On a construction of weak solutions to non-stationary Stokes type equations by minimizing variational functionals and their regularity

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Abstract. In this paper, we prove that the regularity property, in the sense of Gehring-Giaquinta-Modica, holds for weak solutions to non-stationary Stokes type equations. For the construction of solutions, Rothe's scheme is adopted by way of introducing variational functionals and of making use of their minimizers. Local estimates are carried out for time-discrete approximate solutions to achieve the higher integrability. These estimates for gradients do not depend on approximation.

Keywords: non-stationary Stokes type equations, higher integrability of gradients, Caccioppoli type estimate, Gehring theory, Rothe's scheme

Classification: 35Q30, 76D05, 35J50, 39A12

1. Introduction

There has been studied the higher integrability, in the sense of Gehring-Giaquinta-Modica ([1], [2], [3], [5] and [15]), for the gradients of weak solutions to elliptic and parabolic partial differential equations and minimizers of variational functionals.

This paper is motivated by the paper [4] due to Giaquinta and Modica, which has successfully studied the higher integrability for the gradients of weak solutions to stationary Navier-Stokes equations with bounded and measurable coefficients in the second order differential terms. They called these equations *stationary Navier-Stokes type equations*.

The main objective of this paper is to establish the regularity theory for weak solutions to *non-stationary Stokes type equations*. Based on this paper, we will discuss this regularity results for weak solutions to *non-stationary Navier-Stokes type equations* in the forthcoming paper (Kawabi [10]).

Here we remark that this type non-stationary problem has been studied by many authors. Especially Kaplický-Málek-Stará [9] constructed weak solutions of Stokes type equations and gave global estimates under quite similar assumptions to ours. However their approach is different from ours and we obtain local estimates for weak solutions. Hence it seems that our result is not included in [9].

Research partially supported by JSPS Research Fellowships for Young Scientists.

Now we introduce our approach. To construct weak solutions for partial differential equations with time-variable, the method of semi-discretization in time variable, so-called Rothe's method has been used since about 70 years ago. Our method is based on the concept of the discrete Morse semiflows, which was first proposed by Rektorys [16] and which was rediscovered by Kikuchi [11]. This method is Rothe's method taking into considerations the variational structure, in other words it is the semi-discretization in time variable of gradient flows (Morse flows) of some functionals. Recently Nagasawa [14] constructed weak solutions for non-stationary Navier-Stokes equations using this method. But he did not study their regularity property. On the other hand, Kikuchi [12] studied the higher integrability of the gradients of time-discrete approximate solutions for parabolic systems corresponding to a certain variational functional. The obtained estimates are independent of approximation. It enables him to construct Morse flows with the higher integrability of the gradients. In this paper, we adopt this argument.

First we give some notations and formulate the problem. For simplicity, we assume the external force term $f = 0$. Let Ω be a bounded domain in $\mathbb{R}^m$, $m \geq 2$, with Lipschitz boundary $\partial\Omega$ and T a positive real number. We deal with non-stationary Stokes type equations with initial-boundary conditions:

$$(1.1) \qquad \begin{cases} \partial_t u^i = \nabla_\alpha(A_{ij}^{\alpha\beta}(x)\nabla_\beta u^j) + \nabla_i p, & \text{in } (0,T)\times\Omega, \\ \nabla\cdot u = 0, & \text{in } (0,T)\times\Omega, \\ u = u_0, & \text{on } (0,T)\times\partial\Omega, \\ u = u_0, & \text{on } \{0\}\times\Omega, \end{cases}$$

where $u = (u^1, \dots, u^m)$ is a velocity, p is a pressure, $i, j, \alpha, \beta = 1, \dots, m$, $x = (x^1, \dots, x^m)$, $\partial_t = \partial/\partial t$, $\nabla_\alpha = \partial/\partial x_\alpha$, $\nabla\cdot u = \operatorname{div} u$. Here and hereafter, the summation convention is used, Greek and Latin letters running from 1 to m.

The coefficients $\{A_{ij}^{\alpha\beta}(x)\}$ are assumed to satisfy the following conditions:

(**A1**) $A_{ij}^{\alpha\beta}(x) : \Omega \to \mathbb{R}$ are bounded and measurable in Ω, i.e.,
$|A_{ij}^{\alpha\beta}(x)| \leq L$ for almost every $x \in \Omega$.

(**A2**) $A_{ij}^{\alpha\beta}(x)$ are symmetric, i.e.,
$A_{ij}^{\alpha\beta}(x) = A_{ji}^{\beta\alpha}(x)$ for almost every $x \in \Omega$.

(**A3**) $A_{ij}^{\alpha\beta}(x)$ satisfy the ellipticity condition
$A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \geq \lambda|\xi|^2$ for any $\xi = (\xi_\alpha^i) \in \mathbb{R}^{m^2}$ and almost every $x \in \Omega$ with a uniform constant $\lambda > 0$.

In the case of $A_{ij}^{\alpha\beta}(x) = \nu\delta_{\alpha\beta}\delta_{ij}$, the equations (1.1) are the usual non-stationary Stokes equations. Here δ_{ij} is the Kronecker delta. We emphasize the fact that Rothe's method is well adapted to such kind of problems with non-smooth coefficients.

We denote by $W^{k,p}(\Omega, \mathbb{R}^m)$ the usual Sobolev space. In this paper we write $H^k(\Omega, \mathbb{R}^m) := W^{k,2}(\Omega, \mathbb{R}^m)$ for simplicity. We also denote by $H_0^1(\Omega, \mathbb{R}^m)$ the closure of $C_0^\infty(\Omega, \mathbb{R}^m)$ in $H^1(\Omega, \mathbb{R}^m)$, which is equipped with the scalar product

$$(u, v)_{(1)} := \int_\Omega (\nabla u(x), \nabla v(x))\, dx = \int_\Omega \nabla_\alpha u^i(x) \nabla_\alpha v^i(x)\, dx.$$

The space $V(\Omega)$ denotes the closure of the space

$$C_{0,\sigma}^\infty(\Omega) := \{u \in C_0^\infty(\Omega, \mathbb{R}^m) \mid \nabla \cdot u = 0\}$$

in the space $H_0^1(\Omega, \mathbb{R}^m)$.

Let u_0 be a map from $H_\sigma^1(\Omega)$, the closure of the space

$$C_\sigma^\infty(\Omega) := \{u \in C^\infty(\Omega, \mathbb{R}^m) \mid \nabla \cdot u = 0\}$$

in $H^1(\Omega, \mathbb{R}^m)$, which plays a role of the initial-boundary data. We use the set of functions

$$V_{u_0}(\Omega) := \left\{u \in H^1(\Omega, \mathbb{R}^m) \mid u - u_0 \in V(\Omega)\right\}.$$

Abridged notations of bilinear forms

$$A(x)(\nabla u, \nabla v) = A_{ij}^{\alpha\beta}(x) \nabla_\alpha u^i \nabla_\beta u^j$$
$$A(x)(\nabla \eta u, \nabla \eta v) = A_{ij}^{\alpha\beta}(x) \nabla_\alpha \eta \nabla_\beta \eta \cdot u^i u^j,$$

will be adopted for a scalar-valued function η, which may arouse no confusion of the understanding.

A weak solution to the problem (1.1) is defined as a mapping

$$u \in L^\infty(0, T; V_{u_0}(\Omega)) \cap H^1(0, T; L^2(\Omega, \mathbb{R}^m))$$

such that

$$\lim_{t \searrow 0} u(t) = u_0 \quad \text{strongly in } \ L^2(\Omega, \mathbb{R}^m)$$

and for any $\Phi(x)$ in $V(\Omega)$,

$$\int_\Omega \{(\partial_t u(t, x), \Phi(x)) + A(x)(\nabla u(t, x), \nabla \Phi(x))\}\, dx = 0$$

holds for almost every $t \in (0, T)$.

To construct such a weak solution, we use Rothe's method for parabolic differential equations. Take a positive integer N such that $N > T$ and put $h = \frac{T}{N}$ and $t_n = nh$, $n = 0, 1, \dots, N$.

An approximate solution to the problem (1.1) is, by definition, an $V_{u_0}(\Omega)$-valued function $u_h(t)$, $-h < t \le T$ constructed by

$$u_h(t) := \begin{cases} u_n, & \text{for } \ t_{n-1} < t \le t_n, \ n = 1, \dots, N, \\ u_0, & \text{for } \ -h < t \le 0, \end{cases}$$

where $\{u_n\}_{n=1}^N \subset V_{u_0}(\Omega)$ is a family of functions such that

$$\int_\Omega \left\{ \Big(\frac{u_n(x) - u_{n-1}(x)}{h}, \Phi(x) \Big) + A(x)(\nabla u_n(x), \nabla \Phi(x)) \right\} dx = 0 \tag{1.2}$$

for any $\Phi \in V(\Omega)$ and $n = 1, 2, \dots, N$. We note that $\{u_n\}_{n=0}^N$ is a weak solution in $H^1(\Omega, \mathbb{R}^m)$ to difference partial differential systems of elliptic-parabolic type:

$$\begin{cases} \frac{u_n^i - u_{n-1}^i}{h} = \nabla_\alpha (A_{ij}^{\alpha\beta}(x) \nabla_\beta u_n^j) + (\nabla_i p_n), \\ \nabla \cdot u_n = 0, \\ u_n \mid_{\partial\Omega} = 0, \end{cases}$$

where $n = 1, \dots, N$.

Here we shall display the notations used in this paper:

$$\begin{aligned} \overline{\partial_t} u_n &:= \frac{u_n - u_{n-1}}{h}. \\ \overline{\partial_t} u_h(t) &:= \overline{\partial_t} u_n, \qquad t_{n-1} < t \le t_n, \ n = 1, \dots, N. \\ \tilde{u}_h(t) &:= u_h(t-h), \qquad t \in (0, T]. \\ Q &:= (0, T) \times \Omega = \{ z = (t, x) \in \mathbb{R} \times \mathbb{R}^m \mid t \in (0, T), \ x \in \Omega \}. \end{aligned}$$

For $z_0 = (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^m$,

$$\begin{aligned} \Lambda_R(t_0) &:= \left\{ t \in (0, T) \mid t_0 - R^2 < t < t_0 \right\}, \\ B_R(x_0) &:= \{ x \in \mathbb{R}^m \mid |x - x_0| < R \}, \\ Q_R(z_0) &:= \Lambda_R(t_0) \times B_R(x_0). \end{aligned}$$

For $v \in L^1(Q, \mathbb{R}^m)$, we define

$$\overline{v}_A := \fint_A v(z)\, dz = \frac{1}{m(A)} \int_A v(z)\, dz,$$

where $m(A)$ is the Lebesgue measure of $A \subset Q$.

The symbol $[a]$ means, by convention, the greatest integer not greater than the number a, Gauss' symbol of a. $C = C(*)$ denotes a positive constant depending on the quantities $*$ appearing in the parenthesis. The constants C appearing in the argument may depend on m, λ, L, T or Ω, particularly not on h, unless otherwise stated.

We are now in a position to state our main results.

Theorem 1.1. *Let $u_0 \in H^1_\sigma(\Omega)$ and u_h be an approximate solution to the problem* (1.1)*. Then, there exist positive constants ε and C depending only on m, λ and L, not on h, such that*

$$(1.3)\quad \left(\fint_{Q_R(z_0)} |\nabla u_h|^{2+\varepsilon}\, dz\right)^{1/(2+\varepsilon)} \le C\left(\fint_{Q_{4R}(z_0)} |\nabla u_h|^2\, dz\right)^{1/2} + Ch^{(\overline{p}-1)/2}\left(\fint_{Q_{4R}(z_0)} |\overline{\partial_t} u_h|^{(1+\varepsilon/2)\overline{p}} |u_h - \tilde{u}_h|^{(1+\varepsilon/2)(2-\overline{p})}\, dz\right)^{1/(2+\varepsilon)}$$

holds for any $Q_{4R}(z_0) \subset Q$, $z_0 = (t_{n_0}, x_0)$, $n_0 = 1, \dots, N$ and $1 < \overline{p} < 2$.

Theorem 1.2. *For $u_0 \in H^1_\sigma(\Omega)$, there exists the unique weak solution u to the problem* (1.1) *such that*

$$(1.4)\quad \left(\fint_{Q_R(z_0)} |\nabla u|^{2+\varepsilon}\, dz\right)^{1/(2+\varepsilon)} \le C\left(\fint_{Q_{4R}(z_0)} |\nabla u|^2\, dz\right)^{1/2}$$

holds for any $Q_{4R}(z_0) \subset Q$, $z_0 = (t_0, x_0)$, where C and ε are positive constants depending only on m, λ and L.

2. Construction of an approximate solution and preliminary facts

First we construct an approximate solution u_h to the problem (1.1) with the initial-boundary data $u_0 \in H^1_\sigma(\Omega)$. We inductively construct two sets of maps $\{u_n\}_{n=1}^N$ and $\{F_n\}_{n=1}^N$ as follows: A variational functional

$$F_n(u) = \int_\Omega \left(\frac{1}{h}|u - u_{n-1}|^2 + A(x)(\nabla u, \nabla u)\right) dx$$

is introduced and u_n is fixed as a minimizer of F_n in $V_{u_0}(\Omega)$, the existence of which is assured by the weak lower semi-continuity of F_n in $V(\Omega)$, $V_{u_0}(\Omega)$ being convex. Note that u_n thus constructed satisfies the identities (1.2) which are the Euler-Lagrange equations for $n = 1, \dots, N$.

Upon comparing u_{n-1} in F_n with the minimizers u_n, $n = 1, \dots, N$, we infer

$$\int_\Omega A(x)(\nabla u_n, \nabla u_n)\, dx + h\int_\Omega |\overline{\partial_t} u_n|^2\, dx \le \int_\Omega A(x)(\nabla u_{n-1}, \nabla u_{n-1})\, dx$$

and thus have the following lemma. This plays a key role in the proof of Theorem 1.2.

Lemma 2.1. *Let $u_h : [-h, T] \to V_{u_0}(\Omega)$ be the approximate solution to the problem* (1.1)*. Then we have the estimates*

$$\sup_{-h \le t \le T} \int_\Omega A(x)(\nabla u_h(t), \nabla u_h(t))\, dx \le \int_\Omega A(x)(\nabla u_0, \nabla u_0)\, dx,$$

$$\int_0^T \int_\Omega |\overline{\partial_t} u_h|^2\, dx\, dt \le \int_\Omega A(x)(\nabla u_0, \nabla u_0)\, dx.$$

Next we state several preliminary facts which will play a central role in the sequel.

In the study of Navier-Stokes equations, Leray's projection operator is important. In this paper, we use the similar projection operator P_ω defined in Giaquinta-Modica [4]. Let ω be an open subset of Ω. We define the continuous linear operator $L_\omega : H_0^1(\omega, \mathbb{R}^m) \to L^2(\omega, \mathbb{R}^m)$ by $L_\omega \Phi := \nabla \cdot \Phi$ for $\Phi \in H_0^1(\omega, \mathbb{R}^m)$. Then we note that $V(\omega)$ is equal to $\operatorname{Ker} L_\omega$. See Ladyzhenskaya [13] for the proof. We define the projection operator $P_\omega : H_0^1(\omega, \mathbb{R}^m) \to V^\perp(\omega)$ where $V^\perp(\omega)$ is the orthogonal complement of $V(\omega)$.

The following proposition plays a fundamental role to prove Caccioppoli type estimate in Section 3.

Proposition 2.2. *For all $\Phi \in H_0^1(\omega, \mathbb{R}^m)$, we have*

$$\int_\omega |\nabla(P_\omega \Phi)|^2\, dx \le C \int_\omega |\nabla \cdot \Phi|^2\, dx,$$

where the constant C depends on the domain ω; if ω is a ball, then C is an absolutely constant.

Let $x_0 \in \Omega$ and r be a positive number satisfying $B_r(x_0) \subset \Omega$ and $\eta \in C_0^\infty(B_r(x_0))$ a usual cut-off function such that $\eta \equiv 1$ in $B_{r/2}(x_0)$, $0 \le \eta \le 1$, and $|\nabla \eta| \le 4/r$. For a mapping $v \in L^1(\Omega, \mathbb{R}^m)$ and positive number $s < r$, we set

$$\hat{v}_s := \Big(\int_{B_s(x_0)} \eta^2(x) v(x)\, dx \Big) \Big/ \Big(\int_{B_s(x_0)} \eta^2(x)\, dx \Big). \tag{2.3}$$

The following property has been effectively used in Struwe [17].

Proposition 2.3. *There exists a positive constant C depending only on m such that*

$$\int_{B_r(x_0)} |v - \hat{v}_s|^2\, dx \le C \int_{B_r(x_0)} |v - \hat{v}_r|^2\, dx$$

holds for any $v \in L^2(\Omega, \mathbb{R}^m)$ and positive number s satisfying $r/2 \le s \le r$.

Finally we recall the fundamental result due to Gehring-Giaquinta-Modica ([1], [2], [3] and [5]). We, however, need to generalize it so as to be applicable to difference-partial differential equation. See Haga-Kikuchi [7] and Hoshino-Kikuchi [8] in this connection.

Proposition 2.4. *Let $f \in L^q(Q)$ and $g \in L^r(Q)$, $r > q > 1$, be nonnegative h-time step functions. Suppose that there exist two constants θ and C_1 with $0 \le \theta < 1$, $C_1 > 1$ such that*

$$(2.4)\quad \fint_{Q_R(z_0)} f^q\,dz \le C_1\left\{\left(\fint_{Q_{4R}(z_0)} f\,dz\right)^q + \fint_{Q_{4R}(z_0)} g^q\,dz\right\} + \theta\fint_{Q_{4R}(z_0)} f^q\,dz,$$

holds for every $Q_{4R}(z_0) \subset Q$ with $z_0 = (t_{n_0}, x_0)$, $n_0 = 1, \dots, N$. Then there exist two positive constants C_2 and ε depending only on C_1, θ, q, r, m, such that $g \in L^p_{\mathrm{loc}}(Q)$ for $p \in [q, q+\varepsilon)$ and

$$(2.5)\quad \left(\fint_{Q_R(z_0)} f^p\,dz\right)^{1/p} \le C_2\left\{\left(\fint_{Q_{4R}(z_0)} f^q\,dz\right)^{1/q} + \left(\fint_{Q_{4R}(z_0)} g^p\,dz\right)^{1/p}\right\},$$

holds for every $Q_{4R}(z_0) \subset Q$ with $z_0 = (t_{n_0}, x_0)$, $n_0 = 1, \dots, N$.

3. Local estimates for the approximate solution

In this section, we derive the Caccioppoli type estimate which is the key lemma to get the higher integrability of gradients. In this lemma, we use the cut-off function η defined as follows: Let k and l be positive numbers satisfying $R \le k < l \le 2R$ for any positive number R. We define $\eta(x) := \eta_{k,l}(x) \in C_0^\infty(B_l(x_0))$ by

$$(3.1)\qquad \eta(x) \equiv 1 \ \text{ in } \ B_k(x_0),\ \ 0 \le \eta(x) \le 1, \ \text{ and } \ |\nabla\eta(x)| \le 2(l-k)^{-1}.$$

We also set

$$(3.2)\qquad \hat{u}_{h,2R}(t) := \hat{u}_{n,2R} \ \text{ for } \ t_{n-1} < t \le t_n,\ n = 1, \dots, N.$$

Using the notations (2.3) and (3.2), we have

Lemma 3.1 (Caccioppoli type estimate). *Let u_h be an approximate solution to the problem* (1.1). *Then there exists a positive constant C depending on λ, L such that*

$$(3.3)\qquad \begin{aligned}\int_{Q_R(z_0)} |\nabla u_h|^2\,dz &\le CR^{-2}\int_{Q_{2R}(z_0)} |u_h - \hat{u}_{h,2R}|^2\,dz \\ &\quad + Ch^{\overline{p}-1}\int_{Q_{2R}(z_0)} |\overline{\partial_t} u_h|^{\overline{p}}|u_h - \tilde{u}_h|^{2-\overline{p}}\,dz\end{aligned}$$

holds for any $Q_{2R}(z_0) \subset Q$, $z_0 = (t_{n_0}, x_0)$, $n_0 = 1, \dots, N$ and for any $1 < \overline{p} < 2$. Furthermore, for each $1 < \overline{p} < 2$, $|\overline{\partial_t} u_h|^{\overline{p}}|u_h - \tilde{u}_h|^{2-\overline{p}}$ belongs to $L^p(Q)$ for any $1 < p \le m/(m-2+\overline{p})$.

The proof of the lemma is based on the following assertion.

Lemma 3.2. *For $\{u_n\}_{n=1}^N \subset V_{u_0}(\Omega)$ defined in Section 1, the equality*

$$\int_{B_l(x_0)} \Big(u_n - u_{n-1}, P_{B_l}\{\eta^2(u_n - \hat{u}_{n,l})\}\Big)\, dx = 0 \tag{3.4}$$

holds for any $n = 1, \dots, N$. Here $P_{B_l} : H_0^1(B_l(x_0)) \to V^{\perp}(B_l(x_0))$ is the projection operator defined in Section 2.

Proof: First, we show

$$\int_{B_l(x_0)} \Big(P_{B_l}\{\eta^2(u_n - \hat{u}_{n,l})\}, w\Big)\, dx = 0, \tag{3.5}$$

for any $w \in C_{0,\sigma}^\infty(B_l(x_0))$. Here $\hat{u}_{n,l}$ is defined by (2.3). We want to represent w as ΔU. Let $w \in C_{0,\sigma}^\infty(B_l(x_0))$ be fixed. We consider

$$\begin{cases} \Delta U = w & \text{in } \ B_k(x_0), \\ U = 0 & \text{on } \ \partial B_k(x_0), \end{cases} \tag{3.6}$$

where $\mathrm{supp}(w) \subset\subset B_k(x_0) \subset B_l(x_0)$. Generally, there exists a solution $U \in C^\infty(\overline{B_k(x_0)})$ of (3.6). In the sequel, we extend U to a function defined on $B_l(x_0)$ and vanishing identically outside $B_k(x_0)$. We call it U again. For any $\varepsilon < l - k$, we denote by U_ε and w_ε a mollification of U and w, respectively. Then $U_\varepsilon \in C_0^\infty(B_l(x_0))$ and it satisfies

$$\begin{cases} \Delta U_\varepsilon = w_\varepsilon & \text{in } \ B_{k+\varepsilon}(x_0), \\ U_\varepsilon = 0 & \text{on } \ \partial B_{k+\varepsilon}(x_0). \end{cases} \tag{3.7}$$

We want to show $\nabla \cdot U_\varepsilon = 0$. By operating $\nabla \cdot$ to (3.7), we have

$$\Delta(\nabla \cdot U_\varepsilon) = \nabla \cdot (\Delta U_\varepsilon) = \nabla \cdot (w_\varepsilon) = 0 \quad \text{in } \ B_{k+\varepsilon}(x_0),$$

and

$$\nabla \cdot U_\varepsilon = 0 \quad \text{on } \ \partial B_{k+\varepsilon}(x_0).$$

Using the strongly maximum principle for Laplace equation, we obtain

$$\nabla \cdot U_\varepsilon = 0 \quad \text{in } \ B_l(x_0).$$

Hence we obtain $U_\varepsilon \in C_{0,\sigma}^\infty(B_l(x_0))$. By recalling $\lim_{\varepsilon \downarrow 0} U_\varepsilon = U$ strongly in $H_0^1(B_l(x_0))$, we have shown $U \in V(B_l(x_0))$.

Therefore we have

$$
\begin{aligned}
&\int_{B_l(x_0)} \Big(P_{B_l}\{\eta^2(u_n-\hat{u}_{n,l})\}, w\Big)\,dx \\
&= \int_{B_l(x_0)} \Big(P_{B_l}\{\eta^2(u_n-\hat{u}_{n,l})\}, \Delta U\Big)\,dx \\
&= -\int_{B_l(x_0)} \Big(\nabla P_{B_l}\{\eta^2(u_n-\hat{u}_{n,l})\}, \nabla U\Big)\,dx = 0.
\end{aligned}
$$

We complete the proof of (3.5).

Then by de-Rham's theorem ([13]), there exists a scalar valued function $q \in H^2(B_l(x_0))$ satisfying

$$
P_{B_l}\Big\{\eta^2(u_n-\hat{u}_{n,l})\Big\} = \nabla q. \tag{3.8}
$$

By virtue of $\eta \in C_0^\infty(B_l(x_0))$, we have $P_{B_l}\{\eta^2(u_n-\hat{u}_{n,l})\} = 0$ in $B_l(x_0)\setminus \operatorname{supp}(\eta)$. It follows from (3.8) that q is equal to a certain constant C_* in $B_l(x_0)\setminus \operatorname{supp}(\eta)$. Hence we have $\gamma q = C_*$ on $\partial B_l(x_0)$, where $\gamma : H^1(B_l(x_0)) \to L^2(\partial B_l(x_0);\mathcal{H}^{m-1})$ is the trace operator and $\mathcal{H}^{m-1}$ is the $(m-1)$-dimensional Hausdorff measure. Therefore we have the following calculation for sufficient small $\varepsilon > 0$:

$$
\begin{aligned}
&\int_{B_l(x_0)} \Big((u_n)_\varepsilon-(u_{n-1})_\varepsilon, P_{B_l}\{\eta^2(u_n-\hat{u}_{n,l})\}\Big)\,dx \\
&= \int_{B_l(x_0)} \Big((u_n)_\varepsilon-(u_{n-1})_\varepsilon, \nabla q\Big)\,dx \\
&= -\int_{B_l(x_0)} q\cdot\Big(\nabla\cdot(u_n-u_{n-1})\Big)_\varepsilon\,dx + \int_{\partial B_l(x_0)} \gamma q\cdot\Big((u_n)_\varepsilon-(u_{n-1})_\varepsilon, \nu\Big)d\mathcal{H}^{m-1} \\
&= C_*\int_{\partial B_l(x_0)} \Big((u_n)_\varepsilon-(u_{n-1})_\varepsilon, \nu\Big)d\mathcal{H}^{m-1},
\end{aligned}
$$

where ν is the outer normal vector to the boundary $\partial B_l(x_0)$ and we used the identity $\nabla\cdot(u_n-u_{n-1}) = 0$ in $B_l(x_0)$. Gauss' theorem leads us to

$$
\int_{B_l(x_0)} \Big((u_n)_\varepsilon-(u_{n-1})_\varepsilon, P_{B_l}\{\eta^2(u_n-\hat{u}_{n,l})\}\Big)\,dx = 0.
$$

By letting $\varepsilon \downarrow 0$, we complete the proof. □

PROOF OF LEMMA 3.1: Let k and l be positive numbers satisfying $R < k < l < 2R$. By introducing the cut-off function $\eta \in C_0^\infty(B_l(x_0))$ which was defined by (3.1), we carry out the following estimation:

$$\begin{aligned}
&\lambda h \int_{B_k(x_0)} |\nabla u_n|^2 \, dx \\
&\le h \int_{B_l(x_0)} A(x)\big(\nabla \left\{\eta(u_n - \hat{u}_{n,l})\right\}, \nabla \left\{\eta(u_n - \hat{u}_{n,l})\right\}\big) \, dx \\
&= h \int_{B_l(x_0)} A(x)\big(\nabla\eta(u_n - \hat{u}_{n,l}), \nabla\eta(u_n - \hat{u}_{n,l})\big) \, dx \\
&\quad + h \int_{B_l(x_0)} A(x)\big(\nabla u_n, \nabla\{\eta^2(u_n - \hat{u}_{n,l})\}\big) \, dx \\
&=: I_1 + I_2.
\end{aligned}$$

For the estimation of the term I_1, we use estimate (3.1) to obtain

$$I_1 \le Ch(l-k)^{-2} \int_{B_l(x_0)} |u_n - \hat{u}_{n,l}|^2 \, dx. \tag{3.9}$$

Next we shall carry out the estimation of the term I_2. Consider the function $\phi_n := h\eta^2(u_n - \hat{u}_{n,l})$. Then $\Phi_n := \phi_n - P_{B_l}\phi_n$ can be a test function to the equations (1.2) since this function is of $V(B_l(x_0))$. Hence we write the term I_2 to be

$$\begin{aligned}
I_2 &= h \int_{B_l(x_0)} A(x)\big(\nabla u_n, \nabla P_{B_l}\left\{\eta^2(u_n - \hat{u}_{n,l})\right\}\big) \, dx \\
&\quad + \int_{B_l(x_0)} \big(u_n - u_{n-1}, P_{B_l}\left\{\eta^2(u_n - \hat{u}_{n,l})\right\}\big) \, dx \\
&\quad - \int_{B_l(x_0)} \eta^2 \cdot (u_n - u_{n-1}, u_n - \hat{u}_{n,l}) \, dx \\
&=: I_3 + I_4 + I_5.
\end{aligned}$$

Note that $\nabla \cdot u_n = 0$ implies

$$\nabla \cdot \{\eta^2(u_n - \hat{u}_{n,l})\} = 2\eta(\nabla\eta, u_n - \hat{u}_{n,l}).$$

Then by using Proposition 2.2, we have

$$\begin{aligned}
I_3 &\le Ch \left(\int_{B_l(x_0)} |\nabla u_n|^2 \, dx \right)^{1/2} \left(\int_{B_l(x_0)} |\nabla \cdot \{\eta^2(u_n - \hat{u}_n)\}|^2 \, dx \right)^{1/2} \\
&\le \frac{h}{2} \int_{B_l(x_0)} |\nabla u_n|^2 \, dx + C(l-k)^{-2} h \int_{B_l(x_0)} |u_n - \hat{u}_n|^2 \, dx.
\end{aligned} \tag{3.10}$$

By recalling Lemma 3.2, we easily have

$$I_4 = 0. \tag{3.11}$$

For the estimate of the term I_5, we follow the following two ways. The first estimate is of the form

$$I_5 \le \frac{1}{2}\int_{B_l(x_0)} |u_n - u_{n-1}|^2\,dx + \frac{1}{2}\int_{B_l(x_0)} |u_n - \hat{u}_{n,l}|^2\,dx. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned} I_5 &= -\int_{B_l(x_0)} \eta^2 |u_n - \hat{u}_{n,l}|^2\,dx + \int_{B_l(x_0)} \eta^2 (u^i_{n,l} - \hat{u}^i_{n-1,l})(u^i_n - \hat{u}^i_{n,l})\,dx \\ &\le \frac{1}{2}\int_{B_l(x_0)} |u_{n-1} - \hat{u}_{n-1,l}|^2\,dx - \frac{1}{2}\int_{B_l(x_0)} |u_n - \hat{u}_{n,l}|^2\,dx, \end{aligned} \tag{3.13}$$

by noting

$$\int_{B_l(x_0)} \eta^2 \hat{u}^i_{n,l}(u^i_n - \hat{u}^i_{n,l})\,dx = \hat{u}^i_{n,l}\int_{B_l(x_0)} \eta^2 (u^i_n - \hat{u}^i_{n,l})\,dx = 0.$$

Gathering the estimates (3.9), (3.10), (3.11), (3.12) and (3.9), (3.10), (3.11), (3.13) respectively, we achieve the estimate of two type

$$\begin{aligned} &h\int_{B_k(x_0)} |\nabla u_n|^2\,dx \\ &\le \frac{h}{2}\int_{B_l(x_0)} |\nabla u_n|^2\,dx + C(l-k)^{-2}h\int_{B_l(x_0)} |u_n - \hat{u}_{n,l}|^2\,dx \\ &\quad + \frac{1}{2}\int_{B_l(x_0)} |u_n - \hat{u}_{n,l}|^2\,dx + C\int_{B_l(x_0)} |u_n - u_{n-1}|^2\,dx \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &h\int_{B_k(x_0)} |\nabla u_n|^2\,dx \\ &\le \frac{h}{2}\int_{B_l(x_0)} |\nabla u_n|^2\,dx + C(l-k)^{-2}h\int_{B_l(x_0)} |u_n - \hat{u}_{n,l}|^2\,dx \\ &\quad + \frac{1}{2}\Big(\int_{B_l(x_0)} \eta^2|u_{n-1} - \hat{u}_{n-1,l}|^2\,dx - \int_{B_l(x_0)} \eta^2|u_n - \hat{u}_{n,l}|^2\,dx\Big). \end{aligned} \tag{3.15}$$

For k, l and h, two different situations can occur, namely

$$\text{(I)}\quad (l-k)^2 < 4h, \qquad\qquad \text{(II)}\quad (l-k)^2 \ge 4h.$$

As we shall see, the estimate (3.14) will be applied when treating the case (I) meanwhile in the case (II) we will use the estimate (3.15).

First we deal with the case (I). It reduces (3.14) to

$$
\begin{aligned}
& h\int_{B_k(x_0)} |\nabla u_n|^2\,dx \\
& \le \frac{h}{2}\int_{B_l(x_0)} |\nabla u_n|^2\,dx + C(l-k)^{-2}h\int_{B_l(x_0)} |u_n-\hat{u}_{n,2R}|^2\,dx \\
& \quad + Ch^{\overline{p}}\int_{B_l(x_0)} |\overline{\partial}_t u_n|^{\overline{p}}|u_n-u_{n-1}|^{2-\overline{p}}\,dx
\end{aligned}
\tag{3.16}
$$

for each $1<\overline{p}<2$, where Proposition 2.3 is applied in deriving the second term of the right hand side. Taking summation of the inequalities (3.16) over n from $n_0-[k^2/h]+1$ to n_0 and multiplying both sides of (3.16) by $n=n_0-[k^2/h]$ by $(k^2-[k^2/h]h)h^{-1}$, we sum up the two resultant estimates to have the estimate:

$$
\begin{aligned}
& \int_{Q_k(z_0)} |\nabla u_h|^2\,dz \\
& \le \frac{1}{2}\int_{Q_l(z_0)} |\nabla u_h|^2\,dz + C(l-k)^{-2}\int_{Q_l(z_0)} |u_h-\hat{u}_{h,2R}|^2\,dz \\
& \quad + Ch^{\overline{p}-1}\int_{Q_l(z_0)} |\overline{\partial}_t u_h|^{\overline{p}}|u_h-\hat{u}_{h,2R}|^{2-\overline{p}}\,dz
\end{aligned}
\tag{3.17}
$$

for any $1<\overline{p}<2$ and $R<k<l<2R$.

Next we deal with the case (II). We shall introduce the following time discrete cut-off function ([11]):

(3.18)

$$
\zeta_n := \begin{cases} 1, & \text{for } n > n_0-[k^2/h], \\ \frac{n-(n_0-[l^2/h]+1)}{(n_0-[k^2/h]-1)-(n_0-[l^2/h]+1)}, & \text{for } n_0-[l^2/h]+1 < n \le n_0-[k^2/h]-1, \\ 0, & \text{for } n \le n_0-[l^2/h]. \end{cases}
$$

We note

$$
0 \le \zeta_n-\zeta_{n-1} \le 4h(l-k)^{-2} \quad \text{for} \quad 0<k<l \quad \text{with} \quad (l-k)^2 > 4h. \tag{3.19}
$$

Multiplying (3.15) by ζ_n and taking summation over n from $n_0-[l^2/h]+1$ to n_0, we obtain, noting (3.18) and (3.19), that

$$
\begin{aligned}
& \int_{Q_k(z_0)} |\nabla u_h|^2\,dz + C\int_{B_l(x_0)} \eta^2|u_{n_0}-\hat{u}_{n_0,l}|^2\,dx \\
& \le \frac{1}{2}\int_{Q_l(z_0)} |\nabla u_h|^2\,dz + C(l-k)^{-2}\int_{Q_l(z_0)} |u_h-\hat{u}_{h,2R}|^2\,dz.
\end{aligned}
\tag{3.20}
$$

Combination of two estimates (3.17) and (3.20) implies that

$$\begin{aligned} &\int_{Q_k(z_0)} |\nabla u_h|^2\,dz \\ &\le \frac{1}{2}\int_{Q_l(z_0)} |\nabla u_h|^2\,dz + C(l-k)^{-2}\int_{Q_l(z_0)} |u_h - \hat{u}_{h,2R}|^2\,dz \\ &\quad + Ch^{\overline{p}-1}\int_{Q_l(z_0)} |\overline{\partial_t} u_h|^{\overline{p}}|u_h - \hat{u}_h|^{2-\overline{p}}\,dz \end{aligned} \tag{3.21}$$

holds for any $1 < \overline{p} < 2$ and $R < k < l < 2R$.

Then by using Lemma 1.1 in Giaquinta-Giusti [3], we have the Caccioppoli type estimate: For every $1 < \overline{p} < 2$

$$\begin{aligned} \int_{Q_R(z_0)} |\nabla u_h|^2\,dz &\le CR^{-2}\int_{Q_{2R}(z_0)} |u_h - \hat{u}_{h,2R}|^2\,dz \\ &\quad + Ch^{\overline{p}-1}\int_{Q_{2R}(z_0)} |\overline{\partial_t} u_h|^{\overline{p}}|u_h - \hat{u}_h|^{2-\overline{p}}\,dz, \end{aligned} \tag{3.22}$$

where C should be noticed to be a positive constant independent of h and R.

Finally we have that, for each $1 < \overline{p} < 2$, the term $|\overline{\partial_t} u_h|^{\overline{p}}|u_h - \hat{u}_h|^{2-\overline{p}}$ belongs to $L^p(Q)$ for any $1 < p \le m/(m-2+\overline{p})$, which can be verified by making use of the estimates obtained in Lemma 2.1. See the proof of Lemma 1 in [12] for details. □

Next we present the following lemma to obtain the reverse-Hölder estimates.

Lemma 3.3. *Let u_h be the approximate solution to the problem* (1.1). *Then there exists a positive constant C depending on λ and L such that*

$$\sup_{t\in\Lambda_R(t_0)} \int_{B_R(x_0)} |u_h(t,x) - \hat{u}_{h,R}(t)|^2\,dx \le C\int_{Q_{2R}(z_0)} |\nabla u_h|^2\,dz \tag{3.23}$$

holds for any $Q_{2R}(z_0) \subset Q$, $z_0 = (t_{n_0}, x_0)$, $n_0 = 1,\dots,N$.

Proof: We exactly follow the argument discussed in [12]. We distinguish into two cases between h and R:

$$\text{(I)}\quad 4h > R^2, \qquad\qquad \text{(II)}\quad 4h < R^2.$$

In the case (I), Poincaré's inequality directly implies the required estimate (3.23).

For the treatment of the case (II), we prepare a cut-off function $\eta(x) := \eta_{R,2R}(x) \in C_0^\infty(B_{2R}(x_0))$ with

$$\eta(x) \equiv 1 \;\text{ in } B_R(x_0),\; 0 \le \eta(x) \le 1, \;\text{ and }\; |\nabla\eta(x)| \le 2/R.$$

By putting R, $2R$ instead of k, l in (3.15) and using Poincaré's inequality, we have

$$\int_{B_{2R}(x_0)} \eta^2 |u_n - \hat{u}_{n,2R}|^2\, dx - \int_{B_{2R}(x_0)} \eta^2 |u_{n-1} - \hat{u}_{n-1,2R}|^2\, dx \leq Ch \int_{B_{2R}(x_0)} |\nabla u_n|^2\, dx. \tag{3.24}$$

Let $n_0 - [k^2/h] \leq j \leq n_0$ be fixed. Again, we shall use the time discrete cut-off function ζ_n. By multiplying both sides of (3.24) by ζ_n and summing up over n from $n_0 - [l^2/h]$ to j, we have

$$\int_{B_R(x_0)} |u_j - \hat{u}_{j,2R}|^2\, dx \leq C \int_{Q_{2R}(z_0)} |\nabla u_h|^2\, dz. \tag{3.25}$$

Applying Proposition 2.3 to the left hand side of (3.25), we obtain

$$\int_{B_R(x_0)} |u_j - \hat{u}_{j,R}|^2\, dx \leq C \int_{Q_{2R}(z_0)} |\nabla u_h|^2\, dz$$

for $n_0 - [k^2/h] \leq j \leq n_0$. This completes the proof. □

4. Proof of Theorems

In this section, we will give the proof of Theorem 1.2. First we prove Theorem 1.1, i.e., we get the higher integrability of gradient for the approximate weak solution u_h. Next, using Theorem 1.1 and letting $h \downarrow 0$, we prove Theorem 1.2.

Proof of Theorem 1.1: Since the proof completely follows the argument in [6], we only give a sketch of the proof. We apply Poincaré's inequality and Sobolev-Poincaré's inequality to the Caccioppoli type estimate (3.3) with the help of the estimate (3.23) in Lemma 3.3. Then for each $1 < \overline{p} < 2$, we can choose $0 < \theta < 1$ such that the following reverse-Hölder type estimate holds:

$$\begin{aligned} \fint_{Q_R(z_0)} |\nabla u_h|^2\, dz \leq \theta \fint_{Q_{4R}(z_0)} |\nabla u_h|^2\, dz + C(\theta) \left(\fint_{Q_{4R}(z_0)} |\nabla u_h|^\alpha\, dz \right)^{2/\alpha} \\ + Ch^{\overline{p}-1} \fint_{Q_{4R}(z_0)} |\overline{\partial_t} u_h|^{\overline{p}} |u_h - \tilde{u}_h|^{2-\overline{p}}\, dz, \end{aligned} \tag{4.1}$$

where α is the exponent conjugate to the Sobolev index, i.e., $\alpha := 2m/(m+2)$.

Here we set

$$f := |\nabla u_h|^\alpha, \qquad q := \frac{2}{\alpha}, \qquad g := \left\{ h^{\overline{p}-1} |\overline{\partial_t} u_h|^{\overline{p}} |u_h - \tilde{u}_h|^{2-\overline{p}} \right\}^{\alpha/2}$$

and apply Proposition 2.4. We also recall that $|\overline{\partial_t} u_h|^{\overline{p}} |u_h - \hat{u}_h|^{2-\overline{p}} \in L^p(Q)$ for any $1 < p \le m/(m-2+\overline{p})$. Then we have

$$\left(\fint_{Q_R(z_0)} |\nabla u_h|^{2+\alpha\varepsilon}\, dz \right)^{\alpha/(2+\alpha\varepsilon)} \le C \Bigg\{ \left(\fint_{Q_{4R}(z_0)} |\nabla u_h|^2\, dz \right)^{\alpha/2} + \left(\fint_{Q_{4R}(z_0)} \left\{ h^{\overline{p}-1} |\overline{\partial_t} u_h|^{\overline{p}} |u_h - \tilde{u}_h|^{2-\overline{p}} \right\}^{1+(\alpha\varepsilon)/2} dz \right)^{\alpha/(2+\alpha\varepsilon)} \Bigg\}.$$

This completes the proof. □

Proof of Theorem 1.2: For an approximate solution u_h, we define u_h^* by

$$u_h^*(t) = \frac{t_n - t}{h} u_{n-1} + \frac{t - t_{n-1}}{h} u_n, \qquad \text{for } \ t_{n-1} \le t \le t_n, \ n = 1, 2, \ldots, N.$$

Then it is easily seen that $u_h^* \in L^\infty(0,T; V_{u_0}(\Omega)) \cap H^1(0,T; L^2(\Omega, \mathbb{R}^m))$ and $\overline{\partial_t} u_h = \partial_t u_h^*$. By recalling (1.2), the following equality holds for all $\Phi(x) \in V(\Omega)$ and $\Psi(t) \in C_0^\infty(0,T)$:

$$\int_0^T \left\{ \int_\Omega (\partial_t u_h(t,x), \Phi(x)) + A(x)(\nabla u_h(t,x), \nabla \Phi(x))\, dx \right\} \Psi(t)\, dt = 0. \tag{4.2}$$

On the other hand, Lemma 2.1 leads us to the estimates

$$\int_Q |\partial_t u_h^*|^2\, dz = \int_Q |\overline{\partial_t} u_h|^2\, dz \le C \int_\Omega |\nabla u_0|^2\, dx, \tag{4.3}$$

$$\sup_{t \in (0,T)} \int_\Omega |\nabla u_h^*|^2\, dx = \sup_{t \in (0,T)} \int_\Omega |\nabla u_h|^2\, dx \le C \int_\Omega |\nabla u_0|^2\, dx. \tag{4.4}$$

Then (4.3), (4.4) and Rellich's theorem imply that there exist a subsequence $\{h_k\}_{k=1}^\infty$ tending to zero and a map $u \in L^\infty(0,T; V_{u_0}(\Omega)) \cap H^1(0,T; L^2(\Omega, \mathbb{R}^m))$ such that

$$\lim_{k\to\infty} \nabla u_{h_k}^* = \nabla u, \qquad \text{weakly in } \ L^2(Q, \mathbb{R}^m), \tag{4.5}$$

$$\lim_{k\to\infty} u_{h_k}^* = u, \qquad \text{weakly in } \ H^1(0,T; L^2(\Omega, \mathbb{R}^m)), \tag{4.6}$$

$$\lim_{k\to\infty} u_{h_k}^* = u, \qquad \text{strongly in } \ L^2(Q, \mathbb{R}^m). \tag{4.7}$$

Here by recalling (4.3), we have

$$\begin{aligned} \int_Q |u_{h_k} - u_{h_k}^*|^2\, dz &\le \int_Q |u_{h_k} - \tilde{u}_{h_k}|^2\, dz \\ &\le \int_Q (h_k |\bar{\partial}_t u_{h_k}|)^2\, dz \le C h_k^2 \int_\Omega |\nabla u_0|^2\, dx. \end{aligned} \tag{4.8}$$

Then by virtue of (4.7) and (4.8), we have

$$\lim_{k\to\infty} u_{h_k} = u, \quad \text{strongly in } L^2(Q,\mathbb{R}^m). \tag{4.9}$$

Next we carry out an estimation of the following by dividing it into two terms:

$$\begin{aligned}
&\lambda \int_Q |\nabla(u_{h_k}-u)|^2\,dz \\
&\le \int_Q A(x)\big(\nabla(u_{h_k}-u),\nabla(u_{h_k}-u)\big)\,dz \\
&\le \int_Q A(x)\big(\nabla u_{h_k},\nabla(u_{h_k}-u)\big)\,dz - \int_Q A(x)\big(\nabla u,\nabla(u_{h_k}-u)\big)\,dz \\
&=: I_k + I\!I_k.
\end{aligned} \tag{4.10}$$

By recalling the equation (1.2), we have

$$\begin{aligned}
I_k &= -\int_Q (\bar{\partial}_t u_{h_k}, u_{h_k}-u)\,dz \\
&\le \Big(\int_Q |\bar{\partial}_t u_{h_k}|^2\,dz\Big)^{1/2}\cdot\Big(\int_Q |u_{h_k}-u|^2\,dz\Big)^{1/2}.
\end{aligned}$$

(4.9) leads us to $\lim_{k\to\infty} I_k = 0$ and (4.5) gives directly that $\lim_{k\to\infty} I\!I_k = 0$. So we obtain

$$\lim_{k\to\infty} u_{h_k} = u, \quad \text{strongly in } L^2(0,T;V_{u_0}(\Omega)). \tag{4.11}$$

Therefore, in the equality (4.2) with h replaced by h_k, we can let k to infinity in the equality (4.2), so that

$$\int_0^T \left\{\int_\Omega (\partial_t u(t,x),\Phi(x)) + A(x)(\nabla u(t,x),\nabla\Phi(x))\,dx\right\}\Psi(t)\,dt = 0$$

and hence, for any $\Phi \in V(\Omega)$, the following equality holds for almost every $t \in (0,T)$:

$$\int_\Omega (\partial_t u(t,x),\Phi(x)) + A(x)(\nabla u(t,x),\nabla\Phi(x))\,dx = 0.$$

It remains to verify the initial condition. By (4.3) and Schwarz's inequality, we have

$$\begin{aligned}
\|u^*_{h_k}(s)-u^*_{h_k}(t)\|_{L^2(\Omega,\mathbb{R}^m)} &\le \int_s^t \big\|\partial_t u^*_{h_k}(\tau,\cdot)\big\|_{L^2(\Omega,\mathbb{R}^m)}\,d\tau \\
&\le \Big(\int_s^t d\tau\Big)^{1/2}\cdot\Big(\int_s^t \big\|\partial_t u^*_{h_k}(\tau,\cdot)\big\|^2_{L^2(\Omega,\mathbb{R}^m)}\,d\tau\Big)^{1/2} \\
&\le \sqrt{t-s}\cdot\|\partial_t u^*_{h_k}\|_{L^2(Q,\mathbb{R}^m)} \le C\sqrt{t-s}
\end{aligned}$$

for any $0 \le s \le t \le T$. Letting $k \to \infty$, this inequality takes the form

$$\|u(s) - u(t)\|_{L^2(\Omega,\mathbb{R}^m)} \le C\sqrt{|s-t|}$$

for almost all $s, t \in [0,T]$. This means that the limit function u is equivalent in Q to a function that is continuous in all $t \in [0,T]$ in the norm of $L^2(\Omega,\mathbb{R}^m)$. In the sequel, we call it u again. Hence by recalling this and the estimate $\|u^*_{h_k}(t) - u_0\|_{L^2(\Omega,\mathbb{R}^m)} \le C\sqrt{t}$ for any $t \in [0,T]$, we have

$$\lim_{t \searrow 0} u(t) = u_0, \quad \text{in } \ L^2(\Omega,\mathbb{R}^m). \tag{4.12}$$

To show the uniqueness of the solution in the class $L^\infty(0,T;V_{u_0}(\Omega)) \cap H^1(0,T;L^2(\Omega,\mathbb{R}^m))$, we follow the same argument on page 261 of Temam [18].

Finally, we show the local estimate. For $Q_{4R}(z_0) \subset Q$, $z_0 = (t_0, x_0)$, we choose the subsequence $\{h_k\}_{k=1}^\infty$ decreasing to zero such that $t_0/h_k \in \mathbb{N}$. Then by letting k tend to infinity in the inequality (1.3) and taking (4.11) and Lebesgue's dominated convergence theorem into account, we obtain

(4.13)

$$\begin{aligned}
&\left(\fint_{Q_R(z_0)} |\nabla u|^{2+\varepsilon}\, dz\right)^{1/(2+\varepsilon)} \\
&\le C \liminf_{k\to\infty} \left\{ \left(\fint_{Q_{4R_{h_k}}(z_{h_k})} |\nabla u_{h_k}|^2\, dz\right)^{1/2} \right. \\
&\qquad \left. + h_k^{(\overline{p}-1)/2} \left(\fint_{Q_{4R_{h_k}}(z_{h_k})} |\overline{\partial}_t u_{h_k}|^{(1+\varepsilon/2)\overline{p}} |u_{h_k} - \tilde{u}_{h_k}|^{(1+\varepsilon/2)(2-\overline{p})}\, dz\right)^{1/(2+\varepsilon)} \right\} \\
&\le \ C\left(\fint_{Q_{4R}(z_0)} |\nabla u|^2\, dz\right)^{1/2}.
\end{aligned}$$

This completes the proof. □

Acknowledgment. The author would like to acknowledge Professor Norio Kikuchi's suggestion to this problem. The author is also grateful to Professors Shigeki Aida, Tôru Maruyama, Takeyuki Nagasawa, Doctor Takahiro Akiyama and the anonymous referee for their constant encouragement, useful comments and pointing out some errors during the preparation of this paper.

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(*Received* June 9, 2003, *revised* April 22, 2004)