On loops that are abelian groups over the nucleus and Buchsteiner loops

Piroska Csörgő

Abstract. We give sufficient and in some cases necessary conditions for the conjugacy closedness of $Q/Z(Q)$ provided the commutativity of Q/N. We show that if for some loop Q, Q/N and $\operatorname{Inn} Q$ are abelian groups, then $Q/Z(Q)$ is a CC loop, consequently Q has nilpotency class at most three. We give additionally some reasonable conditions which imply the nilpotency of the multiplication group of class at most three. We describe the structure of Buchsteiner loops with abelian inner mapping groups.

Keywords: conjugacy closed loops, Buchsteiner loops

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1. Introduction

Q is a loop if it is a quasigroup with neutral element 1. The mappings $L_a(x) = ax$ (*left translation*) and $R_a(x) = xa$ (*right translation*) are permutations of Q for every $a \in Q$. The permutation group generated by left and right translations $\operatorname{Mlt}(Q) = \langle L_a, R_a \mid a \in Q\rangle$ is called the *multiplication group* of Q. Denote by $\operatorname{Inn}(Q)$ the stabilizer of the neutral element, and call it the *inner mapping group* of the loop Q.

In this paper we generalize the results obtained in [3] concerning the properties of loops such that the factor loop over the nucleus is an abelian group. The motivation of [3] was the theory of Buchsteiner loops ([2], [6], [7], [9] and partly [10]). We give sufficient and in some cases necessary conditions for the conjugacy closedness of $Q/Z(Q)$ provided the commutativity of Q/N.

Then we study the case of abelian inner mapping group. In 1946 Bruck [1] proved that if Q is a loop of nilpotency class at most two then $\operatorname{Inn} Q$ is abelian. In the nineties Kepka and Niemenmaa [13], [14] showed that a finite loop with abelian inner mapping group must be nilpotent, but they did not establish an upper bound on the nilpotency class of the loop. For a long time the prevailing opinion was that every loop Q with abelian $\operatorname{Inn} Q$ has nilpotency class at most two, i.e. that the converse of Bruck's result is true.

However, in 2004 Csörgő [8] constructed a nilpotent loop of order 128 such that the inner mapping group is abelian and the nilpotency class is equal to three. In this loop the nucleus is a normal subloop, and the factor over the nucleus is isomorphic to an abelian group. Later Drápal and Vojtěchovský [11] by analyzing the loop of this example developed a method by which they could construct many other examples.

In this paper we shall show (Theorem 3.14) that if Q/N is an abelian group and $\operatorname{Inn} Q$ is also an abelian group, then $Q/Z(Q)$ is a group and Q is nilpotent of class at most three. Note that Drápal and Kinyon [9] produced a Buchsteiner loop of order 128 that is of nilpotency class three and possesses an abelian inner mapping group. Let us also remark that recently Nagy and Vojtěchovský [12] constructed a Moufang 2-loop of order 2^{14} of nilpotency class three with abelian inner mapping group.

We shall also show that some conditions that are satisfied by Buchsteiner loops imply that the nilpotency class of the multiplication group is at most three. We shall then apply our results to Buchsteiner loops with abelian inner mapping groups, giving a structural description for both the loops and their multiplication groups.

We prove our results by applying the theory of connected transversals. This concept was introduced by Niemenmaa and Kepka [13]. Using their characterization theorem we can transform loop theoretical problems into group theoretical problems.

2. Basic definitions and results

Let Q be a loop. Set $A = \{L_c \mid c \in Q\}$, $B = \{R_d \mid d \in Q\}$. Then A and B are left transversals to $\operatorname{Inn} Q$ in $\operatorname{Mlt} Q$, $\langle A, B\rangle = \operatorname{Mlt} Q$, $[A, B] \le \operatorname{Inn} Q$ and $\operatorname{core}_{\operatorname{Mlt}(Q)} \operatorname{Inn}(Q) = 1$ (i.e. the largest normal subgroup of $\operatorname{Mlt} Q$ in $\operatorname{Inn} Q$ is trivial).

Conversely, consider a group G with the following properties: H is a subgroup of G, A and B are left transversals to H in G. A and B are *H-connected transversals* by definition, if $[A, B] \le H$.

By a result of Kepka and Niemenmaa [13], the above two situations are equivalent:

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there is a subgroup H, for which there exist H-connected transversals A and B such that $\langle A, B\rangle = G$ and $\operatorname{core}_G H = 1$.*

Let Q be a loop and S be a normal subloop of Q. Put $\mathcal{M}(S) = \langle L_c, R_c \mid c \in S\rangle$. Then $\mathcal{M}(S) \operatorname{Inn} Q \le \operatorname{Mlt} Q$ (this is a standard fact). Put $C(S) = \operatorname{core}_{\operatorname{Mlt} Q} \mathcal{M}(S) \operatorname{Inn} Q$. Denote by f the natural homomorphism of $\operatorname{Mlt} Q$ onto $\operatorname{Mlt} Q/C(S)$. Then $f(A)$ and $f(B)$ are $f(\operatorname{Inn} Q)$-connected transversals in $\operatorname{Mlt} Q/C(S)$ and $\operatorname{Mlt}(Q/S) \cong \operatorname{Mlt} Q/C(S)$.

The permutation group generated by all left translations is called the left multiplication group and we shall denote it by $\mathcal{L} = \mathcal{L}(Q)$. In a similar way the right multiplication group $\mathcal{R} = \mathcal{R}(Q)$ is generated by all right translations. Let $\mathcal{L}_1 = \mathcal{L} \cap \operatorname{Inn} Q$, and $\mathcal{R}_1 = \mathcal{R} \cap \operatorname{Inn} Q$.

Proposition 2.2.

$$\mathcal{L}_1 = \left\langle {L_{xy}}^{-1} L_x L_y \mid x, y \in Q \right\rangle,$$
$$\mathcal{R}_1 = \left\langle {R_{yx}}^{-1} R_x R_y \mid x, y \in Q \right\rangle,$$

and $\operatorname{Inn} Q$ *is generated by* $\mathcal{L}_1 \cup \mathcal{R}_1 \cup \{T_x \mid x \in Q\}$ *where* $T_x = {R_x}^{-1} L_x$ *for all* $x \in Q$.

We say that Q is an A_l-loop (A_r-loop) if $\mathcal{L}_1 \leq \operatorname{Aut} Q$ ($\mathcal{R}_1 \leq \operatorname{Aut} Q$). A loop Q is an $A_{r,l}$-loop if it is both an A_r-loop and an A_l-loop.

The *left*, *middle* and *right* nucleus of a loop Q are defined, respectively, by

$$\begin{aligned} N_\lambda = N_\lambda(Q) &:= \{a \in Q \mid a \cdot xy = a \cdot xy \quad \text{for all} \;\; x, y \in Q\}, \\ N_\mu = N_\mu(Q) &:= \{a \in Q \mid x \cdot ay = xa \cdot y \quad \text{for all} \;\; x, y \in Q\}, \\ N_\varrho = N_\varrho(Q) &:= \{a \in Q \mid x \cdot ya = xy \cdot a \quad \text{for all} \;\; x, y \in Q\}. \end{aligned}$$

The intersection

$$N = N(Q) = N_\lambda \cap N_\mu \cap N_\varrho$$

is called the *nucleus* of Q.

Proposition 2.3. *Let Q be a loop. Then*

i) $C_{\operatorname{Mlt} Q}(\mathcal{R}) = \{L_c \mid c \in N_\lambda\}$,
$C_{\operatorname{Mlt} Q}(\mathcal{L}) = \{R_d \mid d \in N_\varrho\}$;
ii) *if* $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$ *then* $C_{\operatorname{Mlt} Q}(\mathcal{R}) \trianglelefteq \operatorname{Mlt} Q$ *and* $N_\lambda \trianglelefteq Q$;
iii) *if* $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$ *then* $C_{\operatorname{Mlt} Q}(\mathcal{L}) \trianglelefteq \operatorname{Mlt} Q$ *and* $N_\varrho \trianglelefteq Q$;
iv) $A^*A = A$, $B^*B = B$, *where* $A^* = C_{\operatorname{Mlt} Q}(\mathcal{R})$, $B^* = C_{\operatorname{Mlt} Q}(\mathcal{L})$.

Proof: i), ii), iii): see [6, Lemma 1.7]. iv) is trivial. □

Proposition 2.4. *Let Q be a loop and let G_0 be the normal closure of* $\operatorname{Inn} Q$ *in* $\operatorname{Mlt} Q$. *Suppose that* $\operatorname{Inn} Q < K \trianglelefteq \operatorname{Mlt} Q$. *Then*

i) $\operatorname{Mlt} Q/K$ *is abelian;*
ii) $\operatorname{Mlt} Q/G_0$ *is abelian,* $G_0 = (\operatorname{Mlt} Q)' \operatorname{Inn} Q$;
iii) $G_0 \leq K$.

Proof: i) Let aK and bK be arbitrary elements of $\operatorname{Mlt} Q/K$ with $a \in A$, $b \in B$. Our statement follows from $[A, B] \leq \operatorname{Inn} Q < K$.

ii) By i) $\operatorname{Mlt} Q/G_0$ is abelian, whence $(\operatorname{Mlt} Q)' \operatorname{Inn} Q \leq G_0$. Using $(\operatorname{Mlt} Q)' \operatorname{Inn} Q \trianglelefteq \operatorname{Mlt} Q$, our statement follows.

iii) We have $K \geq (\operatorname{Mlt} Q)'$ by i), whence $K \geq (\operatorname{Mlt} Q)' \operatorname{Inn} Q$. $\square$

The center of $Z(Q)$ is defined by $Z(Q) = \{a \in N \mid \ xa = ax \text{ for all } x \in Q\}$. By putting $Z_0 = 1$, $Z_1 = Z(Q)$ and $Z_i/Z_{i-1} = Z(Q/Z_{i-1})$ we obtain a series of normal subloops of Q. If Z_{n-1} is a proper subloop of Q but $Z_n = Q$, then Q is centrally nilpotent of class n.

A loop Q is *left conjugacy closed* (LCC loop) if the left translations re closed under the conjugation, i.e. $L_a L_b {L_a}^{-1} = L_c$ for all $a, b \in Q$, respectively, Q is *right conjugacy closed* (RCC loop) if $R_a R_b {R_a}^{-1} = R_d$ for all $a, b \in Q$. A loop Q is conjugacy closed (CC loop) if it is an LCC and an RCC loop.

3. Buchsteiner loops and loops that are abelian modulo the nucleus

Buchsteiner loops are defined by the identity

$$x \setminus (xy \cdot z) = (y \cdot zx)/x. \tag{B}$$

Here $a \setminus b$ denotes the unique solution x to $ax = b$, while b/a denotes the unique solution y to $ya = b$. We call (B) the Buchsteiner law since Hans-Hennig Buchsteiner initiated their study in [2].

Rewriting the Buchsteiner law (B) in terms of translations immediately yields

Lemma 3.1. *Q is a Buchsteiner loop, the Buchsteiner law is equivalent to each of the following:*

$$\begin{aligned} {L_x}^{-1} R_z L_x &= {R_x}^{-1} R_{zx} \quad \text{for all } \ x, z \in Q, \\ {R_x}^{-1} L_y R_x &= {L_x}^{-1} L_{xy} \quad \text{for all } \ x, y \in Q. \end{aligned}$$

Proposition 3.2. *Let Q be a Buchsteiner loop. Then the following statements are true.*

i) $\mathcal{L} = \langle A \rangle \trianglelefteq \operatorname{Mlt} Q$,
$\mathcal{R} = \langle B \rangle \trianglelefteq \operatorname{Mlt} Q$,
$[A, B] = \mathcal{R}_1 = \mathcal{L}_1$.

ii) *The nucleus $N \trianglelefteq Q$ and*

$$N = N_\lambda = N_\mu = N_\varrho.$$

Put $A_0 = \{L_c \mid c \in N\}$, $B_0 = \{R_d \mid d \in N\}$. Then

$$\begin{aligned} A_0 &= C_{\operatorname{Mlt} Q}(\mathcal{R}), \quad A_0 \trianglelefteq \operatorname{Mlt} Q, \\ B_0 &= C_{\operatorname{Mlt} Q}(\mathcal{L}), \quad B_0 \trianglelefteq \operatorname{Mlt} Q. \end{aligned}$$

iii) *Q/N is an abelian group of exponent four* (*an example in which this exponent is achieved is constructed in* [6]).

iv) *Q is an $A_{r,l}$-loop.*

v) *$Q/Z(Q)$ is a CC loop.*

Proof: i) See [6, Corollary 1.3].

ii) See [6, Corollary 1.6, Corollary 1.8].

iii) See [6, Theorem 7.14].

iv) See [6, Corollary 5.4].

v) See [3, Theorem 3.5]. □

Buchsteiner loops are modulo the nucleus abelian groups. We shall now state their further basic properties.

Lemma 3.3. *Let Q be a loop such that $N \trianglelefteq Q$, Q/N is an abelian group. Set $A_0 = \{L_c \mid c \in N\}$, $B_0 = \{R_d \mid d \in N\}$. Let $G = \operatorname{Mlt} Q$ and $H = \operatorname{Inn} Q$. Then the following statements are true.*

i) $\operatorname{core}_G A_0H \supseteq [A, B] \cup (\langle A\rangle \cap H) \cup (\langle B\rangle \cap H)$.

ii) *Put $G_1 = A_0H = B_0H$.*
Then $G_1 \trianglelefteq G$ and G/G_1 is abelian.

iii) $Z(G_1) = Z(G) \times (Z(G_1) \cap H)$.

iv) $A_0 \trianglelefteq G$, $B_0 \trianglelefteq G$.

v) $A_0B_0 \leq C_G([A, B])$.

vi) *Suppose $h \in H \cap \operatorname{Aut} Q$, $a \in A$, $b \in B$. Then $h^a = h\alpha_0$, $h^b = h\beta_0$ with $\alpha_0 \in A_0$, $\beta_0 \in B_0$.*

vii) *If $h \in H \cap \operatorname{Aut} Q$, then $h \in C_G([A, B])$.*

Proof: i) By $N \trianglelefteq Q$, we have $A_0H \leq G$, $B_0H \leq G$. Using Q/N is abelian it follows $\operatorname{core}_G A_0H \supseteq [A, B] \cup (\langle A\rangle \cap H) \cup (\langle B\rangle \cap H)$.

ii) By i) clearly $\langle A\rangle \cap A_0H \trianglelefteq \langle A\rangle$. Let $a \in A$, $b \in B \cap aH$. Then using $[A, B] \leq H$ we get $(a^{-1}b)^{a^*} \in A_0H$ for every $a^* \in A$, in similar way $(a^{-1}b)^{b^*} \in B_0H(= A_0H)$ for every $b^* \in B$. Since $G = \langle A, B\rangle$ and $H = \big\langle a^{-1}b,\ \langle A\rangle \cap H,\ \langle B\rangle \cap H \mid a \in A,\ b \in B \cap aH\big\rangle$ by Proposition 2.2 we can conclude that $G_1 \trianglelefteq G$.

iii) Using $Z(G_1) \leq N_G(H)$ and $N_G(H) = Z(G) \times H$ (see [13, Proposition 2.7]) it follows easily.

iv) By [3, Lemma 1.7] and by i) $A_0 \trianglelefteq \langle A\rangle$. Since $A_0 \leq C_G(B)$ (see Proposition 2.3) and $\langle A, B\rangle = G$ it follows $A_0 \trianglelefteq G$. In a similar way $B_0 \trianglelefteq G$ holds.

v) Using $A_0 \trianglelefteq G$ and $A_0 \leq C_G(B)$ we can see easily $A_0 \leq C_G([A, B])$, and similarly $B_0 \leq C_G([A, B])$.

vi) By [3, Lemma 1.2] $a^h \in A$, $b^h \in B$. Since $G_1 \trianglelefteq G$, $A_0 \trianglelefteq G$, $B_0 \trianglelefteq G$, and $A_0A = A$, $B_0B = B$ we get our statement.

vii) Using vi) and $A_0 \leq C_G(\langle B\rangle)$, $B_0 \leq C_G(\langle A\rangle)$ it follows easily. □

The conjugacy closed loops (CC loops) Q satisfy the following properties:

$$\langle A\rangle \trianglelefteq \operatorname{Mlt} Q, \quad \langle B\rangle \trianglelefteq \operatorname{Mlt} Q,$$

Q is an $A_{r,l}$-loop, furthermore $N \trianglelefteq Q$, Q/N is an abelian group.

In [3] we studied the converse of this result, i.e. those loops satisfying these conditions and we got that they are very close to the CC loops:

Proposition 3.4 ([3, Theorem 3.1]). *Let Q be a loop such that $N \trianglelefteq Q$, $\langle A\rangle \trianglelefteq \operatorname{Mlt} Q$, $\langle B\rangle \trianglelefteq \operatorname{Mlt} Q$, Q is an $A_{l,r}$-loop. If Q/N is an abelian group, then $Q/Z(Q)$ is conjugacy closed.*

In fact, we have proved a somewhat stronger result as well:

Proposition 3.5 ([3, Proposition 3.2]). *Let Q be an $A_{r,l}$-loop in which the nucleus is normal and Q/N is an abelian group. If $[A,B] \leq \operatorname{Aut} Q$, then $Q/Z(Q)$ is a conjugacy closed loop.*

As the Buchsteiner loops satisfy these conditions we get

Corollary 3.6 ([3, Theorem 3.5]). *Let Q be a Buchsteiner loop. Then $Q/Z(Q)$ is a conjugacy closed loop.*

In Proposition 3.5 the requirement that Q is an $A_{r,l}$-loop seems to be too strong. In case of $[A,B] \leq \operatorname{Aut} Q$ we shall obtain an exact description when $Q/Z(Q)$ is conjugacy closed. For this aim we need the following subsets for a loop Q:

$$L_F(Q) = \{{L_z}^{-1}{L_x}^{L_y} \mid \ {L_z}^{-1}{L_x}^{L_y} \in \operatorname{Inn} Q,\ x,y \in Q\},$$
$$R_F(Q) = \{{R_w}^{-1}{R_x}^{R_y} \mid {R_w}^{-1}{R_x}^{R_y} \in \operatorname{Inn} Q,\ x,y \in Q\}.$$

In the following statements A_0, B_0 are defined as in Lemma 3.3.

Proposition 3.7. *Let Q be a loop such that $N \trianglelefteq Q$ and Q/N is an abelian group. Suppose $[A,B] \leq \operatorname{Aut} Q$. Then $Q/Z(Q)$ is a CC loop if and only if $L_F(Q) \subseteq \operatorname{Aut} Q$ and $R_F(Q) \subseteq \operatorname{Aut} Q$.*

Proof: Let $G = \operatorname{Mlt} Q$, $H = \operatorname{Inn} Q$.

Let $h^* \in L_F(Q)$ be arbitrary. Lemma 3.3 i), ii) give that there exist $\alpha_1, \alpha_2 \in A$ such that $\alpha_1^{\alpha_2} = \alpha_1 \delta h^*$ with $\delta \in A_0$. Then $\alpha_1^{\alpha_2^{-1}} = \alpha_1((h^*)^{-1})^{\alpha_2^{-1}}(\delta^{-1})^{\alpha_2^{-1}}$. Using $A_0 \trianglelefteq G$ (see Lemma 3.3 iv)) we get $\alpha_1^{\alpha_2^{-1}} = \alpha_1\gamma((h^*)^{-1})^{\alpha_2^{-1}}$ with $\gamma \in A_0$.

i) First suppose $L_F(Q) \subseteq \operatorname{Aut} Q$ and $R_F(Q) \subseteq \operatorname{Aut} Q$. Since $h^* \in L_F(q)$ it follows $h^* \in \operatorname{Aut} Q$, whence using Lemma 3.3 vi) we can conclude $((h^*)^{-1})^{\alpha_2^{-1}} = \gamma_0 (h^*)^{-1}$ with $\gamma_0 \in A_0$, consequently $\alpha_1^{\alpha_2^{-1}} = \alpha_1\alpha_0(h^*)^{-1}$ with $\alpha_0 \in A_0$. Set $h = (h^*)^{-1}$, clearly $h \in \operatorname{Aut} Q$. Let $\beta \in B$. We have ${\alpha_1}^{\beta} = \alpha_1 h_1$, $\beta^{\alpha_2} = \beta h_0$ with $h_1, h_0 \in [A,B]$. Then ${\alpha_1}^{\beta} = {\alpha_1}^{\beta^{\alpha_2} h_0^{-1}} = \alpha_1 h_1$. Thus ${\alpha_1}^{\beta} = {\alpha_1}^{\alpha_2^{-1}\beta\alpha_2 h_0^{-1}} = (\alpha_1\alpha_0 h)^{\beta\alpha_2 h_0^{-1}}$. Using Lemma 3.3 vi), $h^{\beta} = h\beta_0$ holds with $\beta_0 \in B_0$. Hence ${\alpha_1}^{\beta} = (\alpha_1 h_1 \alpha_0 h \beta_0)^{\alpha_2 h_0^{-1}} = (\alpha_1 h_1^{h\alpha_2}\beta_0)^{h_0^{-1}}$. As $h_1, h_0 \in [A,B]$, Lemma 3.3 vii), v), vi) imply ${\alpha_1}^{\beta} = \alpha_1\alpha^* h_1 \alpha^{**}\beta_0 = \alpha_1 h_1$, where $\alpha^*, \alpha{**} \in A_0$. Using

Lemma 3.3 v), we can again conclude that $\beta_0 \in B_0 \cap A_0$. Since $A_0 \cap B_0 \subseteq Z(G)$, whence $h^\beta \in hZ(G)$. As ${\alpha_1}^{\alpha_2^{-1}} = \alpha_1\alpha_0 h$ and $\beta \in B$ is arbitrary we get $h \in \operatorname{core}_G Z(G)H$. Thus $Q/Z(Q)$ is left conjugacy closed. In a similar way $Q/Z(Q)$ is an RCC loop, consequently $Q/Z(Q)$ is a CC loop.

ii) Suppose $Q/Z(Q)$ is a CC loop. Then $R_F(Q) \cup L_F(Q) \subseteq \operatorname{core}_G Z(G)H$. Since $h^* \in L_F(Q)$ we have $((h^*)^{-1}) \in \operatorname{core}_G Z(G)H$, consequently $((h^*)^{-1})^{\alpha_2^{-1}} = (h^*)^{-1}\widetilde{h}z$ with $\widetilde{h} \in H$, $z \in Z(G) \cap (A_0 H)$. Thus $\alpha_1^{\alpha_2^{-1}} = \alpha_1\gamma z(h^*)^{-1}\widetilde{h}$. Put $\gamma z = \alpha_0 \in A_0$, $h = (h^*)^{-1}\widetilde{h}$, so $\alpha_1^{\alpha_2^{-1}} = \alpha_1\alpha_0 h$. Clearly $h \in \operatorname{core}_G(Z(G))H$. Given $\beta \in B$, we have ${\alpha_1}^\beta = \alpha_1 h_1$, $\beta^{\alpha_2} = \beta h_0$ with $h_1, h_0 \in H \cap [A, B]$. Then ${\alpha_1}^\beta = {\alpha_1}^{\beta^{\alpha_2} h_0^{-1}} = \alpha_1 h_1$. Let us use the same notation and repeat the steps of part i), then we get $\alpha_1^\beta = {\alpha_1}^{h_0^{-1}}{h_1}^{h\alpha_2 h_0^{-1}}(h^{-1}h^\beta)^{\alpha_2 h_0^{-1}}$. Since $h^{-1} \in \operatorname{core}_G Z(G)H$, Lemma 3.3 vi) implies $h^\beta = hzh_2$ with $z \in Z(G)$, $h_2 \in H$, whence $(h^{-1}h^\beta)^{\alpha_2 h_0^{-1}} = (zh_2)^{\alpha_2 h_0^{-1}}$ with $\alpha_{01} \in A_0$. As $h_1 = {\alpha_1}^{-1}{\alpha_1}^\beta \in [A, B]$ it follows $h_1 \in \operatorname{Aut} Q$, using Lemma 3.3 vii) we can conclude ${h_1}^h = h_1$, whence ${h_1}^{\alpha_2 h_0^{-1}} = (h_1\widetilde{\alpha})^{h_0^{-1}}$ with $\widetilde{\alpha} \in A_0$ by Lemma 3.3 vi). Hence ${\alpha_1}^\beta = \alpha_1\alpha_{01}h_1\widetilde{\alpha}{h_2}^{\alpha_2 h_0^{-1}} = \alpha_1 h_1$ with $\alpha_{01} \in A_0$. Since $A_0 \trianglelefteq G$ we can conclude $h_2 = e$, i.e. $h^\beta = hz$. As $\beta \in B$ is arbitrary we get $h \in \operatorname{Aut} Q$. We have $\alpha_1^{\alpha_2^{-1}} = \alpha_1\alpha_0 h$, whence $\alpha_1^{\alpha_2} = \alpha_1(h^{-1})^{\alpha_2}(\alpha_0^{-1})^{\alpha_2}$. Using $A_0 \triangleleft G$ and $h^{-1} \in \operatorname{Aut} Q \cap \operatorname{core}_G(Z(G))H$ it follows $\alpha_1^{\alpha_2} = \alpha_1\xi z h^{-1}$ with $\xi \in A_0$. On the other hand we have $\alpha_1^{\alpha_2} = \alpha_1\delta h^*$, whence $h^* = h^{-1}$, and we can conclude $L_F(Q) \subseteq \operatorname{Aut} Q$. We can show similarly $R_F(Q) \subseteq \operatorname{Aut} Q$. $\square$

We give another sufficient condition for the conjugacy closedness of $Q/Z(Q)$.

Proposition 3.8. *Let Q be a loop such that $N \trianglelefteq Q$, Q/N is an abelian group. Suppose $L_F(Q) \cup R_F(Q) \subseteq Z(\operatorname{Inn} Q)$. Then $Q/Z(Q)$ is a CC loop.*

PROOF: Let $G = \operatorname{Mlt} Q$ and $H = \operatorname{Inn} Q$. We have $B_0 \leq C_G(\langle A\rangle)$ whence $B_0 \leq C_G(L_F(Q))$ whence $L_F(Q) \subseteq Z(H)$ implies $L_F(Q) \subseteq Z(B_0 H)$. Since $B_0 H \trianglelefteq G$ (see Lemma 3.3 i)) it follows $Z(B_0 H) \trianglelefteq G$. By Lemma 3.3 iii) $Z(B_0 H) = Z(G) \times (Z(B_0 H) \cap H)$, consequently $L_F(Q) \subseteq Z(B_0 H) \cap H \leq \operatorname{core}_G Z(G)H$, i.e. $Q/Z(Q)$ is left conjugacy closed. In a similar way we get that $Q/Z(Q)$ is an RCC loop too. $\square$

In case $[A, B] \leq \operatorname{Aut} Q$ the sufficient condition for the conjugacy closedness of $Q/Z(Q)$ in the previous proposition can be proved to be necessary.

Proposition 3.9. *Let Q be a loop such that $N \trianglelefteq Q$, Q/N is abelian group. Suppose that $[A, B] \leq \operatorname{Aut} Q$. Then $Q/Z(Q)$ is conjugacy closed if and only if $L_F(Q) \cup R_F(Q) \subseteq Z(\operatorname{Inn} Q)$.*

PROOF: Let $G = \operatorname{Mlt} Q$, $H = \operatorname{Inn} Q$.

i) Suppose first $L_F(Q) \cup R_F(Q) \subseteq Z(\operatorname{Inn} Q)$. Then Proposition 3.8 implies our statement.

ii) Suppose $Q/Z(Q)$ is a CC loop. Let $\alpha_1, \alpha_2 \in A$. Then using Lemma 3.3 ii) we get ${\alpha_1}^{\alpha_2} = \alpha_1\alpha_0 h$ with $\alpha_0 \in A_0$, $h \in H \cap \langle A\rangle$. Clearly $h \in L_F(Q)$, since $L_F(Q) \subseteq \operatorname{Aut} Q$ by Proposition 3.7, it follows $h^a \in hA_0$ for every $a \in A$ (see Lemma 3.3 vi)). The conjugacy closedness of $Q/Z(Q)$ implies $h \in \operatorname{core}_G Z(G)H$, whence $h^a \in hZ(G)$. Similarly $h^b \in hZ(G)$ for every $b \in B$. As $G = \langle A, B\rangle$ we can conclude $h \in Z(\operatorname{Inn} Q)$, whence clearly $L_F(Q) \subseteq Z(\operatorname{Inn} Q)$. In a similar way we get $R_F(Q) \subseteq Z(\operatorname{Inn} Q)$. □

In case of Buchsteiner loops we have a necessary and sufficient condition that $Q/Z(Q)$ is a group:

Proposition 3.10 ([9, Lemma 7.2]). *Let Q be a Buchsteiner loop. Then $Q/Z(Q)$ is a group, i.e. $A(Q) \leq Z(Q)$ if and only if $[A, B] \leq Z(\operatorname{Inn} Q)$.*

We generalize this result in the following way:

Proposition 3.11. *Let Q be a loop such that $N \trianglelefteq Q$, Q/N is an abelian group and $[A, B] \leq Z(\operatorname{Inn} Q)$. Then $Q/Z(Q)$ is a group, i.e. $A(Q) \leq Z(Q)$.*

Proof: Let $G = \operatorname{Mlt} Q$, $H = \operatorname{Inn} Q$. By Lemma 3.3 ii) G/A_0H is abelian. We show $[A, B] \leq Z(A_0H)$. By Lemma 3.3 v) $A_0 \leq C_G([A, B])$. The condition $[A, B] \leq Z(H)$ implies $[A, B] \leq Z(A_0H)$. Since $Z(A_0H) = (Z(G) \cap A_0) \times (Z(A_0H) \cap H)$ and $A_0H \trianglelefteq G$, by Lemma 3.3 iii), ii) it follows $Z(A_0H) \trianglelefteq G$. Thus we get $Z(A_0H) \leq \operatorname{core}_G Z(G)H$, consequently $Q/Z(Q)$ is a group. □

In case $[A, B] \leq \operatorname{Aut} Q$ the above mentioned sufficient condition can be proved to be necessary.

Proposition 3.12. *Let Q be a loop such that $N \trianglelefteq Q$, Q/N is an abelian group and $[A, B] \leq \operatorname{Aut} Q$. Then $Q/Z(Q)$ is a group if and only if $[A, B] \leq Z(\operatorname{Inn} Q)$.*

Proof: Let $G = \operatorname{Mlt} Q$ and $H = \operatorname{Inn} Q$.

i) First suppose $[A, B] \leq Z(\operatorname{Inn} Q)$. Then our statement follows by Proposition 3.11.

ii) Suppose $Q/Z(Q)$ is a group. Then $[A, B] \leq \operatorname{core}_G Z(G)H$. Since $[A, B] \leq \operatorname{Aut} Q \cap H$, using Lemma 3.3 v) we get $t^a \in tA_0$ for every $t \in [A, B]$ and $a \in A$. Consequently $t^a \in tZ(G)$. Similarly we get $t^b \in tZ(G)$ for every $b \in B$. As $G = \langle A, B\rangle$ we can conclude that $t \in Z(H)$, i.e. $[A, B] \leq Z(H)$. □

In the following we study the case of abelian inner mapping group.

Proposition 3.13. *Let Q be a Buchsteiner loop with abelian inner mapping group. Then*

i) *$Q/Z(Q)$ is a group;*
ii) *Q is nilpotent of class at most three.*

Proof: i) See Proposition 3.10.

ii) Since a CC loop with abelian inner mapping group is nilpotent of class at most two [5, Proposition 2.5], our statement follows. □

We analyze the general case:

Theorem 3.14. *Let Q be a loop with abelian inner mapping group such that $N \trianglelefteq Q$ and Q/N is an abelian group. Then the following statements are true.*

i) *$Q/Z(Q)$ is a group.*
ii) *Q is nilpotent of class at most three.*

Proof: i) See Proposition 3.11.
ii) See the proof of Proposition 3.13 ii). □

In case of abelian inner mapping group under the conditions of Proposition 3.5 we can prove more, namely the nilpotency of class at most three of the multiplication group.

For this aim we need the following

Lemma 3.15. *Let Q be a loop with abelian inner mapping group such that $N \trianglelefteq Q$ and Q/N is an abelian group. Let $G = \mathrm{Mlt}\, Q$, $H = \mathrm{Inn}\, Q$, and G_0 is the normal closure of H in G. Then*

i) *$h^a \in h(Z(G) \cap G_0)$, $h^b \in h(Z(G) \cap G_0)$ for every $h \in H \cap \mathrm{Aut}\, Q$ and $a \in A$, $b \in B$;*
ii) *${a_1}^a \in a_1(Z(G) \cap G_0)$ for every $a_1 \in A_0 \cap G_0$, $a \in A$.*

Proof: i) Let $h \in H \cap \mathrm{Aut}\, Q$. Then $h^{a^{-1}} \in hA_0$ by Lemma 3.3 vi). Since $A_0 \trianglelefteq G$ (see Lemma 3.3 iv)) and $G_0 \trianglelefteq G$ we get $h^{a^{-1}} \in hA_0 \cap G_0$. Clearly $A_0 H \geq G_0$, whence $G_0 = (A_0 \cap G_0)H$, consequently $h^{a^{-1}} \in h(A_0 \cap G_0)$. Let $b \in B \cap aH$, in a similar way we can show $h^{b^{-1}} \in h(B_0 \cap G_0)$. The commutativity of H implies $h^{a^{-1}b} = h$, whence $h^{a^{-1}} = h^{b^{-1}} \in h(A_0 \cap B_0)$. Since $A_0 \cap B_0 \subseteq Z(G)$ we get $h^a \in h(Z(G) \cap G_0)$, $h^b \in h(Z(G) \cap G_0)$ for every $a \in A$, $b \in B$.

ii) By Theorem 3.14 we have that $Q/Z(Q)$ is of nilpotency class at most two. Hence clearly $Q/Z(Q)/Z(Q/Z(Q))$ is an abelian group. Let $U = \mathrm{core}_G\, Z(G)H$. Then $\mathrm{Mlt}(Q/Z(Q)) = \mathrm{Mlt}\, Q/U$. Let Z^* be the inverse image of $Z(\mathrm{Mlt}\, Q/U)$. Since $Q/Z(Q)/Z(Q/Z(Q))$ is an abelian group it follows $Z^* \,\mathrm{Inn}\, Q \trianglelefteq \mathrm{Mlt}\, Q$. By Proposition 2.4 iii), $G_0 \leq Z^* \,\mathrm{Inn}\, Q$. Applying $Z(\mathrm{Mlt}\, Q) \subseteq A \cap B$ for $Z(\mathrm{Mlt}\, Q/Z(Q))$ we get $b_1^{-1} a_1 \in U \cap H$ for every $a_1 \in G_0 \cap A_0$, $b_1 \in B \cap a_1 H$. Thus $(b_1^{-1} a_1)^a \in b_1^{-1} a_1 U$ for every $a \in A$. We have $b_1 \in G_0 \cap B_0 \leq C_G(A)$, whence ${a_1}^a \in a_1 U$. Since $U = \mathrm{core}_G\, Z(G)H$ it follows ${a_1}^a = a_1 z_1 h_1$ with $z_1 \in Z(G)$, $h_1 \in U \cap H$. As $A_0 \trianglelefteq G$ (see Lemma 3.3 iv)), ${a_1}^a = a_1 z_1$ holds. $G_0 \trianglelefteq G$ implies $z_1 \in Z(G) \cap G_0$. □

We return to our statement.

Theorem 3.16. *Let Q be an $A_{r,l}$-loop with abelian* $\operatorname{Inn} Q$ *such that $N \trianglelefteq Q$, Q/N is an abelian group. Suppose $[A,B] \leq \operatorname{Aut} Q$. Then Q and $\operatorname{Mlt} Q$ are nilpotent of class at most three.*

Proof: By Theorem 3.14 Q is nilpotent of class at most three.

Let $G = \operatorname{Mlt} Q$, $H = \operatorname{Inn} Q$. Let $M = \langle A\rangle[A,B]$. We show $M \trianglelefteq G$. Using that $[A,B] \leq \operatorname{Aut} Q$ and that H is abelian, Lemma 3.15 i) implies $Z(G)[A,B] \trianglelefteq G$. Since $Z(G) \leq \langle A\rangle$ it follows $M \leq G$. We have $\langle A\rangle \cap H \leq \operatorname{Aut} Q$, $[A,B] \leq \operatorname{Aut} Q$. Using $\langle A,B\rangle = G$ and Lemma 3.15 i), we get $M \trianglelefteq G$.

Let $Z_1 = Z(G) \cap G_0$ (G_0 is the normal closure of H in G), $D = G_0 \cap M$ and $A_1 = G_0 \cap A_0$. We show $D/Z_1 \leq Z(G/Z_1)$. Using Lemma 3.15 ii), $A_1 \leq C_G(B)$ and $G = \langle A,B\rangle$ we can conclude $A_1Z_1/Z_1 \leq Z(G/Z_1)$. As $D \cap H = (\langle A\rangle \cap H)[A,B]$ and $(\langle A\rangle \cap H)[A,B] \leq \operatorname{Aut} Q$, Lemma 3.15 i) implies $D/Z_1 \leq Z(G/Z_1)$. Since $G/M \cong H/H \cap M$ it follows G/M is abelian. Using G/G_0 is abelian too we get $G' \leq M \cap G_0 = D$, consequently G/D is abelian. Thus G is nilpotent of class at most three. □

Using the previous result we describe the structure of Buchsteiner loop with abelian inner mapping groups. For this aim we need the following:

Proposition 3.17 ([9, Lemma 7.2, Proposition 7.3])**.** *If Q is a Buchsteiner loop with abelian inner mapping group, then Q/N is an elementary abelian 2-group.*

Corollary 3.18. *Let Q be a Buchsteiner loop with abelian* $\operatorname{Inn} Q$*, let $A_0 = \{L_c \mid c \in N\}$. Then $\operatorname{Mlt} Q/A_0 \operatorname{Inn} Q$ is an elementary abelian 2-group.*

Proof: The structure of the multiplication group of the factorloop and Proposition 3.17 imply this statement. □

Proposition 3.19. *Let Q be a Buchsteiner loop with abelian* $\operatorname{Inn} Q$*. Then the following statements are true.*

i) *Q and $\operatorname{Mlt} Q$ are nilpotent of class at most three.*
ii) *$aa^b \in A_0$ for every $a \in A$, $b \in B \cap a \operatorname{Inn} Q$, where $A_0 = \{L_c \mid c \in N\}$.*
iii) *$\langle A\rangle \cap \operatorname{Inn} Q$ is an elementary abelian 2-group.*

Proof: i) By Theorem 3.14 Q is nilpotent of class at most three. We have Q is an $A_{r,l}$-loop, $N \trianglelefteq Q$, Q/N is an abelian group, $[A,B] = \langle A\rangle \cap \operatorname{Inn} Q = \langle B\rangle \cap \operatorname{Inn} Q$ (see Proposition 3.2) whence $[A,B] \leq \operatorname{Aut} Q$, so we can apply Theorem 3.16.

ii) See the definition of Buchsteiner loops and Corollary 3.18.

iii) By ii) $aa^b \in A_0$ for every $a \in A$, $b \in B \cap a \operatorname{Inn} Q$. Let $b_1 \in B$ be arbitrary. Clearly $a^b = ah$, $a^{b_1} = ah_1$ with $h, h_1 \in \operatorname{Inn} Q$, whence $aa^b = (aa^b)^{b_1} = ah_1ah_1h^b = a^2{h_1}^a h_1 h^b$. Lemma 3.15 i) implies ${h_1}^a = h_1z_1$, $h^b = hz$ with $z_1z \in Z(\operatorname{Mlt} Q)$. Hence $(aa^b)^{b_1} = a^2hh_1{}^2z_1z = aah$. Thus $h_1{}^2 = e$. □

We give a characterization theorem about the Buchsteiner loops with abelian inner mapping group.

Theorem 3.20. *Let Q be a loop. Then Q is a Buchsteiner loop with abelian inner mapping group if and only if $Q = Q_1 \times Q_2$, where Q_1 is a Buchsteiner loop with abelian inner mapping of order 2^t and Q_2 is a group of odd order with abelian inner mapping group. Additionally* $\operatorname{Mlt} Q = \operatorname{Mlt} Q_1 \times \operatorname{Mlt} Q_2$ *where* $\operatorname{Mlt} Q_1 \in \operatorname{Syl}_2(\operatorname{Mlt} Q)$.

PROOF: i) Clearly, if $Q = Q_1 \times Q_2$ with the given properties, then Q is a Buchsteiner loop with abelian $\operatorname{Inn} Q$.

ii) Conversely suppose Q is a Buchsteiner loop with abelian $\operatorname{Inn} Q$. Let $G = \operatorname{Mlt} Q$, $H = \operatorname{Inn} Q$. By Proposition 3.19 i) G is nilpotent of class at most three. So $G = S \times T$, where $S \in \operatorname{Syl}_2(G)$.

First we show $S = (S \cap A)(S \cap H)$. Let $S_1 \in \operatorname{Syl}_2(\langle A \rangle)$. Since $\langle A \rangle \trianglelefteq G$ (see Proposition 3.2) it follows $S_1 \trianglelefteq G$. Let $S_0 \in \operatorname{Syl}_2(H)$, then $S_1 S_0 \le G$. Using $\langle A \rangle H = G$ we can conclude $S_1 S_0 = S \in \operatorname{Syl}_2(G)$. As $\langle A \rangle \cap H$ is an elementary abelian 2-group (see Proposition 3.19 iii)). $S \ge \langle A \rangle \cap H$ holds, whence $S_1 = S \cap \langle A \rangle = (S \cap A)(\langle A \rangle \cap H)$. We have $S = S_1 S_0$, consequently $S = (S \cap A)(S \cap H)$. In a similar way $S = (S \cap B)(S \cap H)$.

Since $G/A_0 H$ is an elementary abelian 2-group (see Corollary 3.18) and $S \ge \langle A \rangle \cap H$ we can conclude $T \le A_0 H$ and $T_1 = T \cap \langle A \rangle \le A_0$. As G is nilpotent and T is a Hall subgroup of G, by Hall's theorems $T_1 \trianglelefteq G$. We have $H = S_0 \times T_0$ where $T_0 \le T$ is a Hall subgroup of H. Using $T \le A_0 H$, $A_0 \cap H = 1$ and $A_0 \trianglelefteq G$ it follows $T = T_1 \cdot T_0 = (T \cap \langle A_0 \rangle)(T \cap H)$. Similarly $T = (T \cap B_0)(T \cap H)$.

Let

$$A_S = S \cap A, \quad A_T = T \cap A_0,$$
$$B_S = S \cap B, \quad B_T = T \cap B_0.$$

Since $A_0 A = A$, $B_0 B = B$ we have $A_T A_S \subseteq A$, $B_T B_S \subseteq B$. Clearly $|A_T A_S| = |A_T|\,|A_S|$, $|B_T B_S| = |B_T|\,|B_S|$. Using $G = S \times T$, $S = (S \cap A)(S \cap H) = (S \cap B)(S \cap H)$, $T = (T \cap A_0)(T \cap H) = (T \cap B_0)(T \cap H)$ we get $A = A_T A_S$, $B = B_T B_S$.

Let

$$Q_1 = \{c \in Q \mid L_c \in S\},$$
$$Q_2 = \{d \in Q \mid L_d \in T\}.$$

As $SH \le G$ and $TH \le G$ we can conclude Q_1 and Q_2 are normal subloops of Q. We show $\operatorname{Mlt} Q_1 = S$ and $\operatorname{Mlt} Q_2 = T$. By Niemenmaa and Kepka's theorem [13] it is enough to show:

i$_1$) $\langle S \cap A, S \cap B \rangle = S$, i$_2$) $\operatorname{core}_S(S \cap H) = 1$, i$_3$) $[S \cap A, S \cap B] \le S \cap H$,

j$_1$) $\langle T \cap A_0, T \cap B_0 \rangle = T$, j$_2$) $\operatorname{core}_T(T \cap H) = 1$, j$_3$) $[T \cap A_0, T \cap B_0] \le T \cap H$.

We have $\langle A, B\rangle = G$, since $A = A_T A_S$, $B = B_T B_S$ and $G = S \times T$ i$_1$) and j$_1$) are true.

$G = S \times T$ implies $\operatorname{core}_S(S \cap H) \leq S \cap C_G(T)$ and $\operatorname{core}_S(S \cap H) \trianglelefteq G$. Using $\operatorname{core}_G H = 1$ we get $\operatorname{core}_S(S \cap H) = 1$. In a similar way j$_2$) follows.

i$_3$) and j$_3$) are consequences of $[A, B] \leq H$.

Clearly Q_1 is a Buchsteiner loop of order 2^t with abelian $\operatorname{Inn} Q_1$ $(= S \cap H)$.

Since $\operatorname{Mlt} Q_2 = (T \cap A_0)(T \cap H)$, $T \cap \langle A\rangle \leq A_0$, $A_0 \trianglelefteq \operatorname{Mlt} Q$ it follows that Q_2 is a group of odd order with abelian $\operatorname{Inn} Q_2$ $(= T \cap H)$. $\square$

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Eötvös University, Department of Algebra and Number Theory, Pázmány Péter sétány 1/C, H–1117 Budapest, Hungary

E-mail: ska@cs.elte.hu

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