# A Parseval equation and a generalized finite Hankel transformation

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*Abstract.* In this paper, we study the finite Hankel transformation on spaces of generalized functions by developing a new procedure. We consider two Hankel type integral transformations $h_\mu$ and $h^*_\mu$ connected by the Parseval equation

$$\sum_{n=0}^{\infty}(h_\mu f)(n)(h^*_\mu\varphi)(n) = \int_0^1 f(x)\varphi(x)\,dx.$$

A space $S_\mu$ of functions and a space $L_\mu$ of complex sequences are introduced. $h^*_\mu$ is an isomorphism from $S_\mu$ onto $L_\mu$ when $\mu \geq -\frac{1}{2}$. We propose to define the generalized finite Hankel transform $h'_\mu f$ of $f \in S'_\mu$ by

$$\langle (h'_\mu f), ((h^*_\mu\varphi)(n))_{n=0}^{\infty}\rangle = \langle f, \varphi\rangle, \quad \text{for } \ \varphi \in S_\mu.$$

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## 1. Introduction and preliminaries.

Finite Hankel transforms of classical functions were first introduced by I.N. Sneddon [14] and later studied by other authors [3], [4], [7], [15]. More recently, A.H. Zemanian [18], J.N. Pandey and R.S. Pathak [11] and R.S. Pathak [12] extended these transforms to certain spaces of distributions as a special case of the general theory on orthonormal series expansions of generalized functions. L.S. Dube [5], R.S. Pathak and O.P. Singh [13] and J.M. Méndez and J.R. Negrín [10] investigated finite Hankel transformations in other spaces of distributions through a procedure quite different from that one which was developed in [18] and [12]. All previous authors employ a method usually known as the kernel method.

Specifically, L.S. Dube [5] investigated finite Hankel transformation of the first kind given by

$$(h_\mu f)(n) = \int_0^1 xJ_\mu(\lambda_n x)f(x)\,dx, \quad n = 0, 1, 2\ldots$$

for $\mu \geq -\frac{1}{2}$, where $J_\mu$ denotes the Bessel function of the first kind and order $\mu$ and $\lambda_n$, $n = 0, 1, 2\ldots$, represent the positive roots of $J_\mu(x) = 0$ arranged in ascending order of magnitude [17, p. 596].

For $\mu \geq -\frac{1}{2}$ and $\alpha \geq \frac{1}{2}$, he introduced the space $\mathcal{U}_{\mu,\alpha}$ of finitely differentiable functions on $(0,1)$ such that

$$\rho_k^{\mu,\alpha}(\varphi) = \sup_{0<x<1} |x^{\alpha-1} B_\mu^{*k} \varphi(x)| < \infty, \qquad \text{for every} \quad k \in \mathbb{N},$$

where $B_\mu^* = x^{-\mu} D x^{2\mu+1} D x^{-\mu-1}$.

$\mathcal{U}_{\mu,\alpha}$ is equipped with the topology generated by the family of seminorms $\{\rho_k^{\mu,\alpha}\}_{k=0}^\infty$. Thus $\mathcal{U}_{\mu,\alpha}$ is a Fréchet space. The dual space of $\mathcal{U}_{\mu,\alpha}$ is denoted by $\mathcal{U}'_{\mu,\alpha}$ and it is endowed with the weak topology.

For $f \in \mathcal{U}'_{\mu,\alpha}$, the generalized finite Hankel transform of $f$ is defined by

$$F(n) = \langle f(x), x J_\mu(\lambda_n x) \rangle, \qquad \text{for} \quad n = 0, 1, 2 \ldots \; . \tag{1}$$

Our objective in this paper is to define the finite Hankel transformation $h_\mu$ on new spaces of generalized functions by developing a new procedure. The method that we develop in this work can be seen as a finite analogue to the one investigated by J.M. Méndez [8] for the infinite Hankel transformation.

We introduce the finite Hankel type transformation $h_\mu^*$ by

$$(h_\mu^* f)(n) = \frac{2}{J_{\mu+1}^2(\lambda_n)} \int_0^1 J_\mu(\lambda_n x) f(x)\, dx, \quad n = 0, 1, 2 \ldots$$

when $\mu \geq -\frac{1}{2}$.

The transformations $h_\mu$ and $h_\mu^*$ are closely connected. They satisfy the Parseval equation

$$\sum_{n=0}^\infty (h_\mu f)(n)(h_\mu^* \varphi)(n) = \int_0^1 f(x)\varphi(x)\, dx \tag{2}$$

when $\mu \geq -\frac{1}{2}$ and $f$ and $\varphi$ are suitable functions.

We define a space $S_\mu$ of functions and a space $L_\mu$ of sequences and we prove that $h_\mu^*$ is an isomorphism from $S_\mu$ onto $L_\mu$ provided that $\mu \geq -\frac{1}{2}$.

The generalized finite Hankel transformation $h_\mu f$ of $f \in S'_\mu$, the dual space of $S_\mu$, is defined through

$$\langle (h'_\mu f), ((h_\mu^* \varphi)(n))_{n=0}^\infty \rangle = \langle f, \varphi \rangle, \quad \text{for} \quad \varphi \in S_\mu. \tag{3}$$

Notice that (3) appears as a generalization of the Parseval equation (2).

We show that the conventional finite Hankel transformation $h_\mu$ and the generalized finite Hankel transformation given by (1) are special cases of our generalized transformation.

Finally we present some applications of the new generalized finite Hankel transformation.

Throughout this paper, $\mu$ denotes a real number greater or equal to $-\frac{1}{2}$.

Let us take note here of some properties of Bessel functions that we shall use quite a few times in this work (see [17]).

The behaviours of $J_\mu$ near the origin and the infinity are the following ones:

$$J_\mu(x) = O(x^\mu), \quad \text{as } x \to 0^+, \tag{4}$$

$$J_\mu(x) \simeq \left(\frac{2}{\pi x}\right)^{1/2}\left[\cos(x - \frac{1}{2}\mu\pi - \frac{1}{4}\pi)\sum_{m=0}^{\infty}\frac{(-1)^m(\mu,2m)}{(2x)^{2m}} - \right.$$
$$\left. - \sin(x - \frac{1}{2}\mu\pi - \frac{1}{4}\pi)\sum_{m=0}^{\infty}\frac{(-1)^m(\mu,2m+1)}{(2x)^{2m+1}}\right], \quad \text{as } x \to \infty, \tag{5}$$

where $(\mu, k)$ is understood as in [17, p. 198].

The main differentiation formulas for $J_\mu$ are

$$\frac{d}{dx}(x^\mu J_\mu(xy)) = yx^\mu J_{\mu-1}(xy), \tag{6}$$

$$\frac{d}{dx}(x^{-\mu} J_\mu(xy)) = -yx^{-\mu} J_{\mu+1}(xy), \tag{7}$$

for $x, y > 0$. By combining (6) and (7), it can be easily inferred

$$B_\mu J_\mu(x) = -J_\mu(x), \quad \text{for } x > 0, \tag{8}$$

where $B_\mu = x^{-\mu-1}Dx^{2\mu+1}Dx^{-\mu}$.

## 2. The spaces $S_\mu$ and $L_\mu$ and the finite Hankel transformation.

In this section, we introduce a space $S_\mu$ of functions and a space $L_\mu$ of complex sequences and we investigate the finite Hankel transformation $h_\mu^*$ on them.

We define $S_\mu$ as the space of all complex valued functions $\varphi(x)$ on $(0, 1]$ such that $\varphi(x)$ is infinitely differentiable and satisfies for every $k \in \mathbb{N}$

(i) $B_\mu^{*k}\varphi(1) = 0$,

(ii) $x^{\mu+1}B_\mu^{*k}\varphi(x) \to 0$ and $x^{2\mu+1}\frac{d}{dx}(x^{-\mu-1}B_\mu^{*k}\varphi(x)) \to 0$, as $x \to 0^+$,

and

(iii) $x^{-1/2}B_\mu^{*k}\varphi(x) \in L(0, 1)$.

$S_\mu$ is endowed with the topology generated by the family of seminorms $\{\| \ \|_k\}_{k=0}^{\infty}$, where

$$\|\varphi\|_k = \int_0^1 x^{-1/2}|B_\mu^{*k}\varphi(x)|\,dx, \quad \text{for } \varphi \in S_\mu \text{ and } k \in \mathbb{N}.$$

Notice that $\| \ \|_0$ is a norm. $S_\mu$ is a Hausdorff topological linear space that verifies the first countability axiom. Moreover, the operator $B_\mu^*$ defines a continuous mapping from $S_\mu$ into itself. $S'_\mu$ is the dual space of $S_\mu$ and it is equipped with the usual weak topology.

The following result will be useful in the sequel.

**Proposition 1.** *If $f(x)$ is a function defined on $(0,1)$ such that $x^{1/2}f(x)$ is bounded on $(0,1)$, then $f(x)$ generates a member of $S'_\mu$ through the definition*

$$\langle f(x), \varphi(x)\rangle = \int_0^1 f(x)\varphi(x)\,dx, \quad \varphi \in S_\mu.$$

PROOF: The result easily follows from the inequality

$$|\langle f(x), \varphi(x)\rangle| \leq \|\varphi\|_0 \sup_{0<x<1} |x^{1/2} f(x)|, \quad \varphi \in S_\mu.$$

□

The spaces $\mathcal{U}_{\mu,\alpha}$ defined by L.S. Dube [5] are related to $S_\mu$ as follows:

**Proposition 2.** *Let $\mu \geq -\frac{1}{2}$ and $\alpha \geq \frac{1}{2}$. Then $S_\mu \subset \mathcal{U}_{\mu,\alpha}$ and the topology of $S_\mu$ is stronger than that induced on it by $\mathcal{U}_{\mu,\alpha}$.*

PROOF: Let $\varphi \in S_\mu$. In virtue of the conditions (i) and (ii), we can write

$$x^{\alpha-1} B_\mu^{*k}\varphi(x) = x^{\alpha+\mu}\int_1^x t^{-2\mu-1}\int_0^t u^\mu B_\mu^{*k+1}\varphi(u)\,du\,dt$$

for every $x \in (0,1)$ and $k \in \mathbb{N}$.

Therefore

$$|x^{\alpha-1} B_\mu^{*k}\varphi(x)| \leq x^{\alpha+\mu}\int_x^1 t^{-\mu-(1/2)}\,dt \int_0^1 u^{-1/2}|B_\mu^{*k+1}\varphi(u)|\,du \leq$$
$$\leq x^{\alpha-(1/2)}\int_0^1 u^{-1/2}|B_\mu^{*k+1}\varphi(u)|\,du \leq \int_0^1 u^{-1/2}|B_\mu^{*k+1}\varphi(u)|\,du$$

for every $x \in (0,1)$ and $k \in \mathbb{N}$.

Hence, for every $\varphi \in S_\mu$ and $k \in \mathbb{N}$,

$$\sup_{0<x<1} |x^{\alpha-1} B_\mu^{*k}\varphi(x)| \leq \|\varphi\|_{k+1}\,,$$

and $S_\mu$ is contained in $\mathcal{U}_{\mu,\alpha}$ and the inclusion is continuous. □

From Proposition 2, we can deduce that if $f \in \mathcal{U}'_{\mu,\alpha}$, then the restriction of $f$ to $S_\mu$ is a member of $S'_\mu$.

We now define $L_\mu$ as the space of all complex sequences $(a_n)_{n=0}^\infty$ such that $\lim_{n\to\infty} a_n\lambda_n^{2k} = 0$, for every $k \in \mathbb{N}$, where $\lambda_n$, $n = 0,1,2,\ldots$, represent the positive roots of the equation $J_\mu(x) = 0$ arranged in ascending order of magnitude. The topology of $L_\mu$ is that generated by the family of norms $\{\gamma_\mu^k\}_{k=0}^\infty$, where

$$\gamma_\mu^k((a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n|\lambda_n^{2k}, \quad \text{for} \;\; (a_n)_{n=0}^\infty \in L_\mu \;\; \text{and} \;\; k \in \mathbb{N}.$$

Notice that $\gamma_\mu^k((a_n)_{n=0}^\infty) < \infty$ for every $(a_n)_{n=0}^\infty \in L_\mu$. Thus $L_\mu$ is a Hausdorff topological linear space that satisfies the first countability axiom. $L'_\mu$ denotes the dual space of $L_\mu$ and it is endowed with the weak topology.

In the following proposition, we introduce continuous operations in $L_\mu$ and $L'_\mu$.

**Proposition 3.** *Let $(b_n)_{n=0}^{\infty}$ be a complex sequence such that $|b_n| \leq M\lambda_n^{\ell}$ for every $n \in \mathbb{N}$ and for some $\ell \in \mathbb{N}$ and $M > 0$. Then the linear operator*

$$(a_n)_{n=0}^{\infty} \longrightarrow (a_n b_n)_{n=0}^{\infty}$$

*is a continuous mapping from $L_\mu$ into itself.*

*Moreover, the operator in $L'_\mu$, $B \rightarrow (b_n)_{n=0}^{\infty}B$, where*

$$\langle (b_n)_{n=0}^{\infty}B, (a_n)_{n=0}^{\infty}\rangle = \langle B, (a_n b_n)_{n=0}^{\infty}\rangle, \quad \text{for } (a_n)_{n=0}^{\infty} \in L_\mu,$$

*is a continuous mapping from $L'_\mu$ into itself.*

Proof: It is sufficient to see that

$$\gamma_\mu^k((a_n b_n)_{n=0}^{\infty}) \leq M \sum_{n=0}^{\infty} |a_n|\lambda_n^{2k+\ell} \leq M_1 \gamma_\mu^{k+\ell}((a_n)_{n=0}^{\infty}),$$

$$\text{for } (a_n)_{n=0}^{\infty} \in L_\mu \text{ and } k \in \mathbb{N},$$

$M_1$ being a suitable positive constant. □

By proceeding as in the proof of the last proposition, we also can establish following

**Proposition 4.** *If $(b_n)_{n=0}^{\infty}$ is a complex sequence satisfying the same conditions as in Proposition 3, then $(b_n)_{n=0}^{\infty}$ generates a member of $L'_\mu$ by*

$$\langle (b_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\rangle = \sum_{n=0}^{\infty} a_n b_n, \quad \text{for } (a_n)_{n=0}^{\infty} \in L_\mu.$$

The fundamental theorem in our theory of a generalized finite Hankel transformation asserts that the conventional finite Hankel transformation $h_\mu^*$ is an isomorphism from $S_\mu$ onto $L_\mu$. The proof of this fact is the next object:

**Theorem 1.** *For $\mu \geq -\frac{1}{2}$, the finite Hankel transformation $h_\mu^*$ is an isomorphism from $S_\mu$ onto $L_\mu$.*

Proof: Let $\varphi \in S_\mu$. As it is known, $h_\mu^*\varphi = (a_n)_{n=0}^{\infty}$, where

$$a_n = \frac{2}{J_{\mu+1}^2(\lambda_n)} \int_0^1 J_\mu(\lambda_n x)\varphi(x)\, dx, \quad \text{for every } n \in \mathbb{N}.$$

In virtue of the operational rule (6), we can write for every $n \in \mathbb{N}$

$$\begin{aligned} \lambda_n^2 a_n &= \frac{2\lambda_n^2}{J_{\mu+1}^2(\lambda_n)} \int_0^1 J_\mu(\lambda_n x)\varphi(x)\, dx = \\ &= \frac{2\lambda_n}{J_{\mu+1}^2(\lambda_n)} \int_0^1 \frac{d}{dx}(x^{\mu+1} J_{\mu+1}(\lambda_n x))x^{-\mu-1}\varphi(x)\, dx = \\ &= \frac{2\lambda_n}{J_{\mu+1}^2(\lambda_n)} \left\{ J_{\mu+1}(\lambda_n x)\varphi(x)]_0^1 - \int_0^1 x^{\mu+1} J_{\mu+1}(\lambda_n x)\, \frac{d}{dx}(x^{-\mu-1}\varphi(x))\, dx \right\}. \end{aligned}$$

Moreover, according to (4) $J_{\mu+1}(\lambda_n x)\varphi(x)]_0^1 = 0$ since $\varphi(1) = 0$ and $\lim_{x\to O^+} x^{\mu+1}\varphi(x) = 0$.

Hence

$$\lambda_n^2 a_n = -\frac{2\lambda_n}{J_{\mu+1}^2(\lambda_n)} \int_0^1 x^{\mu+1} J_{\mu+1}(\lambda_n x)\, \frac{d}{dx}(x^{-\mu-1}\varphi(x))\, dx. \tag{9}$$

Now, by invoking (7), one has

$$\begin{aligned}\lambda_n \int_0^1 x^{\mu+1} J_{\mu+1}(\lambda_n x)\, \frac{d}{dx}(x^{-\mu-1}\varphi(x))\, dx = \\ = -\int_0^1 \frac{d}{dx}(x^{-\mu} J_\mu(\lambda_n x)) x^{2\mu+1}\, \frac{d}{dx}(x^{-\mu-1}\varphi(x))\, dx = \\ = -J_\mu(\lambda_n x) x^{\mu+1}\, \frac{d}{dx}(x^{-\mu-1}\varphi(x))]_0^1 + \int_0^1 B_\mu^* \varphi(x) J_\mu(\lambda_n x)\, dx.\end{aligned}$$

Also in this case by (4), the limit terms are equal to zero because $J_\mu(\lambda_n) = 0$, $\varphi \in C^\infty((0,1])$, $\lim_{x\to 0^+} x^{2\mu+1} \frac{d}{dx}(x^{-\mu-1}\varphi(x)) = 0$.

Therefore

$$\lambda_n \int_0^1 x^{\mu+1} J_{\mu+1}(\lambda_n x)\, \frac{d}{dx}(x^{-\mu-1}\varphi(x))\, dx = \int_0^1 B_\mu^* \varphi(x) J_\mu(\lambda_n x)\, dx. \tag{10}$$

By combining (9) and (10), we obtain

$$a_n \lambda_n^2 = -\frac{2}{J_{\mu+1}^2(\lambda_n)} \int_0^1 B_\mu^* \varphi(x) J_\mu(\lambda_n x)\, dx, \quad \text{for every} \quad n \in \mathbb{N}.$$

An inductive procedure allows us to establish that

$$\lambda_n^{2k} a_n = (-1)^k \frac{2}{J_{\mu+1}^2(\lambda_n)} \int_0^1 B_\mu^{*k} \varphi(x) J_\mu(\lambda_n x)\, dx, \quad \text{for every} \quad n, k \in \mathbb{N}. \tag{11}$$

From (11), according to Riemann–Lebesgue Lemma ([17, p. 457]), one follows to

$$J_{\mu+1}^2(\lambda_n)\lambda_n^{2k} a_n \to 0, \quad \text{as} \quad n \to \infty.$$

Moreover by (5), there exists a positive constant $M$ such that

$$\lambda_n^{2k}|a_n| \le M J_{\mu+1}^2(\lambda_n)\lambda_n^{2k+1}|a_n|,$$

and then $\lambda_n^{2k} a_n \to 0$, as $n \to \infty$, for every $k \in \mathbb{N}$.

Also, for certain $M_i > 0$, $i = 1, 2$,

$$\sum_{n=0}^{\infty} \lambda_n^{2k} |a_n| = \sum_{n=0}^{\infty} \frac{2}{J_{\mu+1}^2(\lambda_n)\lambda_n^4} \left| \int_0^1 B_\mu^{*k+2} \varphi(x) J_\mu(\lambda_n x)\, dx \right| \leq$$

$$\leq M_1 \sum_{n=0}^{\infty} \lambda_n^{-5/2} \int_0^1 |\sqrt{\lambda_n x} J_\mu(\lambda_n x)| x^{-1/2} |B_\mu^{*k+2} \varphi(x)|\, dx \leq$$

$$\leq M_2 \sum_{n=0}^{\infty} \lambda_n^{-2} \int_0^1 x^{-1/2} |B_\mu^{*k+2} \varphi(x)|\, dx.$$

Hence, since $\sum_{n=0}^{\infty} \lambda_n^{-2} < \infty$, we get

$$\gamma_\mu^k((a_n)_{n=0}^{\infty}) \leq M_3 \|\varphi\|_{k+2}$$

for every $k \in \mathbb{N}$ and $\varphi \in S_\mu$ and for some $M_3 > 0$.

This inequality proves that the linear mapping $h_\mu^*$ is continuous from $S_\mu$ into $L_\mu$.

Let now $(a_n)_{n=0}^{\infty} \in L_\mu$ and define $\tau_\mu((a_n)_{n=0}^{\infty})(x) = \varphi(x) = \sum_{n=0}^{\infty} a_n x J_\mu(\lambda_n x)$, for $x \in (0, 1]$.

By (4) and (5), we have

$$\sum_{n=0}^{\infty} |a_n x J_\mu(\lambda_n x)| \leq M x^{1/2} \sum_{n=0}^{\infty} |a_n|, \ x > 0$$

for a suitable $M > 0$. Therefore $\varphi(x) \in C(0, \infty)$. In a similar way we can prove that $\varphi \in C^\infty(0, \infty)$.

Moreover, by invoking (8), we obtain

$$B_\mu^{*k} \varphi(x) = \sum_{n=0}^{\infty} (-1)^k a_n \lambda_n^{2k} x J_\mu(\lambda_n x), \quad \text{for} \quad x > 0 \quad \text{and} \quad k \in \mathbb{N}.$$

Then $B_\mu^{*k} \varphi(1) = 0$, for each $k \in \mathbb{N}$.

We also can infer

$$|x^{\mu+1} B_\mu^{*k} \varphi(x)| \leq M_1 x^{\mu+(3/2)} \sum_{n=0}^{\infty} |a_n| \lambda_n^{2k}, \quad \text{for} \quad x > 0 \quad \text{and} \quad k \in \mathbb{N},$$

and from (4), (5) and (6),

$$\left|x^{2\mu+1} \frac{d}{dx}(x^{-\mu-1} B_\mu^{*k} \varphi(x))\right| \leq M_2 x^{2\mu+2} \sum_{n=0}^{\infty} |a_n| \lambda_n^{2k+2+\mu}, \quad \text{for} \quad x > 0 \quad \text{and} \quad k \in \mathbb{N}.$$

Here $M_1$ and $M_2$ denote suitable positive constants. Hence

$$\lim_{x \to 0^+} x^{\mu+1} B_\mu^{*k} \varphi(x) = \lim_{x \to 0^+} x^{2\mu+1} \frac{d}{dx}(x^{-\mu-1} B_\mu^{*k} \varphi(x)) = 0, \quad \text{for every} \quad k \in \mathbb{N}.$$

On the other hand, since the series defining $B_\mu^{*k}\varphi(x)$ is uniformly convergent in $x \in (0,1)$, there exists a positive constant $M_3$ such that

$$\int_0^1 x^{-1/2}|B_\mu^{*k}\varphi(x)|\,dx \le M_3 \sum_{n=0}^{\infty} |a_n|\lambda_n^{2k}, \quad \text{for every } \ k \in \mathbb{N}.$$

Therefore $\tau_\mu$ is a continuous mapping from $L_\mu$ into $S_\mu$.

Finally, we infer from [17, p. 591] that $(\tau_\mu \cdot h_\mu^*)\varphi = \varphi$, for $\varphi \in S_\mu$, and $(h_\mu^* \cdot \tau_\mu)(a_n)_{n=0}^\infty = (a_n)_{n=0}^\infty$, for $(a_n)_{n=0}^\infty \in L_\mu$. Thus the proof is finished. □

**3. The generalized finite Hankel transformation.**

We define the generalized finite Hankel transformation $h'_\mu$ on $S'_\mu$ as follows:

$$\langle (h'_\mu f), ((h_\mu^*\varphi)(n))_{n=0}^\infty\rangle = \langle f(x), \varphi(x)\rangle, \quad \text{for every } \ \varphi \in S_\mu. \tag{12}$$

Notice that (12) appears as a generalization of the Parseval equation (2).

From Theorem 1.10–2 in [19] and Theorem 1, we immediately obtain

**Theorem 2.** *For $\mu \ge -\frac{1}{2}$, the generalized finite Hankel transformation $h'_\mu$ is an isomorphism from $S'_\mu$ onto $L'_\mu$.*

In the following proposition, we establish that the conventional finite Hankel transformation $h_\mu$ is a special case of the generalized finite Hankel transformation defined in (12).

**Theorem 3.** *Let $f(x)$ be a function defined on $(0,1)$ such that $x^{1/2}f(x)$ is bounded on $(0,1)$. Then $((h_\mu f)(n))_{n=0}^\infty$ agrees with $(h'_\mu f)$ as members of $L'_\mu$.*

Proof: The conventional finite Hankel transformation of $f$ is defined by

$$(h_\mu f)(n) = \int_0^1 xJ_\mu(\lambda_n x)f(x)\,dx, \quad \text{for } \ n \in \mathbb{N}.$$

Then, since $x^{1/2}f(x)$ is bounded on $(0,1)$, and by (4) and (5) we can write

$$|(h_\mu f)(n)| \le M\lambda_n^{-1/2}\int_0^1 |\sqrt{\lambda_n x}J_\mu(\lambda_n x)|\,dx \le M_1\lambda_n^{-1/2}, \quad \text{for } \ n \in \mathbb{N},$$

where $M$ and $M_1$ are certain positive constants.

Therefore, in virtue of Proposition 4, $((h_\mu f)(n))_{n=0}^\infty$ generates a member of $L'_\mu$ by

$$\langle ((h_\mu f)(n))_{n=0}^\infty, (a_n)_{n=0}^\infty\rangle = \sum_{n=0}^{\infty}(h_\mu f)(n)a_n = \sum_{n=0}^{\infty} a_n \int_0^1 xJ_\mu(\lambda_n x)f(x)\,dx =$$

$$= \int_0^1 f(x)\sum_{n=0}^{\infty} a_n xJ_\mu(\lambda_n x)\,dx, \quad \text{for every } \ (a_n)_{n=0}^\infty \in L_\mu.$$

The last equality is justified since the series $\sum_{n=0}^{\infty} a_n x^{1/2} J_\mu(\lambda_n x)$ is uniformly convergent on $(0,1)$ and $x^{1/2} f(x)$ is bounded on $(0,1)$.

We can also write

$$\langle ((h_\mu f)(n))_{n=0}^{\infty}, ((h_\mu^* \varphi)(n))_{n=0}^{\infty} \rangle =$$
$$= \int_0^1 f(x) \sum_{n=0}^{\infty} (h_\mu^* \varphi)(n) x J_\mu(\lambda_n x)\, dx = \int_0^1 f(x)\varphi(x)\, dx$$

for every $\varphi \in S_\mu$.

Hence, according to Proposition 1, we conclude

$$\langle ((h_\mu f)(n))_{n=0}^{\infty}, ((h_\mu^* \varphi)(n))_{n=0}^{\infty} \rangle = \langle f(x), \varphi(x) \rangle, \quad \text{for} \quad \varphi \in S_\mu,$$

and $((h_\mu f)(n))_{n=0}^{\infty} = (h'_\mu f)$ as members of $L'_\mu$. □

As it was showed in Section 2, if $f \in \mathcal{U}'_{\mu,\alpha}$, then the restriction of $f$ to $S_\mu$ is in $S'_\mu$. Hence, if $f \in \mathcal{U}'_{\mu,\alpha}$, we can define two generalized finite Hankel transformations of $f$. We now prove that the generalized finite Hankel transform of $f$ given by (1) is equal (in the sense of equality in $L'_\mu$) to the generalized finite Hankel transform of $f$ as given by (12).

**Theorem 4.** *Let $\mu \geq -\frac{1}{2}$, $\alpha \geq \frac{1}{2}$ and $f \in \mathcal{U}'_{\mu,\alpha}$. Then*

$$\langle (F(n))_{n=0}^{\infty}, (a_n)_{n=0}^{\infty} \rangle = \langle (h'_\mu f), (a_n)_{n=0}^{\infty} \rangle, \quad \textit{for every} \quad (a_n)_{n=0}^{\infty} \in L'_\mu,$$

*where $F(n) = \langle f(x), x J_\mu(\lambda_n x) \rangle$, for every $n \in \mathbb{N}$.*

Proof: According to Theorem 1.8–1 in [19], since $f \in \mathcal{U}'_{\mu,\alpha}$, there exist $r \in \mathbb{N}$ and $M > 0$ such that

$$|\langle f(x), x J_\mu(\lambda_n x) \rangle| \leq M \max_{0 \leq k \leq r} \sup_{0<x<1} |x^{\alpha-1} B_\mu^{*k}(x J_\mu(\lambda_n x))|, \quad \text{for every} \quad n \in \mathbb{N}.$$

Hence, from (4), (5) and (8), we infer that

$$|F(n)| \leq M \max_{0 \leq k \leq r} \sup_{0<x<1} |x^{\alpha-1} \lambda_n^{2k} x J_\mu(\lambda_n x)| \leq M_1 \lambda_n^{2r} \tag{13}$$

for a certain $M_1 > 0$. By invoking Proposition 4, (13) proves that $(F(n))_{n=0}^{\infty}$ generates a member of $L'_\mu$ through

$$\langle (F(n))_{n=0}^{\infty}, (a_n)_{n=0}^{\infty} \rangle = \sum_{n=0}^{\infty} F(n) a_n, \quad \text{for} \quad (a_n)_{n=0}^{\infty} \in L_\mu.$$

To show our assertion we must establish that

$$\sum_{n=0}^{\infty} F(n) a_n = \langle f(x), \sum_{n=0}^{\infty} a_n x J_\mu(\lambda_n x) \rangle, \quad \text{for} \quad (a_n)_{n=0}^{\infty} \in L_\mu. \tag{14}$$

Let $(a_n)_{n=0}^\infty \in L_\mu$. As it is easy to see,

$$\sum_{n=0}^{\infty} F(n)a_n = \langle f(x), \sum_{n=0}^{m} a_n x J_\mu(\lambda_n x)\rangle + \sum_{n=m+1}^{\infty} a_n \langle f(x), xJ_\mu(\lambda_n x)\rangle \tag{15}$$

for every $m \in \mathbb{N}$.

We can deduce from (13) that

$$|\sum_{n=m+1}^{\infty} a_n\langle f(x), xJ_\mu(\lambda_n x)\rangle| \le M_1 \sum_{n=m+1}^{\infty} |a_n|\lambda_n^{2r}, \quad \text{for every} \quad m \in \mathbb{N}$$

with $M_1 > 0$. Then

$$\lim_{m\to\infty} \sum_{n=m+1}^{\infty} a_n\langle f(x), xJ_\mu(\lambda_n x)\rangle = 0. \tag{16}$$

Moreover, for every $k \in \mathbb{N}$ and $x \in (0,1)$, we get

$$|x^{\alpha-1} B_\mu^{*k}[\sum_{n=m+1}^{\infty} a_n x J_\mu(\lambda_n x)]| \le$$
$$\le x^{\alpha-1} \sum_{n=m+1}^{\infty} |a_n x J_\mu(\lambda_n x)|\lambda_n^{2k} \le M_2 x^{\alpha-(1/2)} \sum_{n=m+1}^{\infty} |a_n|\lambda_n^{2k}$$

for a suitable $M_2 > 0$.

Hence

$$\sup_{0<x<1} |x^{\alpha-1} B_\mu^{*k}[\sum_{n=m+1}^{\infty} a_n x J_\mu(\lambda_n x)]| \le M_2 \sum_{n=m+1}^{\infty} |a_n|\lambda_n^{2k}, \quad \text{for every} \quad k \in \mathbb{N},$$

and $\sum_{n=m+1}^\infty a_n x J_\mu(\lambda_n x) \to 0$, as $m \to \infty$, in $S_\mu$, because $(a_n)_{n=0}^\infty \in L_\mu$.

Therefore, since $f \in S'_\mu$,

$$\lim_{m\to\infty} \langle f(x), \sum_{n=m+1}^{\infty} a_n x J_\mu(\lambda_n x)\rangle = 0. \tag{17}$$

By combining now (15), (16) and (17), we obtain (14).

From (14), we can conclude

$$\langle (F(n))_{n=0}^\infty, ((h_\mu^*\varphi)(n))_{n=0}^\infty\rangle = \langle f(x), \sum_{n=0}^{\infty} (h_\mu^*\varphi)(n) x J_\mu(\lambda_n x)\rangle = \langle f(x), \varphi(x)\rangle =$$
$$= \langle (h'_\mu f), ((h_\mu^*\varphi)(n))_{n=0}^\infty\rangle, \quad \text{for} \quad \varphi \in S_\mu,$$

and the proof is complete. $\square$

## 4. Applications.

We firstly prove an operation-transform formula for the generalized finite Hankel transformation that will be useful in applications.

**Proposition 5.** *Let $P$ be a polynomial and $f$ be in $S'_\mu$. Then*

$$(h'_\mu P(B_\mu)f) = P(-\lambda_n^2)(h'_\mu f).$$

Proof: If $f \in S'_\mu$, we have

$$\begin{aligned}\langle (h'_\mu P(B_\mu)f), ((h^*_\mu\varphi)(n))_{n=0}^\infty\rangle &= \langle P(B_\mu)f, \varphi\rangle = \langle f, P(B^*_\mu)\varphi\rangle = \\ = \langle (h'_\mu f), ((h^*_\mu P(B^*_\mu)\varphi)(n))_{n=0}^\infty\rangle &= \langle (h'_\mu f), (P(-\lambda_n^2)(h^*_\mu\varphi)(n))_{n=0}^\infty\rangle = \\ = \langle P(-\lambda_n^2)(h'_\mu f), ((h^*_\mu\varphi)(n))_{n=0}^\infty\rangle, &\quad \text{for every } \ \varphi \in S_\mu.\end{aligned}$$

We consider the functional equation

$$P(B_\mu)f = g, \tag{18}$$

where $g$ is a given member of $S'_\mu$, $P$ is a polynomial such that $P(-\lambda_n^2) \neq 0$, for every $n \in \mathbb{N}$, and $f$ is unknown generalized function but required to be in $S'_\mu$.

By applying the generalized finite Hankel transform to (18) and according to Proposition 5, we can prove that (18) is equivalent to

$$P(-\lambda_n^2)(h'_\mu f) = (h'_\mu g).$$

Hence it is not difficult to see that the functional $f$ defined by

$$\langle f, \varphi\rangle = \langle g, \sum_{n=0}^\infty \frac{1}{P(-\lambda_n^2)}(h^*_\mu\varphi)(n)xJ_\mu(\lambda_n x)\rangle \quad \text{for} \quad \varphi \in S_\mu,$$

is in $S'_\mu$ and it is the solution for (18). □

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