \in I} \bar{P}_i$ is an open neighborhood of (x_i) and $\bigvee \bar{P}_i \subset P$, where $\bar{P}_j = P_j$ for $j = 1, \dots, n$, $\bar{P}_i = X_i$ otherwise.

(c) $\Rightarrow$ (d): Clear.

(d) $\Rightarrow$ (a): Let A be an irreducible closed set of X. If A is directed, then $\sup A \in A$, $A = \downarrow \sup A$. So we need only to show that A is directed.

Let $a, b \in A$. If $a \vee b \in X \setminus A$ then by (d), we have open sets U, V with $a \in U$, $b \in V$, and $U \vee V \subset X \setminus A$, i.e. $U \cap V \subset X \setminus A$. Thus $U \subset X \setminus A$ or $V \subset X \setminus A$. This shows $a \in X \setminus A$ or $b \in X \setminus A$, a contradiction. So $a \vee b \in A$, A is directed.

□

Corollary 1. *Let L be a complete lattice. Then the following conditions are equivalent:*

(a) *the Scott topology on L is sober;*

(b) *for any $a, b \in L$, $a \vee b = c$, the set $\Psi_c = \{P \vee Q \mid P$ is a prime open neighborhood of a, Q is a prime open neighborhood of $b\}$ is a prime open neighborhood basis of c;*

(c) *for each set I, the I-indexed supremum map $\sup : L^I \to L$ is primal continuous;*

(d) *the supremum map $\sup : L \times L \to L$ is primal continuous.*

Let X be a T_0 topological space. We call X a primal topological complete sup-semi-lattice if X is a complete lattice for its induced partial order and the supremum map $\sup : X^I \to X$ is primal continuous for any indexed set I.

Lemma 2. *Every primal topological complete sup-semi-lattice is sober.*

Proof: Let X be a primal topological complete sup-semi-lattice, A an irreducible closed set of X, $\sup A = a$. If $a \in X \setminus A$, then $\sup^{-1}(X \setminus A)$ is an open neighborhood of $(x)_{x \in A}$ by the primal continuity of supremum map $X^A \to X$, thus there are finitely many members $a_1, \dots, a_n$ of A and open sets $U_1, \dots, U_n$ with $x_i \in U_i$, $i = 1, \dots, n$, such that $U_1 \times \dots \times U_n \times X^{\{x \mid x \in A,\, x \neq a_i,\, i=1,\dots,n\}} \subset \sup^{-1}(X \setminus A)$, so $U_1 \cap \dots \cap U_n = U_1 \vee \dots \vee U_n \subset X \setminus A$, i.e. $A \subset (X \setminus U_1) \cup \dots \cup (X \setminus U_n)$. There must be a U_i such that $A \subset X \setminus U_i$. Then $a_i \in A$ but $a_i \notin U_i$, a contradiction.

□

Let L be a poset. It is well known that if there is a compatible sober topology on L, then L is directed complete. In view of Lemma 2, we have the following result.

Proposition 3. *Let L be a lattice with a compatible topology. Then L is a sober topological space if and only if L is a primal topological complete sup-semi-lattice.*

In the end of this note, we give a sufficient condition for the weak topology on a poset to be sober. This generalizes the corresponding results in [2]. In [5], P.T. Johnstone showed that there is no compatible sober topology on a directed complete poset. In [2], R.-E. Hoffmann showed that the weak topology is sober for a complete lattice.

Let L be a poset. We call L a weakly complete poset if $\forall A \subset L$, $A \neq \emptyset$, there are finite many members $s_1, \ldots, s_n$ of L such that $\bigcap\{\downarrow a \mid a \in A\} = \downarrow s_1 \cup \cdots \cup \downarrow s_n$. A poset with nonempty meets is a weakly complete poset, especially every complete lattice is weakly complete, but the converse is not true.

Example 1. Let $L = \{a, b, c, d, e\}$. The partial order on L is defined by $a \leq a$, $b \leq b, c \leq a, b, c, d \leq a, b, d, e \leq a, b, c, d, e$. Then L is a weakly complete poset, but $a \wedge b$ does not exist.

Proposition 4. *Let L be a weakly complete poset. Then $(L, W(L))$ is sober.*

Proof: Let A be an irreducible closed set of $(L, W(L))$. A can be expressed as $A = \bigcap\{\downarrow s_1 \cup \cdots \cup \downarrow s_{n_s} \mid s \in S, n_s \in \mathbb{Z}\}$. If there is a $p \in S$ with

$$\bigcap_{s \neq p}\{\downarrow s_1 \cup \cdots \cup \downarrow s_{n_s}\} \cap \downarrow p_j \neq A, \quad j = 1, \ldots, n_p,$$

then

$$A = (\bigcap_{s \neq p}\{\downarrow s_1 \cup \cdots \cup \downarrow s_{n_s}\} \cap \downarrow p_1) \cup \cdots \cup (\bigcap_{s \neq p}\{\downarrow s_1 \cup \cdots \cup \downarrow s_{n_s}\} \cap \downarrow p_{n_p}),$$

contradicting the irreducibility of A. So for each $p \in S$, there is a $p_{j_p} \in L$ such that $(\bigcap_{s \neq p}\{\downarrow s_1 \cup \cdots \cup \downarrow s_{n_s}\}) \cap \downarrow p_{j_p} = A$. Then we have

$$A \subset \bigcap_{p \in S} \downarrow p_{j_p} \subset \bigcap\{\downarrow s_1 \cup \cdots \cup \downarrow s_{n_s} \mid n_s \in \mathbb{Z}, s \in S\} = A,$$

so $A = \bigcap_{p \in S} \downarrow p_{j_p}$. If L is weakly complete, then there are finite many members $a_1, \ldots, a_n$ such that $A = \downarrow a_1 \cup \cdots \cup \downarrow a_n$. But A is irreducible, so there must be an a_i, $1 \leq i \leq n$, such that $A = \downarrow a_i$. $\square$

The weak completeness is not necessary for sobriety of posets.

Example 2. Let $A = \coprod_{i \in \mathbb{Z}} 2_i$ be the disjoint union of copies of two-element sets $2 = \{0, 1\}$ and let $B = \mathbb{Z}$ be the set of natural numbers. Let $L = A \cup B \cup \{\perp\}$ be partially ordered by

$$x \leq y \ \text{ if and only if either } \ x \in B,\ y \in \coprod_{i \geq x} 2_i, \ \text{ or } \ x = y, \ \text{ or } \ x = \perp.$$

Then it is not difficult to show that L is a directed complete poset, the weak topology and Scott topology on L are both sober, but L is not weakly complete.

Question. Characterize those posets such that the weak topology on them is sober.

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