Bol loops with a large left nucleus

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Abstract. Possession of a unique nonidentity commutator/associator is a property which dominates the theory of loops whose loop rings, while not associative, nevertheless satisfy an "interesting" identity. Indeed, until now, with the exception of some ad hoc examples, the only known class of Bol loops whose loop rings satisfy the right Bol identity have this property. In this paper, we identify another class of loops whose loop rings are "strongly right alternative" and present various constructions of these loops.

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1. Introduction

Why is a group ring associative? The argument is simple. Let $\alpha = \sum_{g\in G} \alpha_g g$, $\beta = \sum_{h\in G} \beta_h h$, and $\gamma = \sum_{k\in G} \gamma_k k$ be elements of a group ring RG. Then

$$(\alpha\beta)\gamma = \Big(\sum_{g,h\in G} \alpha_g\beta_h gh\Big) \sum_{k\in G} \gamma_k k = \sum_{g,h,k\in G} \alpha_g\beta_h\gamma_k (gh)k,$$

whereas

$$\alpha(\beta\gamma) = \sum_{g,h,k\in G} \alpha_g\beta_h\gamma_k g(hk).$$

Since $(gh)k = g(hk)$ for all $g, h, k \in G$, it follows that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

Does the right alternative law $(yx)x = yx^2$ similarly "lift" from a loop L to a loop ring RL? A small calculation shows that the repeated variable in this

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identity makes a positive answer unlikely. Replacing y by a loop element g, and x by the sum $h+k$ of loop elements h and k, we have

$$(yx)x = (gh+gk)(h+k) = (gh)h + (gh)k + (gk)h + (gk)k$$

and

$$yx^2 = g(h^2 + hk + kh + k^2) = gh^2 + g(hk) + g(kh) + gk^2.$$

After cancelation, $(yx)x = yx^2$ is equivalent to $(gh)k + (gk)h = g(hk) + g(kh)$ and, in the absence of associativity, there seems no reason for this to hold. In loop theory, most interesting identities, including

$$(xy \cdot z)y = x(y \cdot zy) \qquad \text{the (right) Moufang identity}$$

and

$$(xy \cdot z)y = x(yz \cdot y) \qquad \text{the (right) Bol identity,}$$

have a repeated variable and so are unlikely to lift from a loop to any of its loop rings. A half century ago, Lowell Paige proved, with mild restrictions on characteristic, that if a commutative loop ring was even power associative (in most characteristics, this is equivalent to the single identity $x^2x^2 = x^3x$), then that loop ring and hence the underlying loop as well must be associative [Pai55] (see also [GJM96, Theorem III.1.6]). Such observations are perhaps the reason that the loop ring in general remained an almost forgotten object until more recent times. In the mid 1980s, a class of Moufang loops whose loop rings, in characteristic different from 2, also satisfy the Moufang identity was discovered [Goo83], [CG86]. Later, the authors found a larger class of loops whose loop rings satisfy the Moufang identity in any characteristic [CG90] and, in the 1990s, Goodaire and Robinson found a class of loops whose loop rings in characteristic 2 satisfy just the right Bol identity (they are not associative, nor do they satisfy the Moufang identity) [GR95].

Historically, one loop theoretic property has been dominant amongst those classes of loops whose loop rings satisfy an identity with a repeated variable — possession of a unique nonidentity commutator/associator, that is, an element s with the property that for all elements a, b, c in the loop,

$$ab = ba \quad \text{or} \quad ab = (ba)s \tag{1.1}$$

and

$$(ab)c = a(bc) \quad \text{or} \quad (ab)c = [a(bc)]s. \tag{1.2}$$

Until now, the only known classes of (nonassociative[1]) Bol loops[2] whose loop rings satisfy the right Bol identity have this property. Recently, the authors have found a class of Bol loops whose loop rings satisfy the right Bol identity, but which very well may have a commutator/associator subloop of order greater than 2.

If L is a loop and a, b, c are elements of L, we use (a,b) to denote the commutator of a and b (this is the element s which appears in (1.1)) and (a,b,c) to denote the associator of a, b, and c (this is the element s which appears in (1.2)). (In general, these two elements need not be equal.) The *commutator/associator subloop* of L is the subloop L' generated by all commutators and associators. The *centrum* of L is the set

$$C(L) = \{a \in L \mid (a,x) = 1 \ \text{ for all } \ x \in L\}$$

and the left, middle and right *nuclei* of L are, respectively, the sets

$$\begin{aligned} N_\lambda(L) &= \{a \in L \mid (a,x,y) = 1 \ \text{ for all } x,y \in L\} \\ N_\mu(L) &= \{a \in L \mid (x,a,y) = 1 \ \text{ for all } x,y \in L\} \\ N_\rho(L) &= \{a \in L \mid (x,y,a) = 1 \ \text{ for all } x,y \in L\}. \end{aligned}$$

The *nucleus* of L is $N(L) = N_\lambda \cap N_\mu \cap N_\rho$ and the *centre* of L is $\mathcal{Z}(L) = N(L) \cap C(L)$. A good reference for the theory of loops, and especially Bol loops, is the text by Hala Pflugfelder [Pfl90]. Key properties of Bol loops include their *power associativity* (powers of an element are well-defined) and, more generally, their right *power alternativity*: $(ab^i)b^j = ab^{i+j}$ for all a, b and all integers i and j. This implies, in particular, the *right inverse property*: $(ab)b^{-1} = a$ for all a and b.

In this paper we consider the property that the left nucleus is of index 2, show that the loop rings of certain Bol loops of this type satisfy the right Bol identity, and exhibit various classes of loops of the identified type.

2. A construction

Let L be a power associative loop whose left nucleus, N, is an abelian group which, as a subloop of L, has index 2. Then, for every element $u \notin N$, $L = N \cup Nu$. Choose a fixed element u not in N. We can then define bijections $\theta, \phi\colon N \to N$ by

$$un = (n\theta)u \quad \text{and} \quad n\phi = u(nu). \tag{2.1}$$

Clearly, $1\theta = 1$, $1\phi = u^2$, and $u^2\theta = u^2$, the latter since L is power associative. For reasons that will become clearer in Section 5, we will be primarily interested

[1] In this paper, "nonassociative" always means "not associative."

[2] All Bol loops in this paper are right Bol, that is, satisfy the right Bol identity.

in the case that either $\theta = I$, the identity map, or that $\phi = R(u^2)$, the map $N \to N$ which multiplies each element of N on the right by u^2. Notice that $R(u^2) = R(u)^2$ if L is right power alternative.

Since N is the left nucleus, the following equations show how to multiply elements from the cosets N and Nu. For $n_1, n_2 \in N$,

$$n_1(n_2u) = (n_1n_2)u, \tag{2.2}$$

$$(n_1u)n_2 = n_1(un_2) = [n_1(n_2\theta)]u, \tag{2.3}$$

and

$$(n_1u)(n_2u) = n_1[u(n_2u)] = n_1(n_2\phi). \tag{2.4}$$

With this hindsight, we outline a construction of a loop with index 2 left nucleus that plays a prominent role in this paper.

Let N be an abelian group, v a fixed element of N and θ, ϕ bijections of N such that $1\theta = 1$, $1\phi = v$ and $v\theta = v$. Let u be an indeterminate and extend the multiplication from N to $L = N \cup Nu$ by equations (2.2), (2.3) and (2.4). (Note that $u^2 = v$.) It is not hard to see that L is a loop with N a subloop of index 2 and, moreover, that L is power associative, with $(nu)^{2k} = n^k(n\phi)^k$ and $(nu)^{2k+1} = [n^{k+1}(n\phi)^k]u$. Furthermore, N is contained in the left nucleus of L since

$$(nn_1)(n_2u) = (nn_1n_2)u = n[(n_1n_2)u] = n[n_1(n_2u)],$$
$$[n(n_1u)]n_2 = [(nn_1)u]n_2 = [nn_1(n_2\theta)]u = n\{[n_1(n_2\theta)]u\} = n[(n_1u)n_2]$$

and

$$[n(n_1u)](n_2u) = [(nn_1)u](n_2u) = (nn_1)(n_2\phi) = n[n_1(n_2\phi)] = n[(n_1u)(n_2u)].$$

In the next section, (Lemma 3.6 and Lemma 3.2 respectively), we will investigate conditions on θ and ϕ that make L a Bol loop or a group.

3. Properties

Since we will be concerned with the size of the commutator/associator subloop of a loop constructed as in the previous section, the next result will be of use later.

Lemma 3.1. *Let L be a loop constructed as in Section 2 and suppose L is Bol, but not associative. Then for any $n \in N$, $(n\theta)^{-1} = (n^{-1})\theta$, $(n\theta^{-1})^{-1} = n^{-1}\theta^{-1}$, commutators have the form*

$$[n(n^{-1}\theta)]\theta^{-1} \quad \text{and} \quad n_1(n_1\phi)^{-1}n_2^{-1}(n_2\phi),$$

and associators are of four types:

$$\Big\{(n_1\theta)(n_2\theta)[(n_1n_2)^{-1}\theta]\Big\}\theta^{-1}, \quad (n_1\theta)(n_2\phi)[(n_1n_2)\phi]^{-1},$$
$$(n_1\phi)n_2\big\{[n_1(n_2\theta)]\phi\big\}^{-1} \quad \text{and} \quad \Big\{(n_1\phi)n_2\big\{[n_1(n_2\phi)]^{-1}\theta\big\}\Big\}\theta^{-1}.$$

If $\theta = I$, the identity map, then every commutator is of the form

$$n_1(n_1\phi)^{-1}n_2^{-1}(n_2\phi),$$

every associator is of one of the forms

$$n_1(n_2\phi)[(n_1n_2)\phi]^{-1} \;\; \text{or} \;\; n_1(n_1\phi)^{-1}n_2^{-1}(n_2\phi),$$

and every commutator is an associator.

If $\phi = R(u^2)$, then every commutator is of the form

$$[n(n^{-1}\theta)]\theta^{-1},$$

every associator is of one of the forms

$$n(n^{-1}\theta) \text{ or } \{n(n^{-1}\theta)\}\theta^{-1} \;\; \text{or} \;\; \big\{(n_1\theta)(n_2\theta)[(n_1n_2)^{-1}\theta]\big\}\theta^{-1}.$$

and every commutator is an associator.

Proof: For any $n \in N$, the left nucleus of L, $n\theta \in N$, so $[(n\theta)^{-1}u]n = (n\theta)^{-1}(un) = (n\theta)^{-1}[(n\theta)u] = [(n\theta)^{-1}(n\theta)]u = u$. Thus, by the right inverse property, $(n\theta)^{-1}u = un^{-1} = (n^{-1}\theta)u$. Thus, $(n\theta)^{-1} = (n^{-1})\theta$. Replacing n by $n\theta^{-1}$ in this identity gives the second identity of the lemma.

Let $n_1, n_2 \in N$. The commutator (n_1, n_2u) is the element c defined by $n_1(n_2u) = [(n_2u)n_1]c$. Thus c is in N and $(n_1n_2)u = [n_2(n_1\theta)u]c = [n_2(n_1\theta)(c\theta)]u$. So we have $n_1n_2 = n_2(n_1\theta)(c\theta)$ and, since N is an abelian group, $c = [n_1(n_1\theta)^{-1}]\theta^{-1} = [n_1(n_1^{-1}\theta)]\theta^{-1}$. Since $(a,b)^{-1} = (b,a)$ in a Bol loop, the commutator $(n_1u, n_2) = (n_2, n_1u)^{-1} = \{[n_2(n_2^{-1}\theta)]\theta^{-1}\}^{-1} = [n_2^{-1}(n_2\theta)]\theta^{-1}$, which is of the same form as (n_1, n_2u). Similarly, one can show that the commutator $(n_1u, n_2u) = n_1(n_1\phi)^{-1}n_2^{-1}(n_2\phi)$.

Let $n, n_1, n_2 \in N$ and let $x = nu$, $y = n_1$, $z = n_2$. The associator (x, y, z) is the element a satisfying $(xy)z = [x(yz)]a$. We have

$$(xy)z = [(nu)n_1]n_2 = [n(n_1\theta)(n_2\theta)]u$$

and

$$[x(yz)]a = [(nu)(n_1n_2)]a = \{[n(n_1n_2)\theta]u\}a = [n(n_1n_2)\theta a\theta]u,$$

so a is in N, $n(n_1\theta)(n_2\theta) = n(n_1n_2)\theta(a\theta)$ and

$$\begin{aligned}(nu, n_1, n_2) &= a\\ &= \Big\{(n_1\theta)(n_2\theta)[(n_1n_2)\theta]^{-1}\Big\}\theta^{-1} = \Big\{(n_1\theta)(n_2\theta)[(n_1n_2)^{-1}\theta]\Big\}\theta^{-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}(nu, n_1, n_2u) &= (n_1\theta)(n_2\phi)[(n_1n_2)\phi]^{-1},\\ (nu, n_1u, n_2) &= (n_1\phi)n_2\{[n_1(n_2\theta)]\phi\}^{-1},\\ \text{and}\quad (nu, n_1u, n_2u) &= \Big\{[(n_1\phi)n_2]\{[n_1(n_2\phi)]^{-1}\theta\}\Big\}\theta^{-1}.\end{aligned}$$

If $\theta = I$, commutators of the first type become trivial, as do associators of the first type. Associators of the second type become $n_1(n_2\phi)[(n_1n_2)\phi]^{-1}$ and so do associators of the third type (with n_1 and n_2 interchanged). Associators of the fourth type become $n_1(n_1\phi)^{-1}n_2^{-1}(n_2\phi)$. Clearly the commutator $n_1(n_1\phi)^{-1}n_2^{-1}(n_2\phi)$ is also an associator.

If $\phi = R(u^2)$, then commutators of the second type become trivial. Commutators of the first type and associators of the first type are not changed. Associators of the second type become $n_1^{-1}(n_1\theta)$ and those of the third type become $n_2(n_2^{-1}\theta)$. Both of these are of the form $n(n^{-1}\theta)$. Associators of the fourth type become $\{(n_1n_2u^2)[(n_1n_2u^2)^{-1}\theta]\}\theta^{-1}$, which is of the form $[n(n^{-1}\theta)]\theta^{-1}$.

Once again it is clear that every commutator $\{n(n^{-1}\theta)\}\theta^{-1}$ is also an associator. □

In much of what is to follow, it is often necessary to know whether or not a certain loop is a group. In this regard, the next lemma is critical.

Lemma 3.2. *Let L, N, u, θ and ϕ be as described in Section 2 and assume that L is a Bol loop.*

i. *θ is a homomorphism of N if and only if $(u, n_1, n_2) = 1$ for all n_1, n_2 in N.*
ii. *$\phi = \theta R(u^2)$ if and only if $(u, n, u) = 1$ for all $n \in N$.*
iii. *L is a group if and only if θ is a homomorphism of N, $\theta^2 = I$, and $\phi = \theta R(u^2)$.*

Proof: At the outset, we note that if $n \in N$, then nxy is unambiguous without parentheses to denote order of multiplication because $N \subseteq N_\lambda$. We use this fact without further comment below.

i. If θ is a homomorphism, then, for all $n_1, n_2 \in N$, $u(n_1n_2) = (n_1n_2)\theta u = [n_1\theta n_2\theta]u = n_1\theta(n_2\theta u) = n_1\theta(un_2) = (n_1\theta u)n_2 = (un_1)n_2$, so $(u, n_1, n_2) = 1$. Conversely, if $(u, n_1, n_2) = 1$ for all $n_1, n_2 \in N$, then $(n_1\theta n_2\theta)u = (n_1\theta)(un_2) = (n_1\theta u)n_2 = (un_1)n_2 = u(n_1n_2) = (n_1n_2)\theta u$, so that θ is a homomorphism.

ii. If $\phi = \theta R(u^2)$, then, for all $n \in N$, $u(nu) = n\phi = n\theta u^2 = (n\theta u)u = (un)u$, that is, $(u, n, u) = 1$. Conversely, if $(u, n, u) = 1$ for all n, then $n\phi = u(nu) = (un)u = (n\theta u)u = n\theta R(u)^2$ for any n, so $\phi = \theta R(u^2)$.

iii. Suppose L is a group. Then $(u, n_1, n_2) = (u, n, u) = 1$ for any n, n_1, n_2 in N, so θ is a homomorphism and $\phi = \theta R(u^2)$ by parts i and ii. Furthermore, for any $n \in N$, we have $nu^2 = u^2n = u(un) = u[(n\theta)u] = [u(n\theta)]u = n\theta^2u^2$, so $n\theta^2 = n$. Thus $\theta^2 = I$.

Conversely, suppose θ is a homomorphism, $\theta^2 = I$ and $\phi = \theta R(u^2)$. Let x, y and z be three elements of L. We wish to show that $(xy)z = x(yz)$. This is clear if $x \in N$, so it is sufficient to consider just four cases, with $x \in Nu$ and y, z in N or Nu. For $x = nu$, $(xy)z = [(nu)y]z = [n(uy)]z = n[(uy)z]$ and, similarly, $x(yz) = n[u(yz)]$, so $(xy)z = x(yz)$ if and only if $(uy)z = u(yz)$. In other words, there is no loss of generality if we assume that $n = 1$. Our four cases become

Case 1: $x = u$, $y = n_1$, $z = n_2$;
Case 2: $x = u$, $y = n_1$, $z = n_2u$;
Case 3: $x = u$, $y = n_1u$, $z = n_2$;
Case 4: $x = u$, $y = n_1u$, $z = n_2u$.

In Case 1,
$$\begin{aligned}(xy)z &= (un_1)n_2 = (n_1\theta u)n_2 \\ &= (n_1\theta n_2\theta)u = [(n_1n_2)\theta]u = u(n_1n_2) = x(yz).\end{aligned}$$

In Case 2,
$$\begin{aligned}(xy)z &= (un_1)(n_2u) = (n_1\theta u)(n_2u) \\ &= n_1\theta n_2\phi = n_1\theta n_2\theta R(u^2) \\ &= (n_1n_2)\theta R(u^2) = (n_1n_2)\phi = u(n_1n_2u) = x(yz).\end{aligned}$$

In Case 3,
$$\begin{aligned}(xy)z &= [u(n_1u)]n_2 = n_1\phi n_2 \\ &= n_1\theta R(u^2)n_2 = (n_1\theta)u^2n_2 = n_1\theta n_2u^2 \\ &= (n_1\theta)(n_2\theta^2)u^2 = [n_1(n_2\theta)]\theta R(u^2) \\ &= [n_1(n_2\theta)]\phi = u[n_1(n_2\theta)u] \\ &= u[n_1(un_2)] = u[(n_1u)n_2] = x(yz).\end{aligned}$$

In Case 4, we make use of the fact that $R(u^2)$ and θ commute, so that $\phi\theta = \theta R(u^2)\theta = R(u^2)$. To see this, note that $u^2\theta = u^2$ by power associativity. So, for $n \in N$, $nR(u^2)\theta = (nu^2)\theta = n\theta u^2\theta = n\theta u^2 = n\theta R(u^2)$, so $R(u^2)\theta = \theta R(u^2)$ as claimed. Now

$$\begin{aligned}(xy)z &= [u(n_1u)](n_2u) = (n_1\phi n_2)u = [(n_1\theta u^2 n_2)u = [n_1\theta(n_2u^2)]u \\ &= [n_1\theta(n_2R(u^2))]u = [n_1\theta(n_2\phi)\theta]u = \{[n_1(n_2\phi)]\theta\}\, u \\ &= u[n_1(n_2\phi)] = u\,\{n_1[u(n_2u)]\} = u[(n_1u)(n_2u)] = x(yz).\end{aligned}$$

In all cases, $(xy)z = x(yz)$, so L is a group. □

Remark 3.3. A nonassociative loop with a normal nucleus of prime index cannot be power associative [GR82] (Theorem 1.1 and subsequent remarks). In particular, a Moufang loop cannot have a left nucleus of index 2 (in a Moufang loop, the nucleus and left nucleus coincide and this subloop is normal), so there is no hope of adapting techniques of this paper to Moufang loops. Of more significance here, though, is the observation that a Bol loop with left nucleus of index at most 2 is a group if and only if it is Moufang, so part iii of Lemma 3.2 can and will be used in the sequel to provide assurance that certain Bol loops are not Moufang.

Our next lemma gives information about the centre and nucleus of the loops we construct in this paper.

Lemma 3.4. *Suppose L is a Bol loop (but not a group) constructed as in Section 2. Then the centrum of L is*

$$C(L) = \{n \in N \mid n\theta = n\},$$

the nucleus is

$$\begin{aligned}N(L) = \{n \in N \mid \\ &(nx)\theta = n\theta \cdot x\theta, (nx)\phi = n\theta \cdot x\phi, (x \cdot n\theta)\phi = x\phi \cdot n \quad \textit{for all} \quad x \in N\}\end{aligned}$$

and hence the centre of L is

$$\mathcal{Z}(L) = \{n \in N \mid n\theta = n, (nx)\theta = n \cdot x\theta, (nx)\phi = n \cdot x\phi \quad \textit{for all} \quad x \in N\}.$$

Proof: In this proof, we use tacitly many expressions for commutators and associators displayed in the proof of Lemma 3.1.

We first argue that $C(L) \subseteq N$ and, for this, assume $nu \in C(L)$ for some $n \in N$. Then, for any $x \in N$, $(nu, x) = 1 = (x^{-1} \cdot x\theta)\theta^{-1}$. Since $1\theta = 1$, we have $x\theta = x$ for all x, so $\theta = I$. Also, $1 = (nu, xu)$ for all x, so $n(n\phi)^{-1}x^{-1}(x\phi) = 1$ for all x. Setting $x = 1$ and remembering that $1\phi = u^2$ gives $n(n\phi)^{-1}u^2 = 1$, so $n\phi = nu^2$.

Thus $n(nu^2)^{-1}x^{-1}(x\phi) = 1$ for all x, and $x^{-1} \cdot x\phi = u^2$ for all $x \in N$, hence $\phi = R(u^2)$. Then, however, by part iii of Lemma 3.2, L is a group, which is not true. Hence, $C(L) \subseteq N$, as claimed.

Let $n \in C(L)$. Then $(xu, n) = 1$ for all $x \in N$ implies $(n^{-1} \cdot n\theta)\theta^{-1} = 1$ and hence $n\theta = n$. Conversely, if $n\theta = n$ for some $n \in N$, then $(xu, n) = 1$ for all $x \in N$, so n commutes with all elements of Nu, and also with all elements of N (because N is an abelian group), so $C(L)$ is as stated.

Now let $n \in N(L)$, the nucleus of L. (So $n \in N$ because $N(L) \subseteq N_\lambda = N$.) Then $(u, n, x) = 1$ for $x \in N$ gives $\{n\theta \cdot x\theta \cdot (nx)^{-1}\theta\}\theta^{-1} = 1$. Since $1\theta = 1$, we get $n\theta \cdot x\theta \cdot (nx)^{-1}\theta = 1$ and since $y^{-1}\theta = (y\theta)^{-1}$ for any $y \in N$ (Lemma 3.1), $(nx)\theta = n\theta \cdot x\theta$. Also $(u, n, xu) = 1$, so $n\theta \cdot x\phi[(nx)\phi]^{-1} = 1$ and $(nx)\phi = n\theta \cdot x\phi$ for all $x \in N$. Finally, $(u, xu, n) = 1$ for $x \in N$ gives $[x\phi \cdot n][(x \cdot n\theta)\phi]^{-1} = 1$ and $(x \cdot n\theta)\phi = x\phi \cdot n$. Thus $N(L)$ is contained in the set described in the lemma. On the other hand, since N is abelian, it is straightforward to show that if n satisfies the conditions specified in the set alleged to be $N(L)$, then for any $x, y \in N$, each of the associators (xu, n, y), (xu, n, yu), (xu, y, n), (xu, yu, n) is trivial. Thus $N(L)$ is indeed the specified set.

Since the centre of a loop is the intersection of the nucleus and centrum, to find $\mathcal{Z}(L)$, we put $n\theta = n$ in the conditions defining the nucleus to get the result. □

A ring is *strongly right alternative* if it satisfies the right Bol identity which, in general, is stronger than the right alternative law. Kunen has shown that a loop ring can be strongly right (but not left) alternative only in characteristic 2 [Kun98] and Goodaire and Robinson proved the existence of such rings in this case. A (necessarily Bol) loop whose loop rings in characteristic 2 are strongly right alternative is called an *SRAR* loop. Such loops are characterized as follows [GR95].

A Bol loop L is SRAR, if and only if it is not associative and, for every $x, y, z, w \in L$, at least one of the following holds:

$$\begin{aligned} &D(x,y,z,w)\colon [(xy)z]w = x[(yz)w] \text{ and } [(xw)z]y = x[(wz)y] \\ &E(x,y,z,w)\colon [(xy)z]w = x[(wz)y] \text{ and } [(xw)z]y = x[(yz)w] \\ &F(x,y,z,w)\colon [(xy)z]w = [(xw)z]y \text{ and } x[(yz)w] = x[(wz)y] \end{aligned} \tag{3.1}$$

Our recent discovery, and the justification for this article, is this.

Theorem 3.5. *Let L, N, u, θ and ϕ be as described in Section 2 and suppose that L is a Bol loop. Then*

(1) $(u, n) = 1$ *for all* $n \in N$ *if and only if* $\theta = I$.
(2) *The following are equivalent:*
 (a) $n_1(n_2\phi) = (n_1n_2)\phi$ *for all* $n_1, n_2 \in N$.
 (b) $\phi = R(u^2)$.

(c) $(u, nu) = 1$ *for all* $n \in N$.

(3) *If* L *is not associative, then, given* (1) *or* (2), L *is SRAR.*

(As a result of Lemma 3.2, it is not possible to have both (1) and (2), for then L would be a group.)

Proof: 1. The equivalence of the two assertions is a direct consequence of the definition of θ.

2. To justify the equivalence of the assertions, suppose that $n_1(n_2\phi) = (n_1n_2)\phi$ for all $n_1, n_2 \in N$. In particular then, $nu^2 = n(1\phi) = n\phi$, so $\phi = R(u^2)$. Conversely, if $\phi = R(u^2)$, then $n_1(n_2\phi) = n_1(n_2u^2) = (n_1n_2)u^2 = (n_1n_2)\phi$ for all $n_1, n_2 \in N$. Furthermore, if $\phi = R(u^2)$ and $n \in N$, then $u(nu) = n\phi = nu^2 = (nu)u$, giving $(u, nu) = 1$. On the other hand, if $(u, nu) = 1$, then $n\phi = u(nu) = (nu)u = nu^2$, so that $\phi = R(u^2)$.

3. If $x \in N$, then $[(xy)z]w = [x(yz)]w = x[(yz)w]$ and $[(xw)z]y = [x(wz)]y = x[(wz)y]$, so that $D(x, y, z, w)$ holds. If $x = nu$ with $n \in N$, then $[(xy)z]w = \{[(nu)y]z\}w = \{[n(uy)]z\}w = \{n[(uy)z]\}w = n\{[(uy)z]w\}$. Similarly $x[(yz)w] = n\{u[(yz)w]\}$, $[(xw)z]y = n\{[(uw)z]y\}$ and $x[(wz)y] = n\{u[(wz)y]\}$, so that, when attempting to verify conditions D, E or F of (3.1) with $x \in Nu$, there is no loss of generality if we assume that $x = u$. This leaves eight cases:

$$
(3.2)\qquad
\begin{array}{ll}
\text{Case 1:} & x = u,\ y = n_1,\ \ z = n_2,\ \ w = n_3 \\
\text{Case 2:} & x = u,\ y = n_1,\ \ z = n_2,\ \ w = n_3u \\
\text{Case 3:} & x = u,\ y = n_1,\ \ z = n_2u,\ w = n_3 \\
\text{Case 4:} & x = u,\ y = n_1,\ \ z = n_2u,\ w = n_3u \\
\text{Case 5:} & x = u,\ y = n_1u,\ z = n_2,\ \ w = n_3 \\
\text{Case 6:} & x = u,\ y = n_1u,\ z = n_2,\ \ w = n_3u \\
\text{Case 7:} & x = u,\ y = n_1u,\ z = n_2u,\ w = n_3 \\
\text{Case 8:} & x = u,\ y = n_1u,\ z = n_2u,\ w = n_3u.
\end{array}
$$

Since interchanging w and y changes $[(xy)z]w$ to $[(xw)z]y$, and $x[(yz)w]$ to $x[(wz)y]$, and vice versa, Case 5 is essentially the same as Case 2, and Case 7 is essentially the same as Case 4, so we are left with six cases to consider.

We use the fact that the left nucleus is an abelian group to freely eliminate parentheses and commute elements of N, when possible. We will also use the multiplication laws (2.2), (2.3) and (2.4) without further comment.

In Case 1, $$[(xy)z]w = [(un_1)n_2]n_3 = [(n_1\theta u)n_2]n_3 = [(n_1\theta n_2\theta)u]n_3 = (n_1\theta n_2\theta n_3\theta)u,$$

$$x[(yz)w] = u(n_1n_2n_3) = (n_1n_2n_3)\theta u,$$

$$[(xw)z]y = [(un_3)n_2]n_1 = [(n_3\theta u)n_2]n_1 = [(n_2\theta n_3\theta)u]n_1$$
$$= (n_1\theta n_2\theta n_3\theta)u,$$

and
$$x[(wz)y] = u(n_3n_2n_1) = u(n_1n_2n_3) = (n_1n_2n_3)\theta u.$$

Thus, $[(xy)z]w = [(xw)z]y$ and $x[(yz)w] = x[(wz)y]$, and so $F(x,y,z,w)$ holds, regardless of θ and ϕ.

In Case 2,
$$[(xy)z]w = [(un_1)n_2](n_3u)$$
$$= [(n_1\theta u)n_2](n_3u) = [(n_1\theta n_2\theta)u](n_3u)$$
$$= n_1\theta n_2\theta n_3\phi,$$

$$x[(yz)w] = u[(n_1n_2)(n_3u)] = u[(n_1n_2n_3)u] = (n_1n_2n_3)\phi,$$

$$[(xw)z]y = \{[u(n_3u)]n_2\}n_1 = (n_3\phi)n_2n_1 = n_1n_2(n_3\phi),$$

and
$$x[(wz)y] = u\{[(n_3u)n_2]n_1\} = u\{[(n_2\theta n_3)u]n_1\}$$
$$= u[(n_1\theta n_2\theta n_3)u] = (n_1\theta n_2\theta n_3)\phi.$$

If $\theta = I$, $[(xy)z]w = n_1n_2(n_3\phi) = [(xw)z]y$ and $x[(yz)w] = (n_1n_2n_3)\phi = x[(wz)y]$, giving $F(x,y,z,w)$. On the other hand, if $\phi = R(u^2)$, then $n_1(n_2\phi) = (n_1n_2)\phi$, so $[(xy)z]w = (n_1\theta n_2\theta)(n_3\phi) = x[(wz)y]$ and $x[(yz)w] = (n_1n_2n_3)\phi = [(xw)z]y$, giving $E(x,y,z,w)$.

In Case 3,
$$[(xy)z]w = [(un_1)(n_2u)]n_3 = [(n_1\theta u)(n_2u)]n_3$$
$$= (n_1\theta)(n_2\phi)n_3,$$

$$x[(yz)w] = u\{[n_1(n_2u)]n_3\} = u\{[(n_1n_2)u]n_3\}$$
$$= u\{[n_1n_2(n_3\theta)]u\} = [n_1n_2(n_3\theta)]\phi,$$

$$[(xw)z]y = [(un_3)(n_2u)]n_1 = [(n_3\theta u)(n_2u)]n_1$$
$$= n_3\theta n_2\phi n_1 = n_1(n_2\phi)(n_3\theta),$$

and
$$x[(wz)y] = u\{[n_3(n_2u)]n_1\} = u\{[(n_2n_3)u]n_1\}$$
$$= u\{[n_2n_3(n_1\theta)]u\} = [(n_1\theta)n_2n_3]\phi.$$

Again, if $\theta = I$, then $[(xy)z]w = n_1(n_2\phi)n_3 = [(xw)z]y$ and $x[(yz)w] = (n_1n_2n_3)\phi = x[(wz)y]$ and so $F(x,y,z,w)$ holds. And if $\phi = R(u^2)$, then $(n_1n_2)\phi = n_1(n_2\phi)$ and $[(xy)z]w = [(n_1\theta)n_2n_3]\phi = x[(wz)y]$ and $x[(yz)w] =$

$[n_1n_2(n_3\theta)]\phi = [(xw)z]y$, giving $E(x,y,z,w)$.

In Case 4,
$$\begin{aligned}[(xy)z]w &= [(un_1)(n_2u)](n_3u) = [(n_1\theta u)(n_2u)](n_3u)\\ &= (n_1\theta n_2\phi)(n_3u) = [(n_1\theta)(n_2\phi)n_3]u,\end{aligned}$$
$$\begin{aligned}x[(yz)w] &= u\{[n_1(n_2u)](n_3u)\} = u\{[(n_1n_2)u](n_3u)\}\\ &= u[n_1n_2(n_3\phi)] = [n_1n_2(n_3\phi)]\theta u,\end{aligned}$$
$$\begin{aligned}[(xw)z]y &= \{[u(n_3u)](n_2u)\}n_1 = [(n_3\phi)(n_2u)]n_1\\ &= \{[n_2(n_3\phi)]u\}n_1 = [(n_1\theta)n_2(n_3\phi)]u,\end{aligned}$$
and
$$\begin{aligned}x[(wz)y] &= u\{[(n_3u)(n_2u)]n_1\} = u[n_3(n_2\phi)n_1]\\ &= [n_1(n_2\phi)n_3]\theta u.\end{aligned}$$

If $\theta = I$, we find $[(xy)z]w = [n_1(n_2\phi)n_3]u = x[(wz)y]$ and $x[(yz)w] = [n_1n_2(n_3\phi)]u = [(xw)z]y$, so $E(x,y,z,w)$ holds, while if $\phi = R(u^2)$, then $n_1(n_2\phi) = (n_1n_2)\phi$, $[(xy)z]w = [(n_1\theta)n_2n_3]\phi u = [(xw)z]y$ and $x[(yz)w] = (n_1n_2n_3)\phi\theta u = x[(wz)y]$, giving $F(x,y,z,w)$.

In Case 6,
$$\begin{aligned}[(xy)z]w &= \{[u(n_1u)]n_2\}(n_3u) = (n_1\phi n_2)(n_3u)\\ &= (n_1\phi n_2n_3)u,\end{aligned}$$
$$\begin{aligned}x[(yz)w] &= u\{[(n_1u)n_2](n_3u)\} = u[\{[n_1(n_2\theta)]u\}(n_3u)]\\ &= u[n_1(n_2\theta)(n_3\phi)] = [n_1(n_2\theta)(n_3\phi)]\theta u,\end{aligned}$$
$$\begin{aligned}[(xw)z]y &= \{[u(n_3u)]n_2\}(n_1u) = (n_3\phi n_2)(n_1u)\\ &= [n_1n_2(n_3\phi)]u,\end{aligned}$$
and
$$\begin{aligned}x[(wz)y] &= u\{[(n_3u)n_2](n_1u)\} = u[\{[(n_2\theta)n_3]u\}(n_1u)]\\ &= u[(n_2\theta)n_3(n_1\phi)] = [(n_1\phi)(n_2\theta)n_3]\theta u.\end{aligned}$$

If $\theta = I$, we have $[(xy)z]w = [(n_1\phi)n_2n_3]u = x[(wz)y]$ and $x[(yz)w] = [n_1n_2(n_3\phi)]u = [(xw)z]y$, so $E(x,y,z,w)$ holds, while if $\phi = R(u^2)$, then $n_1(n_2\phi) = (n_1n_2)\phi$, $[(xy)z]w = (n_1n_2n_3)\phi u = [(xw)z]y$ and $x[(yz)w] = [n_1(n_2\theta)n_3]\phi\theta u = x[(wz)y]$, so $F(x,y,z,w)$ holds.

In Case 8,
$$\begin{aligned}[(xy)z]w &= \{[u(n_1u)](n_2u)\}(n_3u) = [(n_1\phi)(n_2u)](n_3u)\\ &= \{[(n_1\phi)n_2]u\}(n_3u) = (n_1\phi)n_2(n_3\phi),\end{aligned}$$
$$\begin{aligned}x[(yz)w] &= u\{[(n_1u)(n_2u)](n_3u)\} = u\{[n_1(n_2\phi)](n_3u)\}\\ &= u\{[n_1(n_2\phi)n_3]u\} = [n_1(n_2\phi)n_3]\phi,\end{aligned}$$

$$\begin{aligned}[(xw)z]y &= \{[u(n_3u)](n_2u)\}(n_1u) = [(n_3\phi)(n_2u)](n_1u)\\ &= \{[(n_3\phi)n_2]u\}(n_1u) = (n_3\phi)n_2(n_1\phi)\\ &= (n_1\phi)n_2(n_3\phi),\end{aligned}$$

and
$$\begin{aligned}x[(wz)y] &= u\{[(n_3u)(n_2u)](n_1u)\} = u\{[n_3(n_2\phi)](n_1u)\}\\ &= u\{[n_1(n_2\phi)n_3]u\} = [n_1(n_2\phi)n_3]\phi.\end{aligned}$$

Here, we have $[(xy)z]w = (n_1\phi)n_2(n_3\phi) = [(xw)z]y$ and $x[(yz)w] = [n_1(n_2\phi)n_3]\phi = x[(wz)y]$ regardless of θ and ϕ, and so $F(x,y,z,w)$ holds. [If $\phi = R(u^2)$, then $n_1(n_2\phi) = (n_1n_2)\phi$, all four terms are equal and $D(x,y,z,w)$ and $E(x,y,z,w)$ hold as well.] □

We use Theorem 3.5 to show that many of the loops described in Section 2 are in fact SRAR. The terminology implies that such loops are Bol but not associative and hence not Moufang. (See Remark 3.3.)

Lemma 3.6. *Let L, N, u, θ, and ϕ be as described in Section 2. Then L is a Bol loop if either*

(i) $\theta = I$, *and*

$$(n_1^2n_2)\phi = n_1^2(n_2\phi) \tag{3.3}$$

and

$$(n_1\phi)^2n_2 = n_1^2(n_2\phi^2) \tag{3.4}$$

for all $n_1, n_2 \in N$, or

(ii) $\phi = R(u^2)$, *and*

$$(n_1^2n_2)\theta = (n_1\theta)^2(n_2\theta) \qquad (\textit{thus } \theta \textit{ is a semiautomorphism of } N), \tag{3.5}$$

and

$$[n_1^2(n_2\theta)u^2]\theta = n_1^2n_2u^2 \tag{3.6}$$

for all $n_1, n_2 \in N$.

In fact, if $\theta = I$, then L is a Bol loop if and only if equations (3.3) and (3.4) hold; and if $\phi = R(u^2)$, then L is Bol if and only if equations (3.5) and (3.6) hold.

Proof: As in the proof of Theorem 3.5, if $x = n \in N$, then $[(xy)z]y = [(ny)z]y = [n(yz)]y = n[(yz)y] = x[(yz)y]$, so there is no loss of generality if we assume that $x \notin N$. Also as before, if $x = nu$, then $[(xy)z]y = \{[(nu)y]z\}y = \{[n(uy)]z\}y =$

$\{n[(uy)z]\}y = n\{[(uy)z]y\}$ and, similarly, $x[(yz)y] = n\{u[(yz)y]\}$, so, when checking the Bol identity $[(xy)z]y = x[(yz)y]$, there is no loss of generality if we assume that $x = u$.

This leaves four cases:

Case 1: $y = n_1$, $z = n_2$;
Case 2: $y = n_1$, $z = n_2u$;
Case 3: $y = n_1u$, $z = n_2$;
Case 4: $y = n_1u$, $z = n_2u$.

Suppose that $\theta = I$.

In Case 1,
$$\begin{aligned}[(xy)z]y &= [(un_1)n_2]n_1 = [(n_1u)n_2]n_1 = [(n_1n_2)u]n_1 \\ &= (n_1^2n_2)u\end{aligned}$$

and
$$x[(yz)y] = u[(n_1n_2)n_1] = u(n_1^2n_2) = (n_1^2n_2)u,$$

so the Bol identity holds in this case.

In Case 2,
$$\begin{aligned}[(xy)z]y &= [(un_1)(n_2u)]n_1 = [(n_1u)(n_2u)]n_1 \\ &= [n_1(n_2\phi)]n_1 = n_1^2(n_2\phi)\end{aligned}$$

and
$$\begin{aligned}x[(yz)y] &= u\{[n_1(n_2u)]n_1\} = u\,\{[(n_1n_2)u]n_1\} \\ &= u[(n_1^2n_2)u] = (n_1^2n_2)\phi,\end{aligned}$$

so the Bol identity holds in this case if and only if $n_1^2(n_2\phi) = (n_1^2n_2)\phi$, that is, if and only if equation (3.3) holds for all $n_1, n_2 \in N$.

In Case 3,
$$\begin{aligned}[(xy)z]y &= \{[u(n_1u)]n_2\}(n_1u) = [(n_1\phi)n_2](n_1u) \\ &= [n_1(n_1\phi)n_2]u\end{aligned}$$

and
$$\begin{aligned}x[(yz)y] &= u\{[(n_1u)n_2](n_1u)\} = u\{[(n_1n_2)u](n_1u)\} \\ &= u[(n_1n_2)n_1\phi] = [n_1(n_1\phi)n_2]u,\end{aligned}$$

so the Bol identity holds in this case.

In Case 4,
$$\begin{aligned}[(xy)z]y &= \{[u(n_1u)](n_2u)\}(n_1u) = \{[(n_1\phi)](n_2u)\}(n_1u) \\ &= \{[(n_1\phi)n_2]u\}(n_1u) = (n_1\phi)n_2(n_1\phi) \\ &= (n_1\phi)^2n_2\end{aligned}$$

and
$$\begin{aligned}x[(yz)y] &= u\{[(n_1u)(n_2u)](n_1u)\} = u\{[n_1(n_2\phi)](n_1u)\} \\ &= u\{[n_1^2(n_2\phi)]u\} = [n_1^2(n_2\phi)]\phi,\end{aligned}$$

so the Bol identity holds in this case if and only if $(n_1\phi)^2n_2 = [n_1^2(n_2\phi)]\phi$. But in Case 2, we saw that $(m^2n)\phi = m^2(n\phi)$ is necessary for L to be Bol, so, with this assumption, the condition in Case 4 becomes $(n_1\phi)^2n_2 = n_1^2(n_2\phi^2)$; that is, equation (3.4). This proves the lemma in the case that $\theta = I$.

Now we consider the other possibility, $\phi = R(u^2)$. Again, we have the same cases as above to consider.

In Case 1, $x = u$, $y = n_1$ and $z = n_2$, so

$$\begin{aligned}[(xy)z]y &= [(un_1)n_2]n_1 \\ &= [(n_1\theta u)n_2]n_1 = \{[(n_1\theta)(n_2\theta)]u\}n_1 = [(n_1\theta)^2(n_2\theta)]u\end{aligned}$$

and

$$x[(yz)y] = u[(n_1n_2)n_1] = u(n_1^2n_2) = (n_1^2n_2)\theta u.$$

Thus, the Bol identity holds in this case if and only if $(n_1^2n_2)\theta = (n_1\theta)^2(n_2\theta)$ for all $n_1, n_2 \in N$; that is, if and only if equation (3.5) holds. Since N is abelian, this is equivalent to saying that θ is a semiendomorphism and hence a semiautomorphism of N.

In Case 2, $x = u$, $y = n_1$ and $z = n_2u$, so

$$\begin{aligned}[(xy)z]y &= [(un_1)(n_2u)]n_1 = [(n_1\theta u)(n_2u)]n_1 = [(n_1\theta)(n_2\phi)]n_1 \\ &= n_1(n_1\theta)(n_2\phi) = n_1(n_1\theta)n_2u^2\end{aligned}$$

and

$$\begin{aligned}x[(yz)y] &= u\{[n_1(n_2u)]n_1\} = u\{[(n_1n_2)u]n_1\} = u\{[(n_1n_2)n_1\theta]u\} \\ &= [n_1(n_1\theta)n_2]\phi = n_1(n_1\theta)n_2u^2.\end{aligned}$$

In this case, the Bol identity holds regardless of θ.

In Case 3, $x = u$, $y = n_1u$ and $z = n_2$, so

$$\begin{aligned}[(xy)z]y &= \{[u(n_1u)]n_2\}(n_1u) = [(n_1\phi)n_2](n_1u) = [n_1(n_1\phi)n_2]u \\ &= (n_1^2n_2u^2)u\end{aligned}$$

and

$$\begin{aligned}x[(yz)y] &= u\{[(n_1u)n_2](n_1u)\} = u[\{[n_1(n_2\theta)]u\}(n_1u)] = u[n_1(n_1\phi)(n_2\theta)] \\ &= [n_1(n_1\phi)(n_2\theta)]\theta u = [n_1^2(n_2\theta)u^2]\theta u,\end{aligned}$$

and the Bol identity holds if and only if $[n_1^2(n_2\theta)u^2]\theta = n_1^2n_2u^2$; that is, if and only if equation (3.6) holds for all $n_1, n_2 \in N$.

Finally, in Case 4, $x = u$, $y = n_1u$ and $z = n_2u$, so

$$\begin{aligned}[(xy)z]y &= \{[u(n_1u)](n_2u)\}(n_1u) = [(n_1\phi)(n_2u)](n_1u)\\ &= \{[(n_1\phi)n_2]u\}(n_1u)] = (n_1\phi)^2n_2 = n_1^2n_2u^4\end{aligned}$$

and

$$\begin{aligned}x[(yz)y] &= u\{[(n_1u)(n_2u)](n_1u)\} = u\{[n_1(n_2\phi)](n_1u)\}\\ &= u[n_1^2(n_2\phi)u] = [n_1^2(n_2\phi)]\phi = n_1^2n_2u^4.\end{aligned}$$

Here, the right Bol identity holds regardless of θ. This completes the proof. □

Remarks 3.7. Let L be a loop described by Lemma 3.6. Suppose condition (i) of that lemma is the case.

(1) Setting $n_2 = 1$ in equation (3.3), we get $(n_1^2\phi) = n_1^2(1\phi) = n_1^2u^2$, for all $n_1 \in N$, so the restriction of ϕ to the set $S = \{x^2 \mid x \in N\}$ is always $R(u^2)$.

(2) Setting $n_1 = n_2 = n$ in (3.4) gives $n(n\phi)^2 = n^2(n\phi^2)$, or, since N is a group, $(n\phi)^2 = n(n\phi^2)$. Multiplying both sides of (3.4) by n_2 now gives $(n_1\phi)^2n_2^2 = n_1^2n_2(n_2\phi^2) = n_1^2(n_2\phi)^2$. Thus, $[n^{-1}(n\phi)]^2 = c$ is an invariant. In particular, since $1\phi = u^2$, $c = [1^{-1}(1\phi)]^2 = u^4$. That is, $[n^{-1}(n\phi)]^2 = u^4$ for all $n \in N$.

Now suppose condition (ii) of Lemma 3.6 is the case.

(3) Setting $n_2 = 1$ in equation (3.5), we get $n_1^2\theta = (n_1\theta)^2$. (This is a property of any semiendomorphism.)

(4) Setting $n_1 = n_2 = 1$ in equation (3.6), we get $u^2\theta = u^2$. On the other hand, if we just set $n_2 = 1$, we get $(n_1^2u^2)\theta = n_1^2u^2$. But then, using equation (3.5), we get $n_1^2u^2 = (n_1^2u^2)\theta = (n_1\theta)^2(u^2\theta) = (n_1\theta)^2u^2$, so that

$$(n_1\theta)^2 = n_1^2 \quad \text{for any } n_1 \in N.$$

Together with property (3) above, we see that θ fixes the elements of $S = \{x^2 \mid x \in N\}$. Also, if we apply equation (3.5) to equation (3.6), we get $n_1^2n_2u^2 = [n_1^2(n_2\theta)u^2]\theta = (n_1\theta)^2[(n_2\theta)u^2]\theta = n_1^2[(n_2\theta)u^2]\theta$, so

$$[(n_2\theta)u^2]\theta = n_2u^2 \quad \text{for any } n_2 \in N.$$

(5) If $u^2 = n^2$ for some $n \in N$, then $n_2u^2 = [(n_2\theta)u^2]\theta = [n^2(n_2\theta)]\theta = (n\theta)^2n_2\theta^2$ (by (3.5)) $= n^2n_2\theta^2 = u^2n_2\theta^2$, so $n_2\theta^2 = n_2$. Thus $\theta^2 = I$, the identity map.

4. Examples

In this section, we give examples of SRAR loops with a variety of properties. Of significance is that a number of the loops we exhibit have more than a single nonidentity commutator/associator. Our general approach is to start with an abelian group N and to construct a loop $L = N \cup Nu$ via maps θ and ϕ as in Section 2, then to use Lemmas 3.6 and 3.2 to be assured the loop is Bol but not Moufang, and finally conclude that our loop is SRAR by virtue of Theorem 3.5.

In our first three examples, we assume that $\theta = I$, so that $un = nu$ for all $n \in N$. In the next four, we assume that $\phi = R(u^2)$. Consequently, by Theorem 3.5, each example produces an SRAR loop provided it is Bol, but not associative, properties we check in each case.

Example 4.1. Let N be an elementary abelian 2-group of order at least 8, let $\theta = I$ and let ϕ be any nonidentity permutation on N satisfying $\phi^2 = I$ which is not a right multiplication map. Since the square of any element of N is 1, equation (3.3) reduces to the tautology $n_2\phi = n_2\phi$ and (3.4) reduces to $n_2 = n_2$. By Lemma 3.6, L is a Bol loop. That L is not Moufang follows from part iii of Lemma 3.2 and the subsequent remark, so L is SRAR by Theorem 3.5. In many cases, we have $|L'| > 2$.

For example, suppose $\langle a\rangle = C_2$, $\langle b\rangle = C_2$ and $\langle c\rangle = C_2$ are three factors of N and ϕ interchanges 1 and a, and b and c, and fixes abc. We have

$$n_1(n_1\phi)^{-1}n_2(n_2\phi)^{-1} = n_1(n_1\phi)n_2(n_2\phi) = \begin{cases} 1 & \text{if } n_1 = 1,\ n_2 = a \\ a & \text{if } n_1 = 1,\ n_2 = abc \\ abc & \text{if } n_1 = a,\ n_2 = b. \end{cases}$$

Thus, by Lemma 3.1, $|L'| > 2$.

In this case, $(nu)^2 = (nu)(nu) = n(n\phi)$, so that we can easily determine the order of each element of the loops constructed (thereby making it easy to see that certain loops are not isomorphic). Taking $N = \langle a\rangle \times \langle b\rangle \times \langle c\rangle = C_2 \times C_2 \times C_2$, we have many possible permutations of order 2, each giving rise to a Bol loop of order 16 that is SRAR. Here are three with $|L'| > 2$.

ϕ								u^2	No. of els of	
1	a	b	c	ab	ac	bc	abc		order 2	order 4
a	1	c	b	ab	ac	bc	abc	a	11	4
a	1	c	b	ac	ab	bc	abc	a	9	6
a	1	c	b	ac	ab	abc	bc	1	7	8

Example 4.2. Let N be an abelian group of exponent 4 (but not of exponent 2). Let $\theta = I$ and define $\phi\colon N \to N$ by $n\phi = n^{-1}$. Noting that $\phi^2 = I$ and $n^{-2} = n^2$ for any $n \in N$, we have, for any $n_1, n_2 \in N$, $(n_1^2 n_2)\phi = n_1^{-2}n_2^{-1} = n_1^2(n_2\phi)$,

so (3.3) holds, and $(n_1\phi)^2n_2 = n_1^{-2}n_2 = n_1^2(n_2\phi^2)$, so equation (3.4) holds too. Thus, L is a Bol loop. Since $u^2 = 1\phi = 1$, $\phi \neq R(u^2)$ and L is not Moufang by Lemma 3.2 and Remark 3.3, thus L is SRAR by Theorem 3.5.

Since
$$n_1(n_1\phi)^{-1}n_2(n_2\phi)^{-1} = n_1^2n_2^2 = (n_1n_2)^2,$$
any square in N is in L'. Thus, for example, with $N = \langle a\rangle \times \langle b\rangle = C_4 \times C_4$, we get a nonassociative SRAR Bol loop of order 32 with $|L'| \geq 4$.

The family of loops described here actually coincides with a class of non-Moufang Bol loops containing an abelian group as a subloop of index 2 discussed by P. Vojtěchovský in [Voj04] and denoted $G(\theta_{xy}, \theta_{xy}, \theta_{xy}, \theta_{x^{-1}y})$ in that paper. (We caution the reader, however, that our loops are the opposites of Vojtěchovský's, whose Bol loops are left Bol.)

Example 4.3. Let N be an abelian group with an element a of order 4. Let $e = a^2$ and $S = \{x^2 \mid x \in N\}$ be the set of squares in N. Note that $e \in S$. Let $\theta = I$ and define ϕ by $n\phi = n$ if $n \in S$ and $n\phi = en$ otherwise. Then $u^2 = 1\phi = 1$. Now the product of two elements in S is in S while the product of an element in S and an element not in S is not in S. Thus $(n_1^2n_2)\phi = n_1^2n_2 = n_1^2(n_2\phi)$ if $n_2 \in S$, and $(n_1^2n_2)\phi = e(n_1^2n_2) = n_1^2(en_2) = n_1^2(n_2\phi)$ if $n_2 \notin S$. Thus, equation (3.3) holds. Also, for any $n \in N$, $n^2 = (en)^2 = (n\phi)^2$ and $n\phi^2 = n$, so that equation (3.4) holds. Thus, L is a Bol loop and Lemma 3.1 gives $L' = \{1, e\}$.

To see that L is SRAR, it suffices to show that L is not associative (Remark 3.3), and this follows from Lemma 3.2 because ϕ is not a right multiplication map.

In each of the examples above, if we replace N by $N \times A$, where A is any abelian group, and extend θ and ϕ so that θ is still the identity map on $N \times A$ and so that $(na)\phi = (n\phi)a$ for $a \in A$, then we get a loop L which is just the direct product of the loop described above with the abelian group A. It is not hard to show directly that equations (3.3) and (3.4) still hold, so the extended loop is still Bol. Since ϕ is not a right multiplication map on N, it is not a right multiplication map on $N \times A$, so L is not associative by Lemma 3.2, hence SRAR by Theorem 3.5.

Example 4.4. Let N be an abelian group of exponent 4 (but not of exponent 2), let u^2 be any element of order 2 in N, let $\phi = R(u^2)$ and let $n\theta = n^{-1}$ for all $n \in N$. Then
$$(n_1^2n_2)\theta = n_1^{-2}n_2^{-1} = (n_1^{-1})^2n_2^{-1} = (n_1\theta)^2(n_2\theta),$$
so equation (3.5) holds. Also
$$[n_1^2(n_2\theta)u^2]\theta = [n_1^2n_2^{-1}u^2]^{-1} = n_1^{-2}n_2u^{-2} = n_1^2n_2u^2,$$
so equation (3.6) holds as well and L is a Bol loop. Since $\theta^2 = I$, θ is a homomorphism, and $R(u^2) = \phi \neq \theta R(u^2)$, L is not Moufang by part iii of Lemma 3.2

and Remark 3.3, so L is SRAR by Theorem 3.5. Lemma 3.1 shows that the loops here often have $|L'| > 2$ because $n(n\theta)^{-1} = n^2$, so $S = \{x^2 \mid x \in N\} \subseteq L'$.

The family of loops described in this example is another of those discussed by P. Vojtěchovský in [Voj04], specifically the class Vojtěchovský labels $G(\theta_{xy}, \theta_{xy}, \theta_{x^{-1}y}, \theta_{xy})$. (Again, our loops are the opposite of Vojtěchovský's.)

Example 4.5. Let N, S and e be as in Example 4.3. Define θ by $n\theta = n$ if $n \in S$ and $n\theta = en$ otherwise. Choose $u^2 \in S$ and let $\phi = R(u^2)$. Note that, regardless of whether or not n is in S, $n^2\theta = n^2 = (n\theta)^2$. Also, $n\theta \in S$ if and only if $n \in S$.

Since the product of two elements of S is in S while the product of an element in S and an element not in S is not in S, $(n_1^2 n_2)\theta = n_1^2 n_2 = (n_1\theta)^2 n_2 = (n_1\theta)^2(n_2\theta)$, if $n_2 \in S$; and $(n_1^2 n_2)\theta = en_1^2 n_2 = e(n_1\theta)^2 n_2 = (n_1\theta)^2(n_2\theta)$, if $n_2 \notin S$. Thus, in either case, (3.5) holds. Also, it is easy to see that $\theta^2 = I$. Therefore, since $u^2 \in S$, $[n_1^2(n_2\theta)u^2]\theta = n_1^2 n_2 u^2$ if $n_2 \in S$; and $[n_1^2(n_2\theta)u^2]\theta = en_1^2(n_2\theta)u^2 = n_1^2 e(n_2\theta)u^2 = n_1^2(n_2\theta^2)u^2 = n_1^2 n_2 u^2$ if $n_2 \notin S$. Thus, again, in either case, (3.6) holds and so L is a Bol loop.

To see that L is SRAR, we must see that it is not Moufang, which, by Remark 3.3, is equivalent to seeing that it is not a group. For $c \notin S$, $(uc)u = (c\theta u)u = ecu^2$, while $u(cu) = c\phi = cu^2$, so $(u, c, u) = e \neq 1$. In this case, we also have $L' = \{1, e\}$ of order 2.

Example 4.6. Let $N = \langle a \rangle$ be a cyclic group of order $4m$, let $u^2 = a^{2k}$ for some integer k, let $\phi = R(u^2)$, and define θ by $n\theta = n^{2m+1}$. Then

$$(n_1^2 n_2)\theta = (n_1^2 n_2)^{2m+1} = n_1^{4m+2} n_2^{2m+1} = (n_1^{2m+1})^2 n_2^{2m+1} = (n_1\theta)^2(n_2\theta),$$

so equation (3.5) holds. Also,

$$\begin{aligned}[n_1^2(n_2\theta)u^2]\theta = [n_1^2 n_2^{2m+1} a^{2k}]\theta &= n_1^{4m+2} n_2^{(2m+1)^2} a^{(4m+2)k} \\ &= n_1^2 n_2^{4m^2+4m+1} a^{2k} = n_1^2 n_2 u^2,\end{aligned}$$

so equation (3.6) holds too. Thus, L is a Bol loop. Nonassociativity follows as in Example 4.4. Thus L is SRAR. Here too $|L'| = 2$ because Lemma 3.1 shows $L' = \{1, a^{2m}\}$.

Again, in Examples 4.4, 4.5 and 4.6, we can replace N by $N \times A$, where A is any abelian group, and extend θ and ϕ so that $(na)\theta = (n\theta)a$ and $(na)\phi = (n\phi)a$ for $a \in A$. As before, we get a loop that is just the direct product of the loop described in the appropriate example with the abelian group A and it can be verified that the new loop remains SRAR.

Our final example is motivated by a question of a referee who wondered if the construction of a Bol loop with prescribed left nucleus that we have described in this paper always leads to a nilpotent loop. Such is not the case.

Example 4.7. Let N be an elementary abelian 2-group of order at least 8 and suppose that $\langle a\rangle$, $\langle b\rangle$ and $\langle v\rangle$ are three factors of N. Let L be the loop constructed as in Section 2 (where $u^2 = v$). Define $\alpha : N \to N$ as the map that interchanges 1 and v, interchanges a and b, and fixes all other elements of N. Let $\phi = R(v)$ and $\theta = \alpha\phi$. Equation (3.5) is trivially satisfied because N has exponent 2, while (3.6) reduces to $[(n\theta)v]\theta = nv$, which is $\theta\phi\theta = \phi$. This holds because both $\theta\phi = \alpha$ and ϕ have order 2. Thus L is a Bol loop by Lemma 3.6. Now $a\theta = a\alpha\phi = b\phi = bv$. Since $bv \notin \{1, a, b, v\}$, bv is fixed by α, so $(a\theta)\theta = (bv)\theta = (bv)\alpha\phi = bv^2 = b$. So $\theta^2 \neq I$ and L is not a group by Lemma 3.2. Hence L is SRAR by Theorem 3.5. We claim that the centre of L is trivial. For this, with reference to Lemma 3.4, it is sufficient to show that if $1 \neq n \in N$, then there exists some $x \in N$ with $(nx)\theta \neq n \cdot x\theta$. If $n \neq v$ then $x = n$ satisfies this condition. To see this, $(nx)\theta = n^2\theta = 1\theta = 1$ while $n \cdot x\theta = 1$ if and only if $n\theta = n$. But if $n = n\theta = n\alpha v$, then $n\alpha = nv$. However, for $n \notin \{1, a, b, v\}$, $n\alpha = n \neq nv$ while $a\alpha = b \neq av$ and $b\alpha = a \neq bv$, so the only possible central elements are 1 and v. In the case that $n = v$, let $x = a$ and observe that $(nx)\theta = (av)\alpha\phi = (av)\phi = a$ whereas $n \cdot x\theta = v(a\alpha\phi) = v(b\phi) = v(bv) = b$. As claimed $\mathcal{Z}(L) = \{1\}$. Finally, we remark that the loop L constructed here cannot have $|L'| = 2$ since a unique nonidentity commutator/associator is always central.

While the thrust of this paper has been to show that an index 2 left nucleus is often sufficient to guarantee an SRAR loop with more than a single nonidentity commutator/associator, several of the above examples show that $|L'| = 2$ is also possible. At present, to the best of our knowledge, all known SRAR loops either have $|L'| = 2$ or satisfy the conditions of Theorem 3.5.

5. Motivation

The reader may wonder why throughout Section 3 we assumed that N is abelian and how the conditions $\theta = I$ and $\phi = R(u^2)$ arose. The purpose of this section is to provide answers.

Recall — see (3.1) — that a Bol loop L is SRAR if and only if it is not associative and for every $x, y, z, w \in L$, at least one of the conditions $D(x, y, z, w)$, $E(x, y, z, w)$, $F(x, y, z, w)$ must hold. When L has a left nucleus N of index 2, an investigation of these properties leads naturally to the consideration of the eight cases enumerated in (3.2). Our proof of part (3) of Theorem 3.5 showed that under certain conditions, a loop L is SRAR if and only if D, E and F hold *uniformly* in the sense that, if a condition holds in a certain case for some n_1, n_2, n_3, then it holds in that case for all n_1, n_2, n_3.

It is this assumption of uniformity that forces the hypotheses in Theorem 3.5, as we now show.

Theorem 5.1. *Suppose L is a nonassociative Bol loop with left nucleus N of index 2. Suppose that for each $n_1, n_2, n_3 \in N$ and each of the eight cases listed* *in* (3.2), *one of the conditions* D, E, F *holds uniformly. Then* N *is abelian and either* $\theta = I$ *or* $\phi = R(u^2)$.

Proof: Setting $w = 1$ and $x = u$ in each of $D(x, y, z, w)$, $E(x, y, z, w)$, $F(x, y, z, w)$, we obtain three additional conditions.

- $D'(u, y, z)$: $(uy)z = u(yz)$ and $(uz)y = u(zy)$,
- $E'(u, y, z)$: $(uy)z = u(zy)$ and $(uz)y = u(yz)$,
- $F'(u, y, z)$: $(uy)z = (uz)y$ and $u(yz) = u(zy)$.

Consider the four cases

$$\begin{array}{ll} & \text{Case 1: } y = n_1 \quad \text{and } z = n_2; \\ & \text{Case 2: } y = n_1 \quad \text{and } z = n_2u; \\ (5.1) & \text{Case 3: } y = n_1u \quad \text{and } z = n_2; \\ & \text{Case 4: } y = n_1u \quad \text{and } z = n_2u. \end{array}$$

The assumption of uniformity on D, E, F, implies uniformity of D', E', F' with respect to the four cases listed. That is, if one of D', E', F' holds in a certain case for some n_1, n_2, then it holds in that case for all n_1, n_2.

Throughout the rest of this proof, we make use, without further mention, of the fact that θ and ϕ are bijections.

In Case 1, we have $(xy)z = [(n_1\theta)(n_2\theta)]u$, $x(yz) = [(n_1n_2)\theta]u$, $(xz)y = [(n_2\theta)(n_1\theta)]u$, and $x(zy) = [(n_2n_1)\theta]u$. Thus, D' holds uniformly if and only if $[(n_1\theta)(n_2\theta)]u = [(n_1n_2)\theta]u$ and $[(n_2\theta)(n_1\theta)]u = [(n_2n_1)\theta]u$ for all n_1 and n_2, that is, if and only if θ is a homomorphism.

Similarly, E' holds uniformly if and only if $[(n_1\theta)(n_2\theta)]u = [(n_2n_1)\theta]u$ and $[(n_2\theta)(n_1\theta)]u = [(n_1n_2)\theta]u$ for all n_1 and n_2, that is, if and only if θ is an antihomomorphism.

Also, F' holds uniformly if and only if $[(n_1\theta)(n_2\theta)]u = [(n_2\theta)(n_1\theta)]u$ and $[(n_1n_2)\theta]u = [(n_2n_1)\theta]u$ for all n_1 and n_2, i.e., if and only if N is abelian.

In Case 2, $(xy)z = (n_1\theta)(n_2\phi)$, $x(yz) = (n_1n_2)\phi$, $(xz)y = (n_2\phi)n_1$, and $x(zy) = [n_2(n_1\theta)]\phi$. Thus D' holds uniformly if and only if $(n_1\theta)(n_2\phi) = (n_1n_2)\phi$ and $(n_2\phi)n_1 = [n_2(n_1\theta)]\phi$ for all n_1 and n_2. Taking $n_2 = 1$ and $n_1 = n$ in the first of these equations, we get $(n\theta)u^2 = n\phi$, so that $\phi = \theta R(u^2)$. But then, $(n_1\theta)(n_2\phi) = (n_1\theta)(n_2\theta)u^2$, and $(n_1n_2)\phi = (n_1n_2)\theta u^2$, so θ is a homomorphism. Substituting $\phi = \theta R(u^2)$ into the second equation, we get $(n_2\theta)u^2n_1 = [n_2(n_1\theta)]\theta u^2 = (n_2\theta)(n_1\theta^2)u^2$, so $u^2n_1 = (n_1\theta^2)u^2$, that is, $L(u^2) = \theta^2R(u^2)$, where $L(u^2)$ denotes left multiplication by u^2.

Conversely, if θ is a homomorphism, then $\phi = \theta R(u^2)$ and $L(u^2) = \theta^2R(u^2)$, so $(n_1\theta)(n_2\phi) = (n_1\theta)[(n_2\theta)u^2] = (n_1n_2)\theta u^2 = (n_1n_2)\theta R(u^2) = (n_1n_2)\phi$, and $(n_2\phi)n_1 = (n_2\theta)u^2n_1 = (n_2\theta)(n_1L(u^2)) = (n_2\theta)(n_1\theta)\theta u^2 = [n_2(n_1\theta)]\theta R(u^2) = [n_2(n_1\theta)]\phi$.

Thus, in Case 2, D' holds uniformly if and only if θ is a homomorphism, $\phi = \theta R(u^2)$, and $L(u^2) = \theta^2 R(u^2)$.

Similarly, E' holds uniformly in Case 2 if and only if $(n_1\theta)(n_2\phi) = [n_2(n_1\theta)]\phi$ and $(n_2\phi)n_1 = (n_1n_2)\phi$ for all n_1 and n_2. Taking $n_2 = 1$ and $n_1 = n$ in the second equation, we get $u^2n = n\phi$, so that $\phi = L(u^2)$. But then, the second equation becomes $u^2n_2n_1 = u^2n_1n_2$, so that N must be abelian. This implies that $\phi = L(u^2) = R(u^2)$.

Conversely, if N is abelian and $\phi = L(u^2)$, then $(n_1\theta)(n_2\phi) = (n_1\theta)u^2n_2 = u^2n_2(n_1\theta) = [n_2(n_1\theta)]\phi$, and $(n_2\phi)n_1 = u^2n_2n_1 = u^2n_1n_2 = (n_1n_2)\phi$.

Thus, in Case 2, E' holds uniformly if and only if N is abelian and $\phi = R(u^2)$.

Finally, in Case 2, F' holds uniformly if and only if $(n_1\theta)(n_2\phi) = (n_2\phi)n_1$ and $(n_1n_2)\phi = [n_2(n_1\theta)]\phi$ for all n_1 and n_2. Taking $n_2 = u^{-2}$ and $n_1 = n$, the first equation becomes $n\theta = n$, so that θ is the identity map. But then, the second equation becomes $(n_1n_2)\phi = [n_2n_1]\phi$, so N is abelian.

Conversely, if N is abelian and θ is the identity map, then $(n_1\theta)(n_2\phi) = (n_2\phi)n_1$ and $(n_1n_2)\phi = (n_2n_1)\phi = [n_2(n_1\theta)]\phi$.

Thus, in Case 2, F' holds uniformly if and only if N is abelian and θ is the identity map. Note that if either E' or F' holds uniformly in this case, then N must be abelian, and either $\theta = I$ or $\phi = R(u^2)$.

Case 3 is essentially the same as Case 2. If we simply interchange y and z in Case 3, we get Case 2. But interchanging y and z in the equations for D', E' and F' does not change these equations, so the same conditions apply to Case 3 that applied to Case 2. That is, in Case 3, D' holds uniformly if and only if θ is a homomorphism, $\phi = \theta R(u^2)$ and $L(u^2) = \theta^2 R(u^2)$; E' holds uniformly if and only if N is abelian and $\phi = R(u^2)$; and F' holds uniformly if and only if N is abelian and θ is the identity map.

Once again, note that if either E' or F' holds uniformly in this case, then N must be abelian, and either $\theta = I$ or $\phi = R(u^2)$.

Finally, in Case 4, $(xy)z = [(n_1\phi)n_2]u$, $x(yz) = [n_1(n_2\phi)]\theta u$, $(xz)y = [(n_2\phi)n_1]u$ and $x(zy) = [n_2(n_1\phi)]\theta u$.

Thus D' holds uniformly if and only if $(n_1\phi)n_2 = [n_1(n_2\phi)]\theta$ and $(n_2\phi)n_1 = [n_2(n_1\phi)]\theta$ for all n_1 and n_2. Taking $n_2 = 1$ and $n_1 = n$ in the first equation, we get $n\phi = (nu^2)\theta$, so that $\phi = R(u^2)\theta$. Taking $n_1 = u^{-2}$ and $n_2 = n$ in the second equation, we get $(n\phi)u^{-2} = n\theta$, so that $n\phi = (n\theta)u^2$, and so $\phi = \theta R(u^2)$. Combining these results, θ and $R(u^2)$ commute. Also, $\theta\phi = \theta R(u^2)\theta = R(u^2)\theta\theta = \phi\theta$, so that ϕ and θ commute.

Next, taking $n_1 = 1$ and $n_2 = n$ in the first equation, we get $u^2n = n\phi\theta$, so that $L(u^2) = \phi\theta = R(u^2)\theta^2 = \theta^2 R(u^2)$. Applying these results to the second equation, we see that $(n_2\phi)n_1 = (n_2\theta)u^2n_1 = (n_2\theta)[n_1L(u^2)] = (n_2\theta)[(n_1\theta)\theta]u^2$, while, on the other hand, $[n_2(n_1\phi)]\theta = [n_2(n_1\theta)u^2]\theta = [n_2(n_1\theta)]u^2\theta = [n_2(n_1\theta)]\theta u^2$, so that $(n_2\theta)([(n_1\theta)]\theta) = [n_2(n_1\theta)]\theta$, all showing that θ is a homomorphism.

Conversely, suppose that θ is a homomorphism, that $\phi = \theta R(u^2) = R(u^2)\theta$, and that $L(u^2) = \theta^2 R(u^2)$. We have $(n_1\phi)n_2 = (n_1\theta)u^2n_2 = (n_1\theta)[(n_2\theta)\theta]u^2 = [n_1(n_2\theta)]\theta u^2 = [n_1(n_2\theta)]u^2\theta = [n_1(n_2\theta)u^2]\theta = [n_1(n_2\phi)]\theta$, and, similarly, $(n_2\phi)n_1 = (n_2\theta)u^2n_1 = (n_2\theta)[(n_1\theta)\theta]u^2 = [n_2(n_1\theta)]\theta u^2 = [n_2(n_1\theta)]u^2\theta = [n_2(n_1\theta)u^2]\theta = [n_2(n_1\phi)]\theta$.

Thus, in Case 4, D' holds uniformly if and only if θ is a homomorphism, $\phi = \theta R(u^2) = R(u^2)\theta$ and $L(u^2) = \theta^2 R(u^2)$.

Similarly, in Case 4, E' holds uniformly if and only if $(n_1\phi)n_2 = [n_2(n_1\phi)]\theta$ and $[n_1(n_2\phi)]\theta = (n_2\phi)n_1$ for all n_1 and n_2. Taking $n_2 = 1$ in the first equation, we see that $n_1\phi = (n_1\phi)\theta$, so that θ is the identity map. But then $(n_1\phi)n_2 = n_2(n_1\phi)$, so that N is abelian.

Conversely, if N is abelian and θ is the identity map, then $(n_1\phi)n_2 = n_2(n_1\phi) = [n_2(n_1\phi)]\theta$, and $[n_1(n_2\phi)]\theta = n_1(n_2\phi) = (n_2\phi)n_1$. Thus, in Case 4, E' holds uniformly if and only if N is abelian and θ is the identity map.

Finally, F' holds uniformly in Case 4 if and only if $(n_1\phi)n_2 = (n_2\phi)n_1$ and $[n_1(n_2\phi)]\theta = [n_2(n_1\phi)]\theta$ for all n_1 and n_2. Canceling θ in the second equation and then setting $n_1 = 1$ and $n_2 = n$, we get $n\phi = nu^2$, so that $\phi = R(u^2)$. Next, taking $n_1 = 1$ and $n_2 = n$ in the first equation, $u^2n = nu^2$, so that $\phi = R(u^2) = L(u^2)$. But then, $(n_1\phi)n_2 = n_1u^2n_2 = n_1n_2u^2$ and $(n_2\phi)n_1 = n_2u^2n_1 = n_2n_1u^2$, so N is abelian.

Conversely, if N is abelian and $\phi = R(u^2) = L(u^2)$, then $(n_1\phi)n_2 = n_1u^2n_2 = n_1n_2u^2 = n_2n_1u^2 = n_2u^2n_1 = (n_2\phi)n_1$, and $n_1(n_2\phi) = n_1n_2u^2 = n_2n_1u^2 = n_2(n_1\phi)$ so, applying θ to both sides, $[n_1(n_2\phi)]\theta = [n_2(n_1\phi)]\theta$.

Thus, F' holds uniformly in Case 4 if and only if N is abelian and $\phi = R(u^2) = L(u^2)$. Again note that if either E' or F' holds uniformly in this case, then N must be abelian, and either $\theta = I$ or $\phi = R(u^2)$.

Suppose that N is not abelian. Then D' must hold uniformly in Cases 2, 3 and 4, and either D' or E' must hold uniformly in Case 1. If D' holds uniformly in Case 1, then, since it also holds in Cases 2, 3 and 4, it is easy to see that $(nu, y, z) = 1$ for all $y, z \in L$ and all $n \in N$. Since $(n, y, z) = 1$ for all $n \in N$ and all $y, z \in L$, we obtain $(x, y, z) = 1$ for all $x, y, z \in L$. Thus L is a group, contrary to assumption. Thus, we cannot have D' holding uniformly in Case 1, so E' must hold uniformly in Case 1. Thus θ is an antihomomorphism. But θ is also a homomorphism (since D' holds uniformly in Case 2). We conclude that N must be abelian.

Now suppose that $\theta \neq I$ and $\phi \neq R(u^2)$. Then neither E' nor F' can hold uniformly in Cases 2, 3 or 4. Thus, in these three cases, D' must hold uniformly, so θ is a homomorphism, $\phi = \theta R(u^2)$, and $L(u^2) = \theta\phi = \theta^2 R(u^2)$. Also $L(u^2) = R(u^2)$ since N is abelian, so $\theta^2 = I$. We consider associators of the form (u, x, y).

First,

$$(un_1)n_2 = [(n_1\theta)u]n_2 = (n_1\theta)(un_2) = (n_1\theta)[(n_2\theta)u]$$
$$= [(n_1\theta)(n_2\theta)]u = [(n_1n_2)\theta]u = u(n_1n_2),$$

so $(u, n_1, n_2) = 1$. Similarly,

$$(un_1)(n_2u) = [(n_1\theta)u](n_2u) = (n_1\theta)[u(n_2u)] = (n_1\theta)(n_2\phi)$$
$$= (n_1\theta)(n_2\theta)u^2 = (n_1n_2)\theta u^2 = (n_1n_2)\phi = u[(n_1n_2)u]$$
$$= u[n_1(n_2u)],$$

so $(u, n_1, n_2u) = 1$. Continuing,

$$[u(n_1u)]n_2 = (n_1\phi)n_2 = (n_1\theta)u^2n_2 = (n_1\theta)n_2u^2 = [n_1(n_2\theta)]\theta u^2$$
$$= [n_1(n_2\theta)]\phi = u\{n_1[(n_2\theta)u]\} = u[n_1(un_2)] = u[(n_1u)n_2],$$

so $(u, n_1u, n_2) = 1$. Finally, since $\theta^2 = I$ and $u(n\theta) = n\theta^2u = nu$,

$$[u(n_1u)](n_2u) = (n_1\phi)(n_2u) = [(n_1\phi)n_2]u = [(n_1\theta)u^2n_2]u$$
$$= [(n_1\theta)n_2u^2]u = u\{[(n_1\theta)n_2u^2]\theta\} = u[n_1(n_2\theta)u^2]$$
$$= u[n_1(n_2\phi)] = u\{n_1[u(n_2u)]\} = u[(n_1u)(n_2u)],$$

giving $(u, n_1u, n_2u) = 1$. All this shows that u is in the left nucleus of L, contrary to assumption. We conclude that either $\theta = I$ or $\phi = R(u^2)$ and the proof is complete. □

Theorem 5.1 still leaves open the question as to whether one of the conditions D, E, F must hold uniformly in each of the cases (3.2) in any SRAR loop (with index 2 left nucleus). The answer is no.

For example, the Bol loop L defined by the multiplication table in Figure 1 (this is the loop Moorhouse labels 16.1.2.29 [Moo]) is an SRAR loop because $|L'| = 2$. However, in at least one of the four cases (5.1), none of D', E', F' holds uniformly.[3] Among the 2033 non-Moufang Bol loops of order 16, there are only a few in which this occurs. In these loops, the only reason we know that L is SRAR is that $|L'| = 2$.

[3]This loop has a left nucleus of order 8, so uniformity in Case 2 would imply 64 triples satisfying D' or 64 triples satisfying E' or 64 triples satisfying F', whereas the respective numbers are 40, 16 and 40.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	10	1	11	12	13	15	16	14	3	4	5	9	6	8	7
3	1	10	12	11	14	16	15	13	2	5	4	6	9	7	8
4	12	11	10	1	15	14	13	16	5	3	2	8	7	6	9
5	11	12	1	10	16	13	14	15	4	2	3	7	8	9	6
6	14	13	16	15	10	11	12	1	9	8	7	2	3	5	4
7	16	15	13	14	12	10	1	11	8	6	9	4	5	2	3
8	15	16	14	13	11	1	10	12	7	9	6	5	4	3	2
9	13	14	15	16	1	12	11	10	6	7	8	3	2	4	5
10	3	2	5	4	9	8	7	6	1	12	11	14	13	16	15
11	4	5	3	2	7	9	6	8	12	10	1	15	16	14	13
12	5	4	2	3	8	6	9	7	11	1	10	16	15	13	14
13	6	9	8	7	3	4	5	2	14	16	15	10	1	11	12
14	9	6	7	8	2	5	4	3	13	15	16	1	10	12	11
15	7	8	6	9	5	3	2	4	16	13	14	12	11	10	1
16	8	7	9	6	4	2	3	5	15	14	13	11	12	1	10

Figure 1

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