L^p-approximation of Jacobians

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Abstract. The paper investigates the nonlinear function spaces introduced by Giaquinta, Modica and Souček. It is shown that a function from $\mathrm{Cart}^p(\Omega, \mathbf{R}^m)$ is approximated by $\mathcal{C}^1$ functions strongly in $\mathcal{A}^q(\Omega, \mathbf{R}^m)$ whenever $q < p$. An example is shown of a function which is in $\mathrm{cart}^p(\Omega, \mathbf{R}^2)$ but not in $\mathrm{cart}^p(\Omega, \mathbf{R}^2)$.

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1. Introduction.

Some integrals in the calculus of variation (e.g. arising from nonlinear elasticity) require nonlinear function spaces for their investigation. The Sobolev spaces with small exponents do not guarantee the weak lower semicontinuity, whereas for great exponents the functional is not coercive. If, for example,

$$\mathcal{F}(u) = \int_\Omega (|Du|^p + |\det Du|^q)\, dx,$$

then an appropriate space for studying this functional is one of the nonlinear function spaces described below.

Let $\Omega \subset \mathbf{R}^N$ be an open set with a finite measure. Consider a function u belonging to the Sobolev space $H^{1,1}(\Omega, \mathbf{R}^m)$. Then the distributive gradient

$$Du = \Big(\frac{\partial u^j}{\partial x_i}\Big)_{\substack{i=1,\dots,N \\ j=1,\dots,m}}$$

is defined almost everywhere. If $k \le N$, α is a multiindex from $J_k := \{1, \dots, N\}^k$ and β is a multiindex from $J^k := \{1, \dots, m\}^k$, then $M_\alpha^\beta Du(x)$ denotes the minor

$$\det\Big(\frac{\partial u^{\beta_j}}{\partial x_{\alpha_i}}(x)\Big)_{i,j=1,\dots,k}$$

(of course, $M_\alpha^\beta Du = 0$ if $k > m$). Further, $M_k Du(x) \in \mathbf{R}^{N^k m^k}$ is the multivector of all minors $M_\alpha^\beta Du(x)$, where $\alpha \in \{1, \dots, N\}^k$ and $\beta \in \{1, \dots, m\}^k$. Let $p =$

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$(p_1, \dots, p_N)$ be a multiexponent, $1 < p_i < \infty$ for all $i = 1, \dots, N$. Following [2], [5] we say that $u \in \mathcal{A}^p(\Omega, \mathbf{R}^m)$, if $\|u\|_{\mathcal{A}^p}$ is finite, where

$$\|u\|_{\mathcal{A}^p} = \|u\|_{H^{1,p_1}(\Omega)} + \sum_{k=2}^{N} \Big(\int_\Omega |M_k Du(x)|^{p_k} \, dx \Big)^{1/p_k}.$$

Notice that $\mathcal{A}^p(\Omega, \mathbf{R}^m)$ is not a vector space and $\|.\|_{\mathcal{A}^p}$ is not a norm. Let u, $u_n \in \mathcal{A}^p(\Omega, \mathbf{R}^m)$. We say that $u_n \to u$ weakly in $\mathcal{A}^p$, if $u_n \to u$ weakly in H^{1,p_1} and for each $k = 1, \dots, N$, and each $\alpha \in \{1, \dots, N\}^k$ and $\beta \in \{1, \dots, m\}^k$, $M_\alpha^\beta Du_n \to M_\alpha^\beta Du$ weakly in L^{p_k}. Further, we say that $u_n \to u$ strongly in $\mathcal{A}^p$, if $u_n \to u$ strongly in H^{1,p_1} and for each $k = 1, \dots, N$, and each $\alpha \in \{1, \dots, N\}^k$ and $\beta \in \{1, \dots, m\}^k$, $M_\alpha^\beta Du_n \to M_\alpha^\beta Du$ strongly in L^{p_k}.

The spaces $\mathcal{A}^p$ are too large: they contain elements which are not accessible as weak limits of smooth functions. Denote (for a moment) by S the set of all $\mathcal{C}^1$ functions in $\mathcal{A}^p(\Omega, \mathbf{R}^m)$. Let $\overline{S}$ be the set of all limits of sequences of functions from S which are weakly convergent in $\mathcal{A}^p$. Similarly, let $\overline{\overline{S}}$ be the set of all limits of sequences of functions from $\overline{S}$ which are weakly convergent in $\mathcal{A}^p$. If $u \in \overline{\overline{S}}$, then there are $u_{n,k} \in S$ and $u_n \in \overline{S}$ such that $u_n \to u$ weakly in $\mathcal{A}^p$ and for fixed n, $u_{n,k} \to u_n$ weakly in $\mathcal{A}^p$. As a consequence of the Banach–Steinhaus theorem we obtain that $\|u_n\|_{\mathcal{A}^p} \le C$ and $\|u_{n,k}\|_{\mathcal{A}^p} \le C(n)$. Nevertheless, it does not follow that the whole family $\{\|u_{n,k}\|_{\mathcal{A}^p}\}$ is bounded. Hence, it is not clear whether $\overline{\overline{S}} = \overline{S}$. The weak sequential closure of S needs to be defined in a more careful way (see [1], [2]): the space $\mathrm{Cart}^p(\Omega)$ is defined to be the smallest set in $\mathcal{A}^p$ which contains S and is closed under weak convergence in $\mathcal{A}^p$. If we want to approximate functions from Cart^p by smooth functions, we use transfinite sequences indexed by ordinals. This situation is difficult to handle. We show that if we reduce the exponents, an approximation by ordinary sequences (and even strong) is available.

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^N$ be an open set with $|\Omega| < \infty$ and $p > (1, \dots, 1)$ be a nonincreasing multiexponent. Let $u \in \mathrm{Cart}^p(\Omega)$. Then there exists a sequence $(u_n)_n$ of $\mathcal{C}^1$ functions from $\mathcal{A}^p(\Omega, \mathbf{R}^m)$ with the following property: $u_n \to u$ strongly in $L^{q_1}(\Omega)$ and $M_i Du_n \to M_i Du$ strongly in $L^{q_i}(\Omega)$ for each $i = 1, \dots, N$ and $1 \le q_i < p$.*

We do not claim that the approximating sequence is bounded in $\mathcal{A}^p$ (it would be interesting to have such an estimate). Theorem 1.1 will be proved in Section 2.

In Section 3 we show an example of a function which is in cart^p but not in Cart^p (for the definitions see [2]). It is not straightforward to prove that u is not in Cart^p according to the definition (using the transfinite process). However, using the approximation theorem the proof is relatively easy.

2. Proof of the approximation theorem.

Let us say (an auxiliar terminology for the purpose of this proof) that a function $v \in \mathcal{A}^p$ has the *approximation property* if there is a sequence $(v_n)_n$ of $\mathcal{C}^1$ functions from $\mathcal{A}^p(\Omega, \mathbf{R}^m)$ such that $v_n \to v$ strongly in $L^{q_1}(\Omega)$ and $M_i Dv_n \to M_i Dv$

strongly in $L^{q_i}(\Omega)$ for each $i = 1, \dots, N$ and $1 \le q_i < p$. Obviously, each $\mathcal{C}^1$ function in $\mathcal{A}^p(\Omega, \mathbf{R}^m)$ has the approximation property. It remains to prove that the collection of all functions with the approximation property is closed under $\mathcal{A}^p$-weak convergence.

Lemma 2.1. *Let $(u_n)_n$ be a sequence of functions from $\mathcal{A}^p(\Omega, \mathbf{R}^m)$ with the approximation property, which converges weakly in $\mathcal{A}^p$ to a function $u \in \mathcal{A}^p(\Omega, \mathbf{R}^m)$. Then u has the approximation property.*

Proof: Choose nonincreasing (this is no loss of generality) multiexponents $q < r < p$. By the approximation property, there is a sequence $\tilde{v}_n$ of $\mathcal{C}^1$ functions from $\mathcal{A}^p(\Omega, \mathbf{R}^m)$ such that $\tilde{v}_n \to u$ a.e. and

(2.2)
$$\int_\Omega \big(|\tilde{v}_n - u_n|^{r_1} + |M_1 D\tilde{v}_n - M_1 Du_n|^{r_1} + \dots + |M_N D\tilde{v}_n - M_N Du_n|^{r_N}\big)\, dx < 2^{-n}\,.$$

Obviously $\tilde{v}_n \to u$ weakly in $\mathcal{A}^r(\Omega, \mathbf{R}^m)$. As a consequence of the Banach–Steinhaus theorem we obtain the estimate

$$\|\tilde{v}_n\|_{\mathcal{A}^r} \le C_1\,.$$

Choose $\varepsilon > 0$. We approximate u by a bounded function $v \in \mathcal{A}^p$ with coordinates $v^j = \eta \circ u^j$, where η is a bounded $\mathcal{C}^1$ function on $\mathbf{R}$ with $0 \le \eta' \le 1$. The function η may be found so close to the identity that

$$\|u - v\|_{H^{1,p}} < \varepsilon$$

and

$$\int_\Omega |M_k Dv - M_k Du|^{p_k} < \varepsilon$$

for each $k \in \{1, \dots, N\}$. We write

$$v_n = (\eta \circ \tilde{v}_n^1, \dots, \eta \circ \tilde{v}_n^N).$$

Obviously $v_n \to v$ weakly in $\mathcal{A}^r(\Omega, \mathbf{R}^m)$ and

$$\|\tilde{v}_n\|_{\mathcal{A}^r} \le C_1\,. \tag{2.3}$$

Denote

$$\delta = \frac{1}{2} \min_k \big(2^{-q_k} C_1^{-q_k} \varepsilon\big)^{\frac{r_k}{r_k - q_k}}\,.$$

We pick open sets $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ such that

$$|\Omega \setminus \Omega''| < \delta\,. \tag{2.4 a}$$

Let g be a function from $H^{1,p_1}(\mathbf{R}^N)$ which coincides with v in Ω'. By [6, Theorem 3.10.5], there is a $\mathcal{C}^1$ function $\tilde{f}$ on $\mathbf{R}^N$ with values in $\mathbf{R}^m$ such that for every $k = 1, \dots, N$ we have

$$|\{\tilde{f} \ne g\}| < \delta\,.$$

We find a bounded $\mathcal{C}^1$ function f such that $f = \tilde{f}$ in Ω'' and each coordinate f^j attains the constant value $\inf \eta - 1$ outside Ω'. By the Yegorov theorem, we find a closed set $F \subset \Omega' \cap \{f = g\}$ such that $v_n \to v$ uniformly in F and

$$|\Omega'' \setminus F| < \delta\,. \tag{2.4 b}$$

We denote $G = \Omega \setminus F$. We may assume that

$$\sup_F |v_n - v| < 2^{-n}$$

and

$$|\{x \in \Omega\colon |v_n - v| \geq 2^{-n}\}| \leq 2^{-n}\,, \tag{2.5}$$

otherwise we pass to a subsequence. Denote $\varphi(x) = \operatorname{dist}(x, F \cup (\mathbf{R}^N \setminus \Omega'))$. Fix $j \in \{1, \dots, m\}$. Then the sets

$$\{x \in \Omega' \cap G\colon f^j + \lambda^j \varphi = v^j\}\,, \quad \lambda^j \in \mathbf{R}$$

are pairwise disjoint, so we find λ^j such that $f^j + \lambda^j \varphi \neq v^j$ a.e. in $\Omega' \cap G$. We denote $w^j = f^j + \lambda^j \varphi$, $w = (w^1, \dots, w^m)$. Let us recall that w is a bounded Lipschitz function which coincides with v on F, differs from v in each coordinate a.e. in $\Omega' \cap G$ and equals $\inf \eta - 1$ in each coordinate outside Ω'. The Lipschitz continuity of w means that

$$|Dw(x)| \leq C_2 \qquad \text{for a.e. } x \in \Omega\,. \tag{2.6}$$

The constant C_2 may depend on ε. We set

$$\begin{aligned} w_n^j &= \max\bigl(\min(w^j, v_n^j + 2^{-n}), v_n^j - 2^{-n}\bigr)\,, \quad j = 1, \dots, m, \\ w_n &= (w_n^1, \dots, w_n^m)\,. \end{aligned} \tag{2.7}$$

Then w_n are locally Lipschitz functions on Ω which coincide with v on F. Obviously, $w_n \to v$ a.e. Since the sequence $\{w_n\}$ is bounded in $L^\infty(\Omega)$, it converges to v in $L^{q_1}(\Omega)$. Let us introduce the multiindex set $I = \{-1, 0, 1\}^m$. With every $\xi \in I$ and $n \in \mathbf{N}$ we associate a function $z_n^\xi : \Omega \to \mathbf{R}^m$ by the formula

$$z_n^{\xi,j} = \begin{cases} w^j, & \text{if } \xi_j = 0\,, \\ v_n^j + 2^{-n}, & \text{if } \xi_j = 1\,, \\ v_n^j - 2^{-n}, & \text{if } \xi_j = -1\,. \end{cases}$$

Then the graph of w_n is covered by the graphs of z_n^ξ, $\xi \in I$. For every $\xi \in I$, $\alpha \in J_k$ and $\beta \in J^k$ ($k \in \{1, \dots, N\}$) there is $i \in \{0, \dots, k\}$ and $b \in J^i$ such that

$$|M_\alpha^\beta D z_n^\xi(x)| \leq C_2^{k-i} \sum_{a \in J_i} |M_a^b D v_n(x)| \qquad \text{a.e. in } \Omega\,. \tag{2.8}$$

It easily follows that for every $\xi \in I$ the sequence $(z_n^\xi)_n$ is bounded in $\mathcal{A}^r$. We fix $k \in \{1, \dots, N\}$. We want to estimate

$$\int_\Omega |M_k Dw_n - M_k Dv|^{q_k}\, dx = \int_G |M_k Dw_n - M_k Dv|^{q_k}\, dx\,.$$

We write

$$E_n^\xi = \{x \in G : w_n = z_n^\xi\}\,.$$

The multiindex ξ is called *pure* if $|\xi_j| = 1$ for all j and *mixed* otherwise. We write

$$\begin{aligned} E_n^{\mathbf{p}} &= \bigcup \{E_n^\xi : \xi \text{ is pure}\}\,,\\ E_n^{\mathbf{m}} &= \bigcup \{E_n^\xi : \xi \text{ is mixed}\}\,. \end{aligned}$$

If $x \in E_n^{\mathbf{m}}$, then there is $j \in \{1, \dots, m\}$ such that $w_n^j(x) = w^j(x)$. By (2.7), $|v_n^j(x) - w^j(x)| = 2^{-n}$. This means that either $|v_n^j(x) - v^j(x)| \geq 2^{-n}$ or $|w^j(x) - v^j(x)| \leq |w^j(x) - v_n^j(x)| + |v_n^j(x) - v^j(x)| < 2^{-n} + 2^{-n}$. We see that

$$E_n^{\mathbf{m}} \subset \{x \in G : |v_n(x) - v(x)| \geq 2^{-n}\} \cup \bigcup_{j=1}^{m} \{x \in G : |w^j(x) - v^j(x)| \leq 2^{-n+1}\}\,,$$

and thus by (2.5) and the definition of w,

$$\lim_{n\to\infty} |E_n^{\mathbf{m}}| = 0\,. \tag{2.9}$$

Using (2.3), (2.8) and the Hölder inequality we estimate

$$\begin{aligned} \int_G |M_k Dw_n - M_k Dv|^{q_k}\, dx &\leq \int_{E_n^{\mathbf{p}}} |M_k Dw_n - M_k Dv|^{q_k}\, dx \\ &\quad + \int_{E_n^{\mathbf{m}}} |M_k Dw_n - M_k Dv|^{q_k}\, dx \\ &\leq \Big(\int_G \big(|M_k Dv_n| + |M_k Dv|\big)^{r_k}\, dx\Big)^{q_k/r_k} |G|^{1-q_k/r_k} \\ &\quad + \Big(\int_{E_n^{\mathbf{m}}} \big(|M_k Dw_n| + |M_k Dv|\big)^{r_k}\, dx\Big)^{q_k/r_k} |E_n^{\mathbf{m}}|^{1-q_k/r_k} \\ &\leq 2^{q_k-1} C_1^{q_k} |G|^{1-q_k/r_k} + C_3 |E_n^{\mathbf{m}}|^{1-q_k/r_k}\,, \end{aligned}$$

where the constant C_3 does not depend on n. Now, from (2.4) it follows

$$2^{q_k-1} C_1^{q_k} |G|^{1-q_k/r_k} \leq \frac{1}{2}\varepsilon$$

and by (2.9)

$$C_3 |E_n^{\mathbf{m}}|^{1-q_k/r_k} \to 0$$

as $n \to \infty$. It follows that

$$\int_\Omega |w_n - v|^{q_1}\, dx < \varepsilon$$

and

$$\int_\Omega |M_k Dw_n - M_k Dv|^{q_k}\, dx < \varepsilon$$

for each $k \in \{1, \dots N\}$ if n is big enough. Let such an n be fixed. Since $w_n^j = v_n^j - 2^{-n}$ on $\mathbf{R}^N \setminus \Omega'$ (as $w^j = f^j = \inf \eta - 1 < v_n - 2^{-n}$ on $\mathbf{R}^N \setminus \Omega'$), we see that $w_n - v_n + (2^{-n}, \dots, 2^{-n})$ is a Lipschitz continuous function with a compact support in Ω. Now, if h is a $\mathcal{C}^1$ function with a compact support in Ω which is sufficiently close to $w_n - v_n + (2^{-n}, \dots, 2^{-n})$ in the H^{1,Np_1}-norm, then $v_n - (2^{-n}, \dots, 2^{-n}) + h$ is a $\mathcal{C}^1$ function which is a good approximation associated with a fixed choice of q and ε. Using $\varepsilon_n \searrow 0$ and $q_n \nearrow p$ we obtain the desired approximating sequence.

□

3. Example.

In this section we consider the case $N = m = 2$ and the set $\Omega = B(0,1)$ (the unit disc in $\mathbf{R}^2$). All multiexponents will be constant. We will investigate the function

$$u(x) = \Big(\frac{x_1 |x_2| (x_1^2 - x_2^2)}{|x|^4},\ \frac{x_1 x_2}{|x|^2} \Big).$$

The function u is differentiable in $\Omega \setminus \{0\}$ and, except at zero, $\det Du = 0$ and $|Du(x)| \le C|x|^{-1}$. It follows that $u \in \mathcal{A}^p(\Omega, \mathbf{R}^2)$ for all $p \in (1,2)$. We fix exponents $1 < q < p < 2$. We want to prove that u does not belong to $\mathrm{Cart}^p(\Omega, \mathbf{R}^2)$. To this end, we consider a sequence $\{u_n\}$ of locally Lipschitz functions from $\mathcal{A}^q(\Omega, \mathbf{R}^2)$ such that $u_n \to u$ strongly in $H^{1,q}(\Omega)$; passing if necessary to a subsequence, we may assume that

$$\|u_n - u\|^q_{H^{1,q}(\Omega)} \le 4^{-n}\,. \tag{3.1}$$

We will obtain

Theorem 3.2. *In the above described situation, we have*

$$\lim \int_\Omega |\det Du_n|^q\, dx \to \infty\,.$$

This means that u has not the approximation property of Section 2, and thus by Theorem 1.1 it is not in $\mathrm{Cart}^p(\Omega, \mathbf{R}^2)$. Theorem 3.2 will be proved later in this section.

Remark 3.3. This remark is addressed to the reader who is familiar with the correspondence between functions, graphs and cartesian currents as it is described in [1], [2], [4]. Let T_u be the cartesian current associated with the graph of u. Then obviously $\partial T_u = 0$ (cf. Example 1 on p. 405 in [2]) and thus u belongs to the class $\mathrm{cart}^p(\Omega)$. This solves negatively the problem whether the classes cart^p and Cart^p coincide (cf. [1], [2]). Notice that the inclusion $\mathrm{Cart}^p \subset \mathrm{cart}^p$ always holds and these spaces are equal in certain special situations (see [3]).

Lemma 3.4. *Let $0 < \rho < 1$. Then there is $R \in (0, \rho)$ such that $u_n \to u$ uniformly on $\partial B(0, R)$.*

PROOF: It follows easily from (3.1) using the capacity theory of Sobolev spaces. An elementary argument is the following: By (3.1) and the monotone convergence theorem,

$$\int_\Omega \sum_{n=1}^\infty 2^n (|u_n - u|^q + |Du_n - Du|^q)\, dx < \infty\,.$$

Hence there is $R \in (0, \rho)$ such that

$$\int_{\partial B(0,R)} \sum_{n=1}^\infty 2^n (|u_n - u|^q + |Du_n - Du|^q) ds < +\infty\,,$$

which gives

$$\int_{\partial B(0,R)} (|u_n - u|^q + |Du_n - Du|^q) ds \le C 2^{-n}$$

with C independent of n. Since $\partial B(0, R)$ is one-dimensional, from the Sobolev imbedding theorem we obtain the uniform convergence of $u_n \to u$ on $\partial B(0, R)$. □

Notation 3.5. We denote

$$U^+ = B((0, \frac{1}{4}), \frac{1}{7})\,, \quad U^- = B((0, -\frac{1}{4}), \frac{1}{7})\,, \quad E = \overline{U^+ \cup U^-}\,.$$

A routine calculation shows that the range of u does not meet E.

Lemma 3.6. *Let $R > 0$. Then the mapping u is not homotopic with a constant in the domain $\partial B(0, R)$ and the range $\mathbf{R}^2 \setminus \{(0, \frac{1}{4}), (0, -\frac{1}{4})\}$.*

PROOF: The increment of the multivalued analytic function

$$\zeta \to \ln\Big(\sqrt{\zeta - \frac{1}{4}\,\mathrm{i}} + \sqrt{\zeta + \frac{1}{4}\,\mathrm{i}}\Big)$$

along the closed curve

$$\zeta(t) = u^1(R\cos t,\, R\sin t) + \mathrm{i}\, u^2(R\cos t,\, R\sin t)\,, \quad t \in [0, 2\pi]$$

is different from zero. □

Corollary 3.7. *Let $R \in (0, 1)$. Suppose that $u_n \to u$ uniformly on $\partial B(0, R)$. Then there is $n_0 \in \mathbf{N}$ such that for all $n \ge n_0$*

(a) *$E \cap u_n\big(\partial B(0, R)\big) = \emptyset$ and for any couple of points $y^+ \in U^+$, $y^- \in U^-$, the mapping u_n is not homotopic with a constant in the domain $\partial B(0, R)$ and the range $\mathbf{R}^2 \setminus \{y^+, y^-\}$,*

(b) *either* $U^+ \subset u(B(0,R))$, *or* $U^- \subset u(B(0,R))$.

Proof: (a) is an immediate consequence of Lemma 3.6. Now, if u_n is continuous, then

$$H(x,s) = u_n\big((1-s)x\big)\,, \quad x \in \partial B(0,R),\ s \in [0,1]$$

is a homotopy and thus by (a) its range contains either U^+ or U^-. □

Proof of Theorem 3.2: Choose $\rho \in (0,1)$. By Lemma 3.4 there is a radius $R \in (0,\rho)$ such that $u_n \to u$ uniformly on $\partial B(0,R)$. By Corollary 3.7 there is $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ we have $|u_n(B(0,R))| \geq |U^+| = |U^-| = \frac{\pi}{49}$. Hence using the Hölder inequality we obtain

$$\begin{aligned}\frac{\pi}{49} \leq \int_{B(0,R)} |\det Du_n|\,dx &\leq \Big(\int_{B(0,R)} |\det Du_n|^q\,dx\Big)^{1/q} (\pi R^2)^{1-1/q}\\ &\leq \Big(\int_\Omega |\det Du_n|^q\,dx\Big)^{1/q} (\pi\rho^2)^{1-1/q}\,.\end{aligned}$$

Since we may choose ρ arbitrarily close to zero, we have proved that

$$\lim_{n\to\infty} \int_\Omega |\det Du_n|^q\,dx = \infty\,.$$

□

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