Comment.Math.Univ.Carolin. 43,4 (2002)755–771

Lattice points in some special three-dimensional convex bodies with points of Gaussian curvature zero at the boundary

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Abstract. We investigate the number of lattice points in special three-dimensional convex bodies. They are called convex bodies of pseudo revolution, because we have in one special case a body of revolution and in another case even a super sphere. These bodies have lines at the boundary, where all points have Gaussian curvature zero. We consider the influence of these points to the lattice rest in the asymptotic representation of the number of lattice points.

Keywords: convex bodies, lattice points, points with Gaussian curvature zero

Classification: 11P21, 11H06

1. Introduction and statement of result

Let F denote the distance function of the convex body PR_3. That is

$$F(t_1,t_2,t_3) = \left\{(|t_1|^\kappa + |t_2|^\kappa)^{\frac{k}{\kappa}} + |t_3|^k\right\}^{\frac{1}{k}},$$
$$\mathrm{PR}_3 = \{(t_1,t_2,t_3) \in \mathbb{R}^3 : F(t_1,t_2,t_3) \le 1\}.$$

It is assumed that $\kappa, k \in \mathbb{N}$, $2 \le \kappa \le k$, $k > 3$, κ a divisor of k. Then we have a body of revolution for $\kappa = 2$ and a super sphere for $\kappa = k$. Therefore, we call PR_3 a body of pseudo revolution in general.

We consider the points (t_1,t_2,t_3) at the boundary and we are confined to the points $t_1,t_2,t_3 \ge 0$ without loss of generality. We put

$$F(t_1,t_2,t_3) = 1, \quad t_3 = f(t_1,t_2),$$

where f is given by

$$f(t_1,t_2) = \left(1 - (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}}\right)^{\frac{1}{k}}.$$

The Gaussian curvature in such a point is defined by

$$K = \frac{H(f(t_1,t_2))}{(1 + f_{t_1}^2(t_1,t_2) + f_{t_2}^2(t_1,t_2))^2}$$

where $H(f(t_1,t_2))$ denotes the HESSIAN

$$H(f(t_1,t_2)) = f_{t_1t_1}(t_1,t_2)f_{t_2t_2}(t_1,t_2) - f^2_{t_1t_2}(t_1,t_2).$$

In the present case we find by means of long but simple calculations

$$K = \frac{(k-1)(\kappa-1)(t_1t_2)^{\kappa-2}(t_1^\kappa+t_2^\kappa)^{2\frac{k}{\kappa}-2}t_3^{k-2}}{(t_3^{2k-2}+(t_1^{2\kappa-2}+t_2^{2\kappa-2})^{2\frac{k}{\kappa}-2})^2}.$$

From this it is seen that we have the following points with GAUSSIAN curvature $K=0$ at the boundary:
(1) Body of revolution ($\kappa=2$): The curve

$$t_1^2+t_2^2=1,\ t_3=0$$

and the isolated points $(t_1,t_2,t_3)=(0,0,\pm1)$.
(2) Super sphere ($\kappa=k$): The curves

$$t_2^k+t_3^k=1,\ t_1=0;\quad t_1^k+t_3^k=1,\ t_2=0;\quad t_1^k+t_2^k=1,\ t_3=0.$$

(3) Body of pseudo revolution ($2<\kappa<k$): The curves

$$t_2^k+t_3^k=1,\ t_1=0;\quad t_1^k+t_3^k=1,\ t_2=0;\quad t_1^\kappa+t_2^\kappa=1,\ t_3=0.$$

The flat points $(t_1,t_2,t_3)=(0,0,\pm1)$ are of exceptional importance in all three cases and the points $(t_1,t_2,t_3)=(\pm1,0,0)$, $(t_1,t_2,t_3)=(0,\pm1,0)$ are meaningful as well in the cases (2) and (3).

The aim of the paper is to investigate the number of lattice points in the dilated body of pseudo revolution $x\,\mathrm{PR}_3$, that is:

$$A_{k,\kappa}(x;\mathrm{PR}_3) = \#\left\{(n_1,n_2,n_3)\in\mathbb{Z}^3 : (|n_1|^\kappa+|n_2|^\kappa)^{\frac{k}{\kappa}}+|n_3|^k\le x\right\}. \tag{1}$$

Especially we study the influence of the points with GAUSSIAN curvature zero to the asymptotic representation of $A_{k,\kappa}(x;\mathrm{PR}_3)$.

In [3] a detailed description is given for the case of super spheres. See also the paper [5]. Therefore, we are here in the first place interested for the case $\kappa<k$, but we do not exclude the case $\kappa=k$.

It is not too hard to obtain the following asymptotic representation for $A_k(x;\mathrm{PR}_3)$ from the results of the paper [7]:

$$A_{k,\kappa}(x;\mathrm{PR}_3) = \mathrm{vol}(\mathrm{PR}_3)x^3+H_{k,\kappa,1}(x)+H_{k,\kappa,2}(x)+O\left(x^{\frac{5}{3}-\frac{2}{3k}}\right)+O\left(x^{\frac{3}{2}}\log^3 x\right).$$

The second main term $H_{k,\kappa,1}(x)$ is a certain function of x coming from the flat points $(t_1, t_2, t_3) = (0, 0, \pm 1)$ and can be estimated by

$$H_{k,\kappa,1}(x) \ll x^{2-\frac{2}{k}}.$$

Analogously, the third main term $H_{k,\kappa,2}(x)$ is a certain function of x coming from the flat points $(t_1, t_2, t_3) = (\pm 1, 0, 0), (0, \pm 1, 0)$ and can be estimated by

$$H_{k,\kappa,2}(x) \ll x^{2-\frac{1}{\kappa}-\frac{1}{k}}.$$

The first error term results from the other points with Gaussian curvature zero and the second error term results from the points with Gaussian curvature non-zero.

In this paper we will give explicit representations of the second and third main terms which automatically show that the above upper bounds are at the same time lower bounds. Further we give an improved estimation of the first error term.

Let the generalized Bessel functions $J_\nu^{(k)}(x)$ be defined by

$$J_\nu^{(k)}(x) = \frac{2}{\sqrt{\pi}\Gamma(\nu + 1 - \frac{1}{k})} \left(\frac{x}{2}\right)^{\frac{k\nu}{2}} \int_0^1 \left(1 - t^k\right)^{\nu - \frac{1}{k}} \cos xt \, dt, \tag{2}$$

where Γ is the gamma function, $k,\ \nu$ are real numbers with $k \geq 1$, $\nu > \frac{1}{k}$. Further let

$$\psi_\nu^{(k)}(x) = 2\sqrt{\pi}\Gamma\left(\nu + 1 - \frac{1}{k}\right) \sum_{n=1}^{\infty} \left(\frac{x}{\pi n}\right)^{\frac{k\nu}{2}} J_\nu^{(k)}(2\pi n x), \tag{3}$$

which is absolutely convergent for $\nu > \frac{1}{k}$. For a proof see [3].

Theorem 1. *Let $\kappa, k \in \mathbb{N}$, $2 \leq \kappa \leq k$, $k > 3$, κ a divisor of k. Then*

$$A_{k,\kappa}(x; \mathrm{PR}_3) = \mathrm{vol}(\mathrm{PR}_3) x^3 + H_{k,\kappa,1}(x) + H_{k,\kappa,2}(x) + \Delta_{k,\kappa}(x) \tag{4}$$

with

$$H_{k,\kappa,1}(x) = \frac{2\Gamma^2(\frac{1}{\kappa})}{\kappa\Gamma(\frac{2}{\kappa})} \psi_{3/k}^{(k)}(x) = O,\ \Omega\left(x^{2-\frac{2}{k}}\right), \tag{5}$$

$$H_{k,\kappa,2}(x) = 8x \int_0^1 t^k \left(1 - t^k\right)^{\frac{1}{k}-1} \psi_{2/\kappa}^{(\kappa)}(xt) \, dt = O,\ \Omega\left(x^{2-\frac{1}{\kappa}-\frac{1}{k}}\right), \tag{6}$$

$$\Delta_{k,\kappa}(x) \ll x^{\frac{119}{73} - \frac{165}{146k}} (\log x)^{\frac{315}{146}} + x^{\frac{3}{2}} \log^3 x. \tag{7}$$

2. Preparation of the problem

We find, by symmetry,

$$A_{k,\kappa}(x;\mathrm{PR}_3) = 16(S_{1,2,3} + S_{1,3,2} + S_{3,1,2}) + O(x),$$

where $S_{i,j,k}$ are triple sums

$$S_{i,j,k} = \sum_{n_1}\sum_{n_2}\sum_{n_3} 1$$

with the summation conditions

$$SC(S_{i,j,k}) : 0 \le n_i \le n_j \le n_k, \quad (n_1^\kappa + n_2^\kappa)^{\frac{k}{\kappa}} + n_3^k \le x^k,$$
$$n_i = 0,\ n_i = n_j,\ n_j = n_k \quad \text{get a factor} \quad \frac{1}{2}\,.$$

We begin the summation process in each sum with n_i.

For example, summing in $S_{1,2,3}$ over n_1, we obtain

$$\left[\left(\left(x^k - n_3^k\right)^{\frac{\kappa}{k}} - n_2^\kappa\right)^{\frac{1}{\kappa}}\right] + \frac{1}{2} \quad \text{for} \quad \left(\left(x^k - n_3^k\right)^{\frac{\kappa}{k}} - n_2^\kappa\right)^{\frac{1}{\kappa}} < n_2$$

and n_2 otherwise. If we use $[y]+\frac{1}{2} = y-\psi(y)$, we get a term which can be written as an integral, and a term with the ψ-function as a remainder. Hence

$$16S_{1,2,3} = S_{1,2,3}^{(1)} + \Delta_{2,3}(x) + O(x).$$

Here

$$S_{1,2,3}^{(1)} = 16\sum_{n_2}\sum_{n_3}\int_{t_1} dt_1$$

with the summation-integration conditions

$$SIC\left(\sum_{n_2}\sum_{n_3}\int_{t_1}\right) : 0 \le t_1 \le n_2 \le n_3, \quad (t_1^\kappa + n_2^\kappa)^{\frac{k}{\kappa}} + n_3^k \le x^k,$$
$$n_2 = n_3 \quad \text{gets a factor} \quad \frac{1}{2}$$

and

$$\Delta_{2,3}(x) = -16\sum_{(n_1,n_2)\in D_{2,3}} \psi\left(\left(\left(x^k - n_3^k\right)^{\frac{\kappa}{k}} - n_2^\kappa\right)^{\frac{1}{\kappa}}\right), \tag{8}$$

where $D_{2,3}$ denotes the domain

$$D_{2,3} = \left\{ (t_2, t_3) \in \mathbb{R}^2 : 0 \le \left(x^k - t_3^k \right)^{\frac{\kappa}{k}} - t_2^\kappa < t_2^\kappa \le t_3^\kappa \right\}.$$

In the next step we sum over n_2 in $S_{1,2,3}^{(1)}$ and obtain similarly

$$16 S_{1,2,3} = S_{1,2,3}^{(2)} + P_{2,3}(x) + \Delta_{2,3}(x) + O(x)$$

with

$$S_{1,2,3}^{(2)} = 16 \sum_{n_3} \int_{t_1} \int_{t_2} dt_1\, dt_2,$$

$$SIC \left(\sum_{n_3} \int_{t_1} \int_{t_2} \right) : 0 \le t_1 \le t_2 \le n_3, \quad (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}} + n_3^k \le x^k$$

and

$$P_{2,3}(x) = -16 \sum_{n_3} \int_{t_1} \psi \left(\left(\left(x^k - n_3^k \right)^{\frac{\kappa}{k}} - t_1^\kappa \right)^{\frac{1}{\kappa}} \right) dt_1 \tag{9}$$

$$SIC \left(\sum_{n_3} \int_{t_1} \right) : \ 0 \le t_1 \le n_3,$$

$$\left(x^k - n_3^k \right)^{\frac{\kappa}{k}} - n_3^\kappa < t_1^\kappa \le \frac{1}{2} \left(x^k - n_3^k \right)^{\frac{\kappa}{k}}.$$

In the last step we sum over n_3 in $S_{1,2,3}^{(2)}$ and finally we obtain

$$16 S_{1,2,3} = S_{1,2,3}^{(3)} + H_{2,3}(x) + P_{2,3}(x) + \Delta_{2,3}(x) + O(x), \tag{10}$$

where

$$S_{1,2,3}^{(3)} = 16 \int_{t_1} \int_{t_2} \int_{t_3} dt_1\, dt_2\, dt_3, \tag{11}$$

$$IC \left(\int_{t_1} \int_{t_2} \int_{t_3} \right) : 0 \le t_1 \le t_2 \le t_3, \quad (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}} + t_3^k \le x^k,$$

$$H_{2,3}(x) = -16 \int_{t_1} \int_{t_2} \psi \left(\left(x^k - (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}} \right)^{\frac{1}{k}} \right) dt_1\, dt_2, \tag{12}$$

$$IC \left(\int_{t_1} \int_{t_2} \right) = 0 \le t_1^k \le t_2^k \le x^k - (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}}.$$

In the same way we obtain for the other triple sums:

$$16S_{1,3,2} = S^{(3)}_{1,3,2} + H_{3,2}(x) + P_{3,2}(x) + \Delta_{3,2}(x) + O(x), \tag{13}$$

where

$$S^{(3)}_{1,3,2} = 16 \int_{t_1} \int_{t_2} \int_{t_3} dt_1\, dt_2\, dt_3, \tag{14}$$
$$IC\left(\int_{t_1} \int_{t_2} \int_{t_3}\right) :\ 0 \le t_1 \le t_3 \le t_2, \quad (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}} + t_3^k \le x^k,$$

$$H_{3,2}(x) = -16 \int_{t_1} \int_{t_3} \psi\left(\left(\left(x^k - t_3^k\right)^{\frac{\kappa}{k}} - t_1^\kappa\right)^{\frac{1}{\kappa}}\right) dt_1\, dt_3, \tag{15}$$
$$IC\left(\int_{t_1} \int_{t_3}\right) :\ 0 \le t_1^\kappa \le t_3^\kappa \le \left(x^k - t_3^k\right)^{\frac{\kappa}{k}} - t_1^\kappa,$$

$$P_{3,2}(x) = -16 \sum_{n_2} \int_{t_1} \psi\left(\left(x^k - (t_1^\kappa + n_2^\kappa)^{\frac{k}{\kappa}}\right)^{\frac{1}{k}}\right) dt_1, \tag{16}$$
$$SIC\left(\sum_{n_2} \int_{t_1}\right) :\ 0 \le t_1^k \le x^k - (t_1^\kappa + n_2^\kappa)^{\frac{k}{\kappa}} < n_2^k,$$

$$\Delta_{3,2}(x) = -16 \sum_{(n_3,n_2)\in D_{3,2}} \psi\left(\left(\left(x^k - n_3^k\right)^{\frac{\kappa}{k}} - n_2^\kappa\right)^{\frac{1}{\kappa}}\right), \tag{17}$$
$$D_{3,2} = \left\{(t_3, t_2) \in \mathbb{R}^2 : 0 \le \left(x^k - t_3^k\right)^{\frac{\kappa}{k}} - t_2^\kappa < t_3^\kappa \le t_2^\kappa\right\}.$$

Finally, we obtain for $S_{3,1,2}$

$$16S_{3,1,2} = S^{(3)}_{3,1,2} + H_{1,2}(x) + P_{1,2}(x) + \Delta_{1,2}(x) + O(x), \tag{18}$$

where

$$S^{(3)}_{3,1,2} = 16 \int_{t_1} \int_{t_2} \int_{t_3} dt_1\, dt_2\, dt_3, \tag{19}$$
$$IC\left(\int_{t_1} \int_{t_2} \int_{t_3}\right) :\ 0 \le t_3 \le t_1 \le t_2, \quad (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}} + t_3^k \le x^k,$$

$$H_{1,2}(x) = -16 \int_{t_3} \int_{t_1} \psi \left(\left(\left(x^k - t_3^k \right)^{\frac{\kappa}{k}} - t_1^\kappa \right)^{\frac{1}{\kappa}} \right) dt_1 \, dt_3, \tag{20}$$

$$IC \left(\int_{t_3} \int_{t_1} \right) : \; 0 \le t_3^\kappa \le t_1^\kappa \le \frac{1}{2} \left(x^k - t_3^k \right)^{\frac{\kappa}{k}},$$

$$P_{1,2}(x) = -16 \sum_{n_2} \int_{t_3} \psi \left(\left(\left(x^k - t_3^k \right)^{\frac{\kappa}{k}} - n_2^\kappa \right)^{\frac{1}{\kappa}} \right) dt_3, \tag{21}$$

$$SIC \left(\sum_{n_2} \int_{t_3} \right) : \; \frac{1}{2} \left(x^k - t_3^k \right)^{\frac{\kappa}{k}} < n_2^\kappa \le \left(x^k - t_3^k \right)^{\frac{\kappa}{k}} - t_3^\kappa,$$

$$\Delta_{1,2}(x) = -16 \sum_{(n_1,n_2) \in D_{1,2}} \psi \left(\left(x^k - (n_1^\kappa + n_2^\kappa)^{\frac{k}{\kappa}} \right)^{\frac{1}{k}} \right), \tag{22}$$

$$D_{1,2} = \left\{ (t_1, t_2) \in \mathbb{R}^2 : 0 \le x^k - (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}} < t_1^k \le t_2^k \right\}.$$

3. Representation of the triple integrals

It is clear that it follows from (11), (14) and (19)

$$S^{(3)}_{1,2,3} + S^{(3)}_{1,3,2} + S^{(3)}_{3,1,2} = \mathrm{vol}(\mathrm{PR}_3) x^3,$$

which is the main term in (4).

4. Representation and estimation of the double integrals

We begin with $H_{2,3}(x)$. We write the integration condition in (12) in the form

$$0 \le t_1 \le t_2, \quad t_1^\kappa + t_2^\kappa \le \left(x^k - t_2^k \right)^{\frac{\kappa}{k}}.$$

We have

$$\left(x^k - t_2^k \right)^{\frac{\kappa}{k}} \ge x^\kappa 2^{-\frac{\kappa}{k}}.$$

We integrate only up to $x^\kappa \frac{1}{2}$. The remainder is of order x. Hence, by substituting $t_1 \to t_1 x$, $t_2 \to t_2 x$, we obtain

$$H_{2,3}(x) = -16 x^2 \int_{t_1} \int_{t_2} \psi \left(x \left(1 - (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}} \right)^{\frac{1}{k}} \right) dt_1 \, dt_2 + O(x),$$

$$IC \left(\int_{t_1} \int_{t_2} \right) : \; 0 \le t_1 \le t_2, \quad t_1^\kappa + t_2^\kappa \le \frac{1}{2}.$$

Putting $t_1^\kappa + t_2^\kappa = z^\kappa$ we get, by symmetry,

$$\begin{aligned} H_{2,3}(x) &= -8x^2 \int_{z^\kappa \le \frac{1}{2}} z^{\kappa-1} \psi\left(x\left(1-z^k\right)^{\frac{1}{k}}\right) dz \int_0^z (z^\kappa - t_1^\kappa)^{\frac{1}{\kappa}-1}\, dt_1 + O(x) \\ &= -\frac{8\Gamma^2(\frac{1}{\kappa})x^2}{\kappa\Gamma(\frac{2}{\kappa})} \int_{z^\kappa \le \frac{1}{2}} z\psi\left(x\left(1-z^k\right)^{\frac{1}{k}}\right) dz + O(x) \\ &= -\frac{8\Gamma^2(\frac{1}{\kappa})x^2}{\kappa\Gamma(\frac{2}{\kappa})} \int_0^1 z^{k-1}\left(1-z^k\right)^{\frac{2}{k}-1} \psi(xz)\, dz + O(x), \end{aligned}$$

where again the integral from 0 up to $(1-z^{-k/\kappa})^{1/k}$ is of order $\frac{1}{x}$. By means of the Fourier representation of the ψ-function,

$$\psi(t) = -\frac{1}{\pi}\sum_{n=1}^\infty \frac{1}{n}\sin(2\pi n t)\, dt, \tag{23}$$

we obtain

$$\begin{aligned} H_{2,3}(x) &= \frac{8\Gamma^2(\frac{1}{\kappa})x^2}{\pi\kappa\Gamma(\frac{2}{\kappa})} \sum_{n=1}^\infty \frac{1}{n}\int_0^1 z^{k-1}\left(1-z^k\right)^{\frac{2}{k}-1} \sin(2\pi nxz)\, dz + O(x) \\ &= \frac{8\Gamma^2(\frac{1}{\kappa})x^3}{\kappa\Gamma(\frac{2}{\kappa})} \sum_{n=1}^\infty \int_0^1 \left(1-z^k\right)^{\frac{2}{k}} \cos(2\pi nxz)\, dz + O(x) \end{aligned}$$

and, by (2) and (3),

$$\begin{aligned} H_{2,3}(x) &= \frac{4\sqrt{\pi}\Gamma^2(\frac{1}{\kappa})\Gamma(\frac{2}{\kappa}+1)x^3}{\kappa\Gamma(\frac{2}{\kappa})} \sum_{n=1}^\infty (\pi n x)^{-\frac{3}{2}} J_{3/k}^{(k)}(2\pi n x) + O(x) \\ &= \frac{2\Gamma^2(\frac{1}{\kappa})}{\kappa\Gamma(\frac{2}{\kappa})}\psi_{3/k}^{(k)}(x) + O(x). \end{aligned}$$

Hence

$$H_{2,3}(x) = H_{k,\kappa,1}(x) + O(x)$$

and the representation (5) is obtained. The asymptotic representation of the generalized Bessel functions is given in Lemma 3.11 of [3]. We obtain

$$J_{3/k}^{(k)}(2\pi n x) = \sqrt{nx}\left\{\left(\frac{k}{2\pi n x}\right)^{\frac{2}{k}} \cos\left(2\pi n x - \frac{\pi}{2}\left(\frac{2}{k}+1\right)\right) + O\left(\frac{1}{nx}\right)\right\}.$$

Hence, it follows with a positive constant c

$$H_{k,\kappa,1}(x) = cx^{2-\frac{2}{k}} \sum_{n=1}^{\infty} n^{-1-\frac{2}{k}} \sin\left(2\pi nx - \frac{\pi}{k}\right) + O(x).$$

Thus, the estimations in (5) are clear.

In order to obtain the term $H_{k,\kappa,2}(x)$ we add $H_{3,2}(X)$ and $H_{1,2}(x)$. Then we get from (15) and (20)

$$H_{3,2}(x) + H_{1,2}(x) = -16x^2 \int_{t_1}\int_{t_3} \psi\left(x\left(\left(1-t_3^k\right)^{\frac{\kappa}{k}} - t_1^\kappa\right)^{\frac{1}{\kappa}}\right) dt_1\, dt_3,$$
$$IC\left(\int_{t_1}\int_{t_3}\right):\ 0 \le t_1^\kappa \le \frac{1}{2}\left(1-t_3^k\right)^{\frac{\kappa}{k}}, \quad 0 \le t_3^\kappa \le \left(1-t_3^k\right)^{\frac{\kappa}{k}} - t_1^\kappa.$$

By means of the substitution $t_1^\kappa = (1-t_3^k)^{\kappa/k} - z^\kappa$ we obtain

$$H_{3,2}(x) + H_{1,2}(x) = 16x^2 \int_z\int_{t_3} z^{\kappa-1}\left(\left(1-t_3^k\right)^{\frac{\kappa}{k}} - z^\kappa\right)^{\frac{1}{\kappa}-1} \psi(xz)\, dt_3\, dz,$$
$$IC\left(\int_z\int_{t_3}\right):\ \frac{1}{2}\left(1-t_3^k\right)^{\frac{\kappa}{k}} \le z^k,\ t_3^k + z^k \le 1,\ 0 \le t_3 \le z.$$

We may extend the domain of integration such that the new integration conditions are given by

$$IC\left(\int_z\int_{t_3}\right): t_3^k + z^k \le 1,\ 0 \le t_3^k \le \frac{1}{2}.$$

The integral over the new domain is of order $\frac{1}{x}$ which can be seen by partial integrating with respect to z. Now we substitute $1 - t_3^k = t^k$. Then

$$\begin{aligned} &H_{3,2}(x) + H_{1,2}(x) = \\ &= -16x^2 \int_{2^{-1/k}}^1 dt \int_0^t t^{k-1}\left(1-t^k\right)^{\frac{1}{k}-1} z^{\kappa-1}\left(t^\kappa - z^\kappa\right)^{\frac{1}{\kappa}-1} \psi(xz)\, dz + O(x). \end{aligned}$$

The substitution $z \to zt$ gives

$$\begin{aligned} &H_{3,2}(x) + H_{1,2}(x) = \\ &\quad = -16x^2 \int_{2^{-1/k}}^1 t^k\left(1-t^k\right)^{\frac{1}{k}-1} dt \int_0^1 z^{\kappa-1}\left(1-z^\kappa\right)^{\frac{1}{\kappa}-1} \psi(xtz)\, dz + O(x). \end{aligned}$$

We take the integral with respect to t from 0 up to 1, since the new part of the integral is of order $\frac{1}{x}$. As in case of $H_{2,3}(x)$ we use the FOURIER representation (23) of the ψ-function and we obtain analogously

$$\begin{aligned} &H_{3,2}(x) + H_{1,2}(x) = \\ &= \frac{16}{\sqrt{\pi}}\Gamma\left(\frac{1}{k}+1\right)x^2\int_0^1 t^{k+1}\left(1-t^k\right)^{\frac{1}{k}-1}\sum_{n=1}^{\infty}\frac{1}{n}J_{2/\kappa}^{(\kappa)}(2\pi nxt)\,dt + O(x) \\ &= 8x\int_0^1 t^k\left(1-t^k\right)^{\frac{1}{k}-1}\psi_{2/\kappa}^{(\kappa)}(xt)\,dt + O(x). \end{aligned}$$

Hence

$$H_{3,2}(x) + H_{1,2}(x) = H_{k,\kappa,2}(x) + O(x)$$

and the representation (6) is obtained. For the asymptotic representation we use (24). Clearly, the integral from 0 up to $\frac{1}{2}$ is of order $\frac{1}{x}$. Therefore, we use the asymptotic representation of the generalized BESSEL function from Lemma 3.11 in [3] for $t \geq \frac{1}{2}$. Then

$$J_{2/\kappa}^{(\kappa)}(2\pi nxt) = \frac{1}{\sqrt{\pi}}\left(\frac{\kappa}{2\pi nxt}\right)^{\frac{1}{\kappa}}\cos\left(2\pi nxt - \frac{\pi}{2}\left(\frac{1}{\kappa}+1\right)\right) + O\left(\frac{1}{nx}\right).$$

Hence, it follows with a positive constant a

$$\begin{aligned} &H_{k,\kappa,2}(x) = \\ &= ax^{2-\frac{1}{\kappa}}\sum_{n=1}^{\infty} n^{-1-\frac{1}{\kappa}}\int_{\frac{1}{2}}^1 t^{k+1-\frac{1}{\kappa}}\left(1-t^k\right)^{\frac{1}{k}-1}\sin\left(2\pi nxt - \frac{\pi}{2\kappa}\right)\,dt + O(x). \end{aligned}$$

The remaining integral has a singularity at $t = 1$. We obtain the asymptotic representation of the integral very easy by means of Chapter 3, Section 11 from [1]. We use a special case of formula (11.6) on page 24: *Let $\phi(t)$ be continuously differentiable in $\alpha \leq t \leq \beta$. Let $\phi(\alpha) = 0$. Then, if* $0 < \mu < 1$,

$$\int_\alpha^\beta e^{ixt}(\beta - t)^{\mu-1}\phi(t)\,dt = \frac{\Gamma(\mu)}{x^\mu}e^{ix\beta - \frac{1}{2}\mu\pi i}\phi(\beta) + O\left(\frac{1}{x}\right).$$

The condition $\phi(\alpha) = 0$ is not necessary. In case of $\phi(\alpha) \neq 0$ the point α yields an error term of order $1/x$. Thus, with a constant $b \neq 0$, we get

$$H_{k,\kappa,2}(x) = bx^{2-\frac{1}{\kappa}-\frac{1}{k}}\sum_{n=1}^{\infty} n^{-1-\frac{1}{\kappa}-\frac{1}{k}}\sin\left(2\pi nx - \frac{\pi}{2}\left(\frac{1}{\kappa}-\frac{1}{k}\right)\right) + O(x).$$

Hence, the estimations (6) follow immediately.

5. Estimations of sums and integrals

The aim of this chapter is to estimate the sums and integrals (9), (16) and (21).

Lemma 1. *Assume that for all z with $\frac{1}{2} \le z \le 1$ and for all τ with $\frac{x^k}{2} \le \tau^k \le x^k$*

$$\sum_{\frac{x^k}{2} < n^k \le \tau^k} \psi\left(\left(x^k - n^k\right)^{\frac{1}{k}} z\right) \ll \Delta_{k,2,3}(x).$$

Then

$$P_{2,3}(x) \ll x^{1-\frac{1}{2\kappa}} \left(\Delta_{k,2,3}(x)\right)^{1-\frac{1}{\kappa}}. \tag{25}$$

Proof: It is easily seen that in (9) the condition $t_1 \le n_3$ is superfluous and that

$$\left(x^k - n_3^k\right)^{\frac{\kappa}{k}} - n_3^\kappa > 0 \quad \Longleftrightarrow \quad n_3^k > \frac{x^k}{2}.$$

Further, consider

$$-16 \sum \int \psi\left(\left(\left(x^k - n_3^k\right)^{\frac{\kappa}{k}} - t_1^\kappa\right)^{\frac{1}{\kappa}}\right) dt_1,$$

$$SIC\left(\sum \int\right):\ 0 \le t_1^\kappa \le \left(x^k - n_3^k\right)^{\frac{\kappa}{k}} - n_3^\kappa,$$

$$\frac{x^k}{2^{k/\kappa}+1} < n_3^k \le \frac{x^k}{2}.$$

It is seen at once that this term is of order x. Now we bring (9) and this term together and obtain

$$P_{2,3}(x) = -16 \sum \int \psi\left(\left(\left(x^k - n_3^k\right)^{\frac{\kappa}{k}} - t_1^\kappa\right)^{\frac{1}{\kappa}}\right) dt_1,$$

$$SIC\left(\sum \int\right):\ 0 \le t_1^\kappa \le \frac{1}{2}\left(x^k - n_3^k\right)^{\frac{\kappa}{k}},$$

$$\frac{x^k}{2^{k/\kappa}+1} < n_3^k \le \frac{x^k}{2}.$$

Now we put $t_1 = (x^k - n_3^k)^{1/k} t$. Then

$$\begin{aligned} P_{2,k}(x) &= \\ &= -16 \int_0^{2^{-1/\kappa}} \sum_{\frac{x^k}{2^{k/\kappa}+1} < n_3^k \le \frac{x^k}{2}} \left(x^k - n_3^k\right)^{\frac{1}{k}} \psi\left(\left(x^k - n_3^k\right)^{\frac{1}{k}} (1 - t^\kappa)^{\frac{1}{\kappa}}\right) dt + O(x) \\ &= -16(I_1 + I_2) + O(x). \end{aligned}$$

Applying partial summation we obtain by means of the condition of the lemma

$$I_1 = \int_0^y \sum_{\frac{x^k}{2^{k/\kappa}+1} < n_3^k \le \frac{x^k}{2}} \left(x^k - n_3^k\right)^{\frac{1}{k}} \psi\left(\left(x^k - n_3^k\right)^{\frac{1}{k}} (1-t^\kappa)^{\frac{1}{\kappa}}\right) dt$$
$$\ll xy\Delta_{k,2,3}(x).$$

In I_2, where the integral is taken from y up to $2^{-1/\kappa}$, we use the FOURIER expansion of the ψ-function (23) and obtain

$$I_2 = \frac{1}{\pi} \sum_{\frac{x^k}{2^{k/\kappa}+1} < n_3^k \le \frac{x^k}{2}} \left(x^k - n_3^k\right)^{\frac{1}{k}} \sum_{m=1}^{\infty} \frac{1}{m} \int_y^{2^{-1/\kappa}} t^{1-\kappa} (1-t^\kappa)^{1-\frac{1}{\kappa}}$$
$$\times\, t^{\kappa-1} (1-t^\kappa)^{\frac{1}{\kappa}-1} \sin\left(2\pi m \left(x^k - n_3^k\right)^{\frac{1}{k}} (1-t^\kappa)^{\frac{1}{\kappa}}\right) dt$$
$$= \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{\frac{x^k}{2^{k/\kappa}+1} < n_3^k \le \frac{x^k}{2}}$$
$$\left\{\left[t^{1-\kappa} (1-t^\kappa)^{1-\frac{1}{\kappa}} \cos\left(2\pi m \left(x^k - n_3^k\right)^{1-\frac{1}{k}} (1-t^\kappa)^{\frac{1}{\kappa}}\right)\right]_y^{2^{-1/k}} -\right.$$
$$\left. - \int_y^{2^{-1/\kappa}} \frac{d}{dt}\left(t^{1-\kappa} (1-t^\kappa)^{1-\frac{1}{\kappa}}\right) \cos\left(2\pi m \left(x^k - n_3^k\right)^{1-\frac{1}{k}} (1-t^\kappa)^{\frac{1}{\kappa}}\right) dt\right\}.$$

We estimate the sum over n_3 with VAN DER CORPUT'S simplest theorem (see Theorem 2.1 in [3]). Then we get exactly in the same way as on page 183 of [3]

$$I_2 \ll x^{\frac{1}{2}} y^{1-\kappa}.$$

Now we put

$$y = \left(x^{\frac{1}{2}} \Delta_{k,2,3}(x)\right)^{-\frac{1}{\kappa}}.$$

Then the estimations of I_1 and I_2 are equal and we obtain (25). □

Lemma 2. *Assume that for all z with $1 \le z \le 2$ and for all u with $\frac{x^k}{1+z^k} < u^k \le x^k$*

$$\sum_{\frac{x^k}{1+z^k} < n^k \le u^k} \psi\left(\left(x^k - n^k z^k\right)^{\frac{1}{k}}\right) \ll \Delta_{k,3,2}(x).$$

Then

$$P_{3,2}(x) \ll x^{1-\frac{1}{2\kappa}} \left(\Delta_{k,3,2}(x)\right)^{1-\frac{1}{\kappa}}. \tag{26}$$

Proof: The proof is quite the same as the proof to Lemma 1 such that we omit it. □

Lemma 3. *Assume that*

$$\sum_{\frac{t^\kappa}{2} < n^\kappa \le t^\kappa} \psi\left((t^\kappa - n^\kappa)^{\frac{1}{\kappa}}\right) \ll \Delta_{\kappa,1,2}(x).$$

Then

$$P_{1,2}(x) \ll x^{1-\frac{1}{2k}} \left(\Delta_{\kappa,1,2}(x)\right)^{1-\frac{1}{k}}. \tag{27}$$

Proof: Consider

$$-16 \int_0^x \sum_{(x^k - t_3^k)^{\kappa/k} - t_3^\kappa < n_2^\kappa \le (x^k - t_3^k)^{\kappa/k}} \psi\left(\left(\left(x^k - t_3^k\right)^{\frac{\kappa}{k}} - n_2^\kappa\right)^{\frac{1}{\kappa}}\right) dt_3.$$

It is easily seen that this term is of order x. Now we bring (21) and this term together and obtain

$$P_{1,2}(x) =$$
$$= -16 \int_0^x \sum_{\frac{1}{2}(x^k - t_3^k)^{\kappa/k} < n_2^\kappa \le (x^k - t_3^k)^{\kappa/k}} \psi\left(\left(\left(x^k - t_3^k\right)^{\frac{\kappa}{k}} - n_2^\kappa\right)^{\frac{1}{\kappa}}\right) dt_3 + O(x).$$

Putting $x^k - t_3^k = t^k$ then

$$P_{1,2}(x) = -16 \int_0^x t^{k-1} \left(x^k - t^k\right)^{\frac{1}{k}-1} \sum_{\frac{t^\kappa}{2} < n_2^\kappa \le t^\kappa} \psi\left((t^\kappa - n_2^\kappa)^{\frac{1}{\kappa}}\right) dt + O(x).$$

Here we have the same situation as in Lemma 4.8 of [3]. Using this result (27) follows immediately.

Estimations of the error terms $\Delta_{k,2,3}(x)$, $\Delta_{k,3,2}(x)$, $\Delta_{\kappa,1,2}(x)$: G. Kuba [8] has pointed out that M.N. Huxley's [2] method is applicable to the above sums. Assume that a, b, c, d are fixed positive real numbers. Then he proved

$$\sum_{(\frac{X}{2b})^{1/k} - d < n \le (\frac{X - ac^k}{b})^{1/k} - d} \psi\left(\left(\frac{X - b(n+d)^k}{a}\right)^{\frac{1}{k}}\right) \ll$$
$$\ll \left(\frac{X}{\sqrt{ab}}\right)^{\frac{46}{73k}} \left(\log\left(\frac{X^2}{ab}\right)^{\frac{1}{k}}\right)^{\frac{315}{146}}.$$

But the proof shows essentially more: All the three cases in the lemmas are included. Hence

$$P_{2,3}(x),\ P_{3,2}(x) \ll x^{\frac{119}{73}-\frac{165}{146\kappa}}(\log x)^{\frac{315}{146}},$$
$$P_{1,2}(x) \ll x^{\frac{119}{73}-\frac{165}{146k}}(\log x)^{\frac{315}{146}}.$$

This gives the first term in the estimation (7). □

6. Estimations of the double sums

In order to estimate the sums (8), (17) and (22) we apply Theorem 3 in [4] or, what is the same, Satz 4.4 in [6]. For this purpose the domains $D_{2,3}$, $D_{3,2}$ and $D_{1,2}$ must be divided into some subdomains. The technical realization is worst of all in case of $D_{1,2}$. The both other cases are somewhat simpler and the calculations are in principle the same, but easier. Therefore, we consider only the estimation of $\Delta_{1,2}(x)$, which is given by (22).

Now let

$$f(t_1,t_2) = -\left(x^k - (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}}\right)^{\frac{1}{k}}$$

such that

$$f_{t_1}(t_1,t_2) = t_1^{\kappa-1}\,(t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}-1}\left(x^k - (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}}\right)^{\frac{1}{k}-1}.$$

Let ν_1, ν_2 be non-negative integers. Then we consider the following subdomains $D_{1,2}(\nu_1,\nu_2)$ of $D_{1,2}$:

$$\begin{aligned} D_{1,2}(\nu_1,\nu_2) = \Big\{(t_1,t_2)\in\mathbb{R}^2:\ &0 \le x^k - (t_1^\kappa + t_2^\kappa)^{\frac{k}{\kappa}} \le t_1^k \le t_2^k,\\ &2^{\nu_1(1-\frac{1}{k})+\frac{k}{\kappa}-1} \le f_{t_1} \le 2^{(\nu_1+1)(1-\frac{1}{k})+\frac{k}{\kappa}-1},\\ &\frac{2^{\nu_2\frac{k}{k-1}}x^k}{2^{\frac{k}{\kappa}}+2^{\nu_2\frac{k}{k-1}}} \le t_2^k \le \frac{2^{(\nu_2+1)\frac{k}{k-1}}x^k}{2^{\frac{k}{\kappa}}+2^{(\nu_2+1)\frac{k}{k-1}}}\Big\}. \end{aligned}$$

It is easily seen that

$$\frac{x^k - t_2^k}{3^{\frac{k}{\kappa}}+1} \le t_1^k \le x^k - t_2^k$$

and from this

$$t_1 \asymp 2^{-\frac{\nu_2}{k-1}}x.$$

If $a_2 \le t_2 \le b_2$ and $c_2 = b_2 - a_2$ then

$$c_2 \ll 2^{-\nu_2\frac{k}{k-1}}x.$$

If $\alpha_1 \leq f_{t_1} \leq \beta_1$ and $\gamma_1 = \alpha_1 - \beta_1$ then

$$\gamma_1 \asymp 2^{\nu_1(1-\frac{1}{k})}.$$

We obtain for the second partial derivative

$$\begin{aligned} f_{t_1t_1} = \ & t_1^{\kappa-2}\left(t_1^\kappa + t_2^\kappa\right)^{\frac{k}{\kappa}-2}\left(x^k - \left(t_1^\kappa + t_2^\kappa\right)^{\frac{k}{\kappa}}\right)^{\frac{1}{k}-2} \cdot \\ & \cdot \left\{(k-1)t_1^\kappa x^k + (\kappa-1)t_2^\kappa\left(x^k - \left(t_1^\kappa + t_2^\kappa\right)^{\frac{k}{\kappa}}\right)\right\} \end{aligned}$$

and for the Hessian

$$\begin{aligned} H(f) &= f_{t_1t_1}f_{t_2t_2} - f_{t_1t_2}^2 \\ &= (t_1t_2)^{\kappa-2}\left(t_1^\kappa + t_2^\kappa\right)^{\frac{2k}{\kappa}-4}\left(x^k - \left(t_1^\kappa + t_2^\kappa\right)^{\frac{k}{\kappa}}\right)^{\frac{2}{k}-4} \\ &\quad \times \left\{\left[(k-1)t_1^\kappa x^k + (\kappa-1)t_2^\kappa\left(x^k - \left(t_1^\kappa + t_2^\kappa\right)^{\frac{k}{\kappa}}\right)\right]\right. \\ &\quad \times \left[(k-1)t_2^\kappa x^k + (\kappa-1)t_1^\kappa\left(x^k - \left(t_1^\kappa + t_2^\kappa\right)^{\frac{k}{\kappa}}\right)\right] - \\ &\quad \left. -(t_1t_2)^\kappa\left[(k-1)\left(t_1^\kappa + t_2^\kappa\right)^{\frac{k}{\kappa}} + (k-\kappa)\left(x^k - \left(t_1^\kappa + t_2^\kappa\right)^{\frac{k}{\kappa}}\right)\right]^2\right\}. \end{aligned}$$

From this we get after some calculations

$$\begin{aligned} f_{t_1t_1} &\asymp \lambda_{11} = 2^{\nu_1(2-\frac{1}{k})+\frac{\nu_2}{k-1}}\frac{1}{x}, \\ H(f) &\asymp \Lambda = 2^{2\nu_1(2-\frac{1}{k})+\frac{\nu_2\kappa}{k-1}}\frac{1}{x^2}. \end{aligned}$$

Now we apply Theorem 3 from [4] or Satz 4.4 from [6]. The following assumptions of that theorem are certainly satisfied in the domain $D = D_{1,2}(\nu_1, \nu_2)$:
(A) *Let D be a compact domain defined by*

$$D = \{(t_1, t_2) : a_1 \leq \sigma(t_2) \leq t_1 \leq \varrho(t_2) \leq b_1,\ a_2 \leq t_2 \leq b_2\},$$

with $c_1 = b_1 - a_1 > 1$, $c_2 = b_2 - a_2 > 1$, where c_1, c_2 are so small as possible. Assume that $\sigma(t)$, $\varrho(t)$ are partly monotonic and two times differentiable in $[a_2, b_2]$.
(B) *Let $f(t_1, t_2)$ be a real-valued function in D with continuous partial derivatives up to the third order.*
(C) *Let*

$$|f_{t_1t_1}| \asymp \lambda_{11}, \quad H(f) = f_{t_1t_1}f_{t_2t_2} - f_{t_1t_2}^2, \quad |H(f)| \asymp \Lambda.$$

(D) *Suppose that* $\alpha_1 \leq f_{t_1} \leq \beta_1$, $\gamma_1 = \beta_1 - \alpha_1$.
(E) *Let the function* $(\varphi(y, t_2), t_2)$ *be defined by*

$$f_{t_1}(\varphi(y, t_2), t_2) = y.$$

Let $\eta(t) = \varrho(t)$, $\sigma(t)$, $\varphi(y, t)$. *Then suppose that the functions*

$$\eta''(t),\ f_{t_1 t_1}(\eta(t), t)\eta'(t) + f_{t_1 t_2}(\eta(t), t),$$
$$f_{t_1 t_1}(\eta(t), t) f_{t_1}(\eta(t), t)\eta''(t)$$

are partly monotonic. Further, let $\varphi_y(y, t_2)$ *be partly monotonic with respect to* t_2. *Then*

$$\sum_{(n_1,n_2)\in D} \psi(f(n_1, n_2)) \ll \frac{\gamma_1}{\lambda_{11}} \left(\Lambda^{\frac{1}{4}} + \lambda^{\frac{1}{2}}\right) c_2 +$$
$$+ \left\{ \left(\frac{\sqrt{\Lambda}}{\lambda_{11}} + 1 \right) c_2 + (\gamma_1 + 1) \left(\frac{1}{\sqrt{\Lambda}} + \frac{\lambda_{11}}{\Lambda} \right) \right\} (|\log \Lambda| + |\log \lambda_{11}| + 1)^2.$$

Now we use this result for our problem and we obtain

$$\Delta_{1,2}(x) = -16 \sum_{(n_1,n_2)\in D_{1,2}(\nu_1,\nu_2)} \psi\left(\left(x^k - (n_1^\kappa + n_2^\kappa)^{\frac{k}{\kappa}}\right)^{\frac{1}{k}}\right) \ll$$
$$\ll 2^{-\frac{\nu_1}{2k} - \frac{\nu_2}{k-1}(k+1-\frac{\kappa}{4})} x^{\frac{3}{2}} + 2^{-\frac{\nu_2}{k-1}(k+1-\frac{\kappa}{2})} x (\log \nu_1 + \log \nu_2 + \log x)^2.$$

We may sum over ν_1 such that $2^{\nu_1} \leq \sqrt{x}$, because the trivial estimation of the remainder gives an error term of order $x^{3/2}$. Then

$$\Delta_{1,2}(x) \ll x^{\frac{3}{2}} \log^3 x.$$

Analogously we obtain the same estimation for $\Delta_{2,3}(x)$ and $\Delta_{3,2}(x)$. This gives the second term in (7).

References

[1] Copson E.T., *Asymptotic Expansions*, At the University Press, Cambridge, 1965.
[2] Huxley M.N., *Area, Lattice Points and Exponential Sums*, Clarendon Press, Oxford, 1996.
[3] Krätzel E., *Lattice Points*, Dt. Verlag d. Wiss., Berlin and Kluwer Academic Publishers, Dordrecht/Boston/London, 1988.
[4] Krätzel E., *Double exponential sums*, Analysis **16** (1996), 109–123.
[5] Krätzel E., *Lattice points in super spheres*, Comment. Math. Univ. Carolinae **40.2** (1999), 373–391.

[6] Krätzel E., *Analytische Funktionen in der Zahlentheorie*, B.G. Teubner, Stuttgart-Leipzig-Wiesbaden, 2000.
[7] Krätzel E., *Lattice points in three-dimensional convex bodies with points of Gaussian curvature zero at the boundary*, Monatsh. Math., 2001.
[8] Kuba G., *On sums of two k-th powers of numbers in residue classes II.*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 87–95.

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(*Received* January 11, 2002, *revised* July 1, 2002)