On central nilpotency in finite loops with nilpotent inner mapping groups

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Abstract. In this paper we consider finite loops whose inner mapping groups are nilpotent. We first consider the case where the inner mapping group $I(Q)$ of a loop Q is the direct product of a dihedral group of order 8 and an abelian group. Our second result deals with the case where Q is a 2-loop and $I(Q)$ is a nilpotent group whose nonabelian Sylow subgroups satisfy a special condition. In both cases it turns out that Q is centrally nilpotent.

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1. Introduction

If Q is a loop, then we have two permutations L_a and R_a on Q defined by $L_a(x) = ax$ and $R_a(x) = xa$ for each $a \in Q$. These permutations generate a permutation group $M(Q)$ which is said to be the multiplication group of Q. The stabilizer of the neutral element of the loop is denoted by $I(Q)$ and we say that $I(Q)$ is the inner mapping group of Q. In 1946 Bruck [2] showed that if a finite loop Q is centrally nilpotent, then $M(Q)$ is a solvable group. He also showed that if $M(Q)$ is a nilpotent group, then Q is a centrally nilpotent loop.

It is reasonable to expect that also the structure of the inner mapping group $I(Q)$ determines the structure of the loop Q to a certain degree. In 1992 Kepka and Niemenmaa [9] showed for finite loops that if $I(Q)$ is an abelian p-group, then Q is centrally nilpotent. In 1994 they managed to prove a more general result, namely if $I(Q)$ is a finite abelian group, then Q is centrally nilpotent (for the details, see [10]). This result also holds in the case that Q is infinite as was shown by Kepka [4] in 1998. One of the main results for nonabelian $I(Q)$ is by Niemenmaa [6] from 1995: if $I(Q)$ is a dihedral 2-group, then Q is centrally nilpotent.

The purpose of this paper is to investigate further those properties of $I(Q)$ which guarantee the central nilpotency of the loop Q. Our two main results are the following:

1) If Q is a finite loop and $I(Q) \cong D \times E$, where D is dihedral of order 8 and E is abelian, then Q is centrally nilpotent.

2) If $|Q| = 2^m$, $I(Q)$ is nilpotent and $q \geq m$ for every odd prime q such that the Sylow q-subgroup of $I(Q)$ is nonabelian, then Q is centrally nilpotent.

The above results are direct consequences of purely group theoretic results and the basic tool in our reasoning is the notion of connected transversals. Thus in Section 2 we introduce this notion and we go through basic results which are needed later. In Section 3 we prove our main group theoretic results (subnormality conditions for subgroups with connected transversals). In the proofs we apply classic results from finite group theory (solvability criteria, Frobenius groups, the structure of noncyclic p-groups etc.). All these results are introduced with references. Finally, in Section 4 we give loop theoretic interpretations of the main group theoretic results given in Section 3.

2. Preliminary results

Let H be a subgroup of G and let A and B be two left transversals to H in G. If $[a, b] = a^{-1}b^{-1}ab \in H$ for every $a \in A$ and $b \in B$, then we say that A and B are H-connected in G. By [8, Lemmas 2.1 and 2.2] H-connected transversals are both left and right transversals. The core of H in G (the largest normal subgroup of G contained in H) will be denoted by H_G. A subgroup L of G is said to be subnormal in G if there exists a chain of subgroups $L = L_0 \leq L_1 \leq \ldots \leq L_r = G$ such that L_i is normal in L_{i+1}. If K and L are subnormal subgroups in G, then $K \cap L$ is subnormal in G, too.

In the following lemmas we assume that H is a subgroup of G and A and B are H-connected transversals in G.

Lemma 2.1. *If $C \subseteq A \cup B$ and $K = \langle H, C\rangle$, then $C \subseteq K_G$.*

Lemma 2.2. *If H is finite and abelian, then G is solvable.*

Lemma 2.3. *If H is a dihedral 2-group, then G is solvable.*

For the proofs, see [8, Lemma 2.5], [10, Theorem 4.1] and [6, Theorem 3.3]. In the following lemmas we further assume that $G = \langle A, B\rangle$.

Lemma 2.4. *If H is cyclic, then $G' \leq H$.*

Lemma 2.5. *If G is finite and H is abelian, then H is subnormal in G.*

Lemma 2.6. *If G is finite and H is a dihedral 2-group, then H is subnormal in G.*

Lemma 2.7. *If G is finite and $H = D \times E$, where D is a dihedral 2-group and E is a nontrivial abelian group of odd order, then H is subnormal in G.*

For the proofs, see [5, Theorem 2.2], [10, Proposition 6.3], [6, Theorem 4.1] and [7, Lemma 3.2]. For the proofs of our main theorems we still need some group theoretical results which are listed below.

Lemma 2.8. *If the finite group $G = KM$ is the product of two nilpotent subgroups K and M, then G is solvable.*

The proof can be found in [1, Theorem 2.4.3, p. 27–32]. The following three results are from Huppert [3, p. 445–446, p. 303–305, p. 499].

Lemma 2.9. *Let G be a finite group with a nilpotent maximal subgroup M. If the Sylow 2-subgroup of M is of nilpotency class at most 2, then G is solvable.*

Lemma 2.10. *Let p be an odd prime and let P be a noncyclic p-group. Then P has a normal subgroup K which is elementary abelian of order p^2 (thus $K \cong C_p \times C_p$).*

Lemma 2.11. *Let P be a Sylow p-subgroup of a Frobenius complement. If $p > 2$, then P is cyclic. If $p = 2$, then P is either cyclic or generalized quaternion.*

As the solvability of a finite group G guarantees the existence of Hall-subgroups and all Hall-subgroups of the same order are conjugate to each other in G, we can easily prove the following generalized version of the Frattini Argument.

Lemma 2.12. *Let G be a finite solvable group with a normal subgroup M. If H is a Hall-subgroup of M, then $G = MN_G(H)$.*

3. Main results

In this section we assume that H is subgroup of a finite group G and A and B are H-connected transversals in G. The following two results are motivated by Lemma 2.7.

Lemma 3.1. *If $H \cong D \times E$, where D is a dihedral group of order 8 and E is abelian, then G is solvable.*

Proof: We argue by induction on $|G|$. The Sylow 2-subgroup of H is of nilpotency class 2 and thus, by Lemma 2.9, we may assume that H is not a maximal subgroup in G. Let $H < M < G$. By Lemma 2.1, $M_G > 1$. Now $H/(H \cap M_G) \cong HM_G/M_G = M/M_G$ is either $D \times F$ (where F is abelian), dihedral or abelian. By using induction (or Lemmas 2.2 and 2.3) we conclude that G/M_G and M are solvable, hence G is solvable. □

Theorem 3.2. *Let $H \cong D \times E$, where D and E are as in the previous lemma. If $G = \langle A, B\rangle$, then H is subnormal in G.*

Proof: Let G be a counterexample of the smallest order. If $H_G > 1$, then we use induction or Lemmas 2.5 and 2.6 and conclude that H/H_G is subnormal in G/H_G. But this implies that H is subnormal in G, so we may assume that $H_G = 1$.

Assume that H is not maximal in G. If $H < M < G$, then $M_G > 1$ by Lemma 2.1. Now $HM_G/M_G = M/M_G$ is subnormal in G/M_G (again by induction or by Lemmas 2.5 and 2.6), hence M is subnormal in G. Therefore we may also assume that $N_G(H) = H$. As the intersection of two subnormal subgroups of G is again subnormal in G, there exists only one subgroup $K < G$ such that H is maximal in K. If P is the Sylow p-subgroup of H without being a Sylow p-subgroup of G, then $K \leq N_G(P)$. As this cannot be true for every prime divisor of $|H|$, there is a Sylow q-subgroup Q of G such that $Q \leq H$ and $N_G(Q) = H$. If $H < M < G$ and M is maximal in G, then M is subnormal and, in fact, normal in G. By using the Frattini Argument, we get $G = MN_G(Q) = MH = M$, a contradiction.

Therefore we may assume that H is maximal in G and $N_G(H) = H$. Let N be a minimal normal subgroup of G. Since G is solvable by Lemma 3.1, N is an elementary abelian p-group. Clearly, $G = HN$, $H \cap N = 1$ and p does not divide $|H|$. Let $1 \neq z \in Z(D)$ and $e \in E$ with $|e| = 2$. Now $S = \langle z, e\rangle$ is a normal subgroup of H. Consider the group $T = NS$. Obviously $N_T(S) = S$ and $S \cap S^x = 1$ for all $x \in T \setminus S$. Thus T is a Frobenius group with a Frobenius complement S and using Lemma 2.11, it follows that S is either cyclic or generalized quaternion, a contradiction. This means that E has no element of order 2. But then the subnormality of H follows from Lemmas 2.6 and 2.7. $\square$

We shall now turn our attention to the case where H is nilpotent and $[G : H]$ is a power of two.

Lemma 3.3. *Let H be a nilpotent subgroup of G. If $[G : H] = 2^m$, then G is solvable.*

Proof: Let S be a Sylow 2-subgroup of G. Then $G = SH$ and as S and H are both nilpotent, our claim follows from Lemma 2.8. $\square$

Theorem 3.4. *Let $H \leq G$ be nilpotent, $[G : H] = 2^m$ and assume further that if q is an odd prime number and the Sylow q-subgroup of H is nonabelian, then $q \geq m$. If $G = \langle A, B\rangle$, then H is subnormal in G.*

Proof: Let G be a minimal counterexample. If $H_G > 1$, then it is clear that our claim is true. Thus we may assume that $H_G = 1$. We now divide the proof in two parts.

1) Assume that H is a maximal subgroup of G and let N be a minimal normal subgroup of G. As G is solvable by Lemma 3.3, it follows that N is an elementary abelian p-group. Clearly, N is not contained in H, hence $G = NH$ and we conclude that $p = 2$. If S is a Sylow 2-subgroup of H, then S is normal in G. Since $H_G = 1$, it follows that $S = 1$ and H is of odd order. By Lemma 2.5, we may assume that H has a nonabelian normal Sylow q-subgroup Q. By Lemma 2.10, Q has a normal subgroup $K \cong C_q \times C_q$. If $E = NK$, then $N_E(K) = K$. From Lemma 2.11 it follows that there exists $1 \neq n \in N$ such that $R = K \cap K^n = \langle x\rangle$, where $|x| = q$. As $|n| = 2$, we get $R^n = R$ and $xn = nx$. Now $N_Q(K)/C_Q(K) = Q/C_Q(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$ (for details, see [3, p. 20]). Thus

$|Q/C_Q(K)|$ divides the order $|\operatorname{GL}(2,q)| = (q^2-1)(q^2-q)$. It follows that either $C_Q(K) = Q$ or $C_Q(K)$ is a maximal subgroup of Q. As $H_G = 1$, we may conclude that $C_Q(K) = C_Q(x) = M$ is maximal in Q.

We may write $C_G(x) = LF$, where $L < N$, $M \le F < H$ and $|H/F| = q$. Now $Z(N\langle x\rangle) = L$ and $N\langle x\rangle$ is normal in NF, hence L is normal in NF. Let $H/F = \langle hF\rangle$, where $h \in Q$ and $h^q \in F$. If $y \in L \cap L^h$, then $y \in C_G(x) \cap C_G(x^h)$ and thus $y \in C_G(K) \cap N = 1$. As N is a minimal normal subgroup of G and $G = NF\langle h\rangle$, it follows that $N = L \times L^h \times \ldots \times L^{h^{q-1}}$. Thus $|N| = 2^m = |L|^q$, hence q divides m. But then $q = m$. Now Q is a Sylow q-subgroup of G, $N_G(Q) = H$ and therefore $2^m \equiv 1$ (q), a contradiction.

2) We assume that H is not a maximal subgroup of G. If $T < G$ and H is a maximal subgroup of T, then $T_G > 1$ by Lemma 2.1. We consider the factor groups G/T_G and $HT_G/T_G = T/T_G$ and conclude by induction that T is subnormal in G. Therefore $N_G(H) = H$ and H cannot be a 2-group. Let $N \le T_G$ be a minimal normal subgroup of G. As G is solvable, it follows that N is an elementary abelian p-group and from $T = NH$ we conclude that $p = 2$.

We write $H = SP$, where S is the Sylow 2-subgroup of H and $P > 1$ is the odd order Hall-subgroup of H. As T is subnormal in G, we have a subgroup F of G such that $T < F$ and T is normal in F. Now $N_T(S) = T$ and $N_T(P) = H$. If $N_F(P) > H$, then we have a subgroup $L \le N_F(P)$ such that H is maximal in L and T is not contained in L. Clearly, L is subnormal in G, hence $H = L \cap T$ is subnormal in G. Thus we may assume that $N_F(P) = H$. As P is a Hall-subgroup of T, we apply the generalized Frattini Argument and get $F = TN_F(P) = TH = T$, a contradiction. We conclude that H is subnormal in G. □

Corollary 3.5. *Let the assumptions be as in the previous theorem. If $G = \langle A, B\rangle$ and $H = S \times Q$, where S is the Sylow 2-subgroup of H and $|Q|$ is odd, then Q is normal in G.*

Proof: By Theorem 3.4, H is subnormal in G. Assume that H is normal in E and $H < E$. Since Q is a characteristic subgroup of H, we conclude that Q is normal in E. As $[E : H]$ divides $[G : H] = 2^m$, it follows that Q is characteristic in E. We may continue similarly and finally Q is normal in G. □

4. Loop theoretic results

Let Q be a loop and define the multiplication group $M(Q)$ and the inner mapping group $I(Q)$ as in the introduction. The relation between multiplication groups of loops and connected transversals in groups is given by the following result that was proved by Kepka and Niemenmaa [8, Theorem 4.1] in 1990.

Theorem 4.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B\rangle$.*

If Q is a loop, then the centre $Z(Q)$ consists of all elements $a \in Q$ which satisfy the equations $ax.y = a.xy$, $xa.y = x.ay$, $xy.a = x.ya$ and $ax = xa$ for all $x, y \in Q$. Clearly, $Z(Q)$ is an abelian group and $Z(Q) \cong Z(M(Q))$. It is also well-known that $N_{M(Q)}(I(Q)) = I(Q) \times Z(M(Q))$. If we put $Z_0 = 1$, $Z_1 = Z(Q)$ and $Z_i/Z_{i-1} = Z(Q/Z_{i-1})$, then we obtain a series of normal subloops of Q. If Z_{n-1} is a proper subloop of Q but $Z_n = Q$, then we say that Q is centrally nilpotent of class n. Following Bruck [2, p. 278–281] we write $I_0 = I(Q)$ and $I_i = N_{M(Q)}(I_{i-1})$ for each $i \geq 1$.

Bruck proved the following criterion for central nilpotency of Q.

Theorem 4.2. *A loop Q is centrally nilpotent of class n if and only if $I_n = M(Q)$ but $I_{n-1} \neq M(Q)$.*

We now link Theorems 2.3 and 3.4 and Corollary 3.5 with the criterion given in Theorem 4.2.

Corollary 4.3. *If Q is a finite loop and $I(Q) \cong D \times E$, where D is a dihedral group of order 8 and E is abelian, then Q is centrally nilpotent.*

Corollary 4.4. *If $|Q| = 2^m$, $I(Q)$ is nilpotent and $q \geq m$ for every odd prime q such that the Sylow q-subgroup of $I(Q)$ is nonabelian, then Q is centrally nilpotent and $I(Q)$ is a 2-group.*

If we combine Lemma 2.4 with Theorem 4.1, we immediately see that $I(Q)$ can never be a nontrivial cyclic group. Thus we pose the following

Problem 1. *Which finite nilpotent groups are possible as inner mapping groups of loops?*

In the light of the results given in Corollaries 4.3 and 4.4 it is natural to introduce

Problem 2. *If Q is a finite loop and $I(Q)$ is nilpotent, does it then follow that Q is centrally nilpotent?*

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