Inductive limit topologies on Orlicz spaces

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Abstract. Let L^φ be an Orlicz space defined by a convex Orlicz function φ and let E^φ be the space of finite elements in L^φ (= the ideal of all elements of order continuous norm). We show that the usual norm topology $\mathcal{T}_\varphi$ on L^φ restricted to E^φ can be obtained as an inductive limit topology with respect to some family of other Orlicz spaces. As an application we obtain a characterization of continuity of linear operators defined on E^φ.

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1. Introduction and preliminaries.

In [1] and [2] Davis, Murray and Weber discussed the spaces

$$L^{p+} = \bigcup_{p<t<\infty} L^t[0,1] \text{ and } l^{p-} = \bigcup_{1\le t<p} l^t \quad (1 < p \le \infty)$$

(endowed with the appropriate inductive limit topologies) which turned out to be distinct from the spaces L^p and l^p, respectively.

Moreover, in [8] it is proved that if $S \subset [0,\infty)$ with $\inf S \notin S$ or $\sup S \notin S$ and μ is an infinite atomless measure (resp. $\sup S \notin S$ and μ is the counting measure on $\mathbb{N}$), there is no Orlicz function φ such that:

$$E^\varphi = \operatorname{Lin} \bigcup_{p\in S} L^p \text{ or } L^\varphi = \operatorname{Lin} \bigcup_{p\in S} L^p.$$

On the other hand, Krasnoselskii and Rutickii [3, p. 60] showed that if μ is the finite Lebesgue measure, then

$$L^1 = \bigcup_{\varphi} L^\varphi,$$

where φ are taken over the family of all N-functions. This equality was a starting point for many results concerning a representation of an Orlicz space L^φ or a space E^φ as the union of some families of Orlicz spaces which they contain properly (see [4], [7], [9], [12]).

In [7] for a convex Orlicz function φ we found the set Ψ^φ of N-functions such that:

$$E^\varphi = \bigcup_{\psi\in\Psi^\varphi} E^\psi = \bigcup_{\psi\in\Psi^\varphi} L^\psi.$$

In this paper we show that the appropriate inductive limit topologies on E^φ defined with respect to these representations coincide with the norm topology $\mathcal{T}_\varphi$ on L^φ restricted to E^φ.

We now recall some notation and terminology concerning Orlicz spaces (see [3], [5], [11] for more details).

By an Orlicz function we mean a function $\varphi : [0,\infty) \to [0,\infty]$ which is non-decreasing, left continuous, continuous at zero with $\varphi(0) = 0$, and not identically equal to zero.

We shall say that an Orlicz function φ jumps to ∞, whenever there is a number $u_0 > 0$ such that $\varphi(u) = \infty$ for $u > u_0$. We shall say that φ vanishes near zero, whenever $\varphi(u) = 0$ for $0 \le u \le u_0$ for some $u_0 > 0$.

An Orlicz function φ is called convex, if $\varphi(\alpha u+\beta v) \le \alpha\varphi(u)+\beta\varphi(v)$ for $\alpha, \beta \ge 0$, $\alpha + \beta = 1$. A convex Orlicz function is usually called a Young function. A convex Orlicz function φ, vanishing only at 0 and taking only finite values is called an N-function if $\varphi(u)/u \to 0$ as $u \to 0$ and $\varphi(u)/u \to \infty$ as $u \to \infty$. By Φ_N we will denote the collection of all N-functions.

For a convex Orlicz function φ we denote by φ^* the function complementary to φ in the sense of Young, i.e.

$$\varphi^*(v) = \sup\{uv - \varphi(u) : u \ge 0\} \quad \text{for} \quad v \ge 0.$$

For a set Ψ of convex Orlicz functions we will write

$$\Psi^* = \{\psi^* : \psi \in \Psi\}.$$

Throughout this paper we will write: $\varphi_p(u) = u^p$ for $u \ge 0$, where $p \ge 1$ and

$$\varphi_0(u) = \begin{cases} 0 & \text{for} \quad 0 \le u \le 1, \\ 1 & \text{for} \quad u > 1 \end{cases} \quad \text{and} \quad \varphi_\infty(u) = \begin{cases} 0 & \text{for} \quad 0 \le u \le 1, \\ \infty & \text{for} \quad u > 1 \end{cases}.$$

We shall say that two Orlicz functions ψ and φ are equivalent for all u (resp. for small u, resp. for large u), in symbols $\psi \overset{a}{\sim} \varphi$ (resp. $\psi \overset{s}{\sim} \varphi$, resp. $\psi \overset{l}{\sim} \varphi$) if there exist constants $a, b, c, d > 0$ such that $a\psi(bu) \le \varphi(u) \le c\psi(du)$ for all $u \ge 0$ (resp. for $0 \le u \le u_0$, resp. for $u \ge u_0$), where $u_0 > 0$.

We say that an Orlicz function φ increases essentially more rapidly than any other ψ for all u (resp. for small u, resp. for large u), in symbols $\psi \overset{a}{\ll} \varphi$ (resp. $\psi \overset{s}{\ll} \varphi$, resp. $\psi \overset{l}{\ll} \varphi$) if for any $c > 0$, $\psi(cu)/\varphi(u) \to 0$ as $u \to 0$ and $u \to \infty$ (resp. as $u \to 0$, resp. $u \to \infty$) (see [3, p. 114]).

It is known that $\psi \overset{a}{\ll} \varphi$ (resp. $\psi \overset{s}{\ll} \varphi$, resp. $\psi \overset{l}{\ll} \varphi$) implies $\varphi^* \overset{a}{\ll} \psi^*$ (resp. $\varphi^* \overset{s}{\ll} \psi^*$, resp. $\varphi^* \overset{l}{\ll} \psi^*$) (see [3, Lemma 13.1]).

Let (Ω, Σ, μ) be a positive measure space, and let L^0 denote the set of equivalence classes of all real valued μ-measurable functions defined and finite a.e. on Ω. An Orlicz function φ determines a functional $m_\varphi : L^0 \to [0,\infty]$ by the formula:

$$m_\varphi(x) = \int_\Omega \varphi(|x(t)|)\, d\mu.$$

The Orlicz space determined by φ is the ideal of L^0 defined by

$$L^\varphi = \{x \in L^0 : m_\varphi(\lambda x) < \infty \ \text{ for some } \ \lambda > 0\}.$$

The functional m_φ restricted to L^φ is an orthogonally additive modular (see [6]).

L^φ can be equipped with the complete metrizable topology $\mathcal{T}_\varphi$ of the Riesz F-norm

$$|x|_\varphi = \inf\{\lambda > 0 : m_\varphi(x/\lambda) \le \lambda\}.$$

Moreover, if φ is convex, then the topology $\mathcal{T}_\varphi$ is generated by the norm

$$\|x\|_\varphi = \inf\{\lambda > 0 : m_\varphi(x/\lambda) \le 1\}.$$

Let

$$E^\varphi = \{x \in L^0 : m_\varphi(\lambda x) < \infty \ \text{ for all } \ \lambda > 0\}.$$

Then E^φ is a closed ideal of L^φ, and it is well known that E^φ coincides with the ideal of all elements of L^φ with order continuous F-norm $|\cdot|_\varphi$. It is known that $L^\varphi = E^\varphi$ if φ satisfies the Δ_2-condition, i.e.

$$\limsup \frac{\varphi(2u)}{\varphi(u)} < \infty \ \text{ as } \ u \to 0 \ \text{ and } \ u \to \infty.$$

If μ is the counting measure on the set $\mathbb{N}$ of all natural numbers, we will write l^φ and h^φ instead of L^φ and E^φ, respectively. By c_0 we will denote the space of all sequences that are convergent to 0.

Given a linear topological space (X, ξ), by $(X, \xi)^*$ we will denote its topological dual.

2. Some equalities among Orlicz spaces.

In this section we present some equalities among Orlicz spaces, obtained in [7], that are of the key importance in the paper.

Let Φ_1 be the set of all convex Orlicz functions φ taking only finite values and such that $\varphi(u)/u \to 0$ as $u \to 0$.

Denote by

$$\begin{aligned}
\Phi_{11} &= \{\varphi \in \Phi_1 : \varphi(u) > 0 \ \text{ for } \ u > 0 \ \text{ and } \ \varphi(u)/u \to \infty \ \text{ as } \ u \to \infty\},\\
\Phi_{12} &= \{\varphi \in \Phi_1 : \varphi(u) > 0 \ \text{ for } \ u > 0 \ \text{ and } \ \varphi(u)/u \to a \ \text{ as } \ u \to \infty, a > 0\},\\
\Phi_{13} &= \{\varphi \in \Phi_1 : \varphi(u) = 0 \ \text{ near zero and } \ \varphi(u)/u \to \infty \ \text{ as } \ u \to \infty\},\\
\Phi_{14} &= \{\varphi \in \Phi_1 : \varphi(u) = 0 \ \text{ near zero and } \ \varphi(u)/u \to a \ \text{ as } \ u \to \infty, a > 0\}.
\end{aligned}$$

Then $\Phi_1 = \bigcup_{i=1}^4 \Phi_{1i}$, where the sets are pairwise disjoint. It is seen that $\Phi_{11} = \Phi_N$.

Theorem 2.1 [7, Theorems 1.1–1.4, Theorem 1.7]. *Let* $\varphi \in \Phi_{1i}$ $(i = 1,2,3,4)$. *Then the following equalities hold:*

$$E^{\varphi} = \bigcup_{\psi \in \Psi^{\varphi}_{1i}} E^{\psi} = \bigcup_{\psi \in \Psi^{\varphi}_{1i}} L^{\psi},$$

where:

$$\begin{aligned}
\Psi^{\varphi}_{11} &= \{\psi \in \Phi_N : \varphi \overset{a}{\ll} \psi\},\\
\Psi^{\varphi}_{12} &= \{\psi \in \Phi_N : \varphi \overset{s}{\ll} \psi\},\\
\Psi^{\varphi}_{13} &= \{\psi \in \Phi_N : \varphi \overset{l}{\ll} \psi\},\\
\Psi^{\varphi}_{14} &= \Phi_N.
\end{aligned}$$

Moreover, if μ is an atomless measure or the counting measure on $\mathbb{N}$, then for each $\psi \in \Psi^{\varphi}_{1i}$, the strict inclusion $L^{\psi} \subsetneq E^{\varphi}$ holds.

Next, let Φ_2 be the set of all convex Orlicz functions φ vanishing only at 0 and such that $\varphi(u)/u \to \infty$ as $u \to \infty$.

Denote by

$$\begin{aligned}
\Phi_{21} &= \{\varphi \in \Phi_2 : \varphi(u) < 0 \ \text{ for } \ u > 0 \ \text{ and } \ \varphi(u)/u \to 0 \ \text{ as } \ u \to 0\},\\
\Phi_{22} &= \{\varphi \in \Phi_2 : \varphi \ \text{ jumps to } \ \infty \text{ and } \ \varphi(u)/u \to 0 \ \text{ as } \ u \to 0\},\\
\Phi_{23} &= \{\varphi \in \Phi_2 : \varphi(u) < 0 \ \text{ for } \ u > 0 \ \text{ and } \ \varphi(u)/u \to a \ \text{ as } \ u \to 0, a > 0\},\\
\Phi_{24} &= \{\varphi \in \Phi_2 : \varphi \ \text{ jumps to } \ \infty \text{ and } \ \varphi(u)/u \to a \ \text{ as } \ u \to 0, a > 0\}.
\end{aligned}$$

Then $\Phi_2 = \bigcup_{i=1}^{4} \Phi_{2i}$ and $\Phi_{21} = \Phi_N$.

Theorem 2.2 [7, Theorems 2.1–2.4, Theorem 2.6]. *Let* $\varphi \in \Phi_{2i}$ $(i = 1,2,3,4)$. *Then the following equalities hold:*

$$L^{\varphi} = \bigcap_{\psi \in \Psi^{\varphi}_{2i}} L^{\psi} = \bigcap_{\psi \in \Psi^{\varphi}_{2i}} E^{\psi},$$

where:

$$\begin{aligned}
\Psi^{\varphi}_{21} &= \{\psi \in \Phi_N : \psi \overset{a}{\ll} \varphi\},\\
\Psi^{\varphi}_{22} &= \{\psi \in \Phi_N : \psi \overset{s}{\ll} \varphi\},\\
\Psi^{\varphi}_{23} &= \{\psi \in \Phi_N : \psi \overset{l}{\ll} \varphi\},\\
\Psi^{\varphi}_{24} &= \Phi_N.
\end{aligned}$$

At last, according to [7, Lemma 3.1, Theorem 3.3] we have

Theorem 2.3. *Let φ_1 and φ_2 be a pair of complementary convex Orlicz functions, i.e. $\varphi_1^* = \varphi_2$. Then $\varphi_1 \in \Phi_{1i}$ iff $\varphi_2 \in \Phi_{2i}$ $(i = 1, 2, 3, 4)$, and moreover, the sets $\Psi_{1i}^{\varphi_1}$ and $\Psi_{2i}^{\varphi_2}$ are mutually related in such a way that:*

$$\left(\Psi_{1i}^{\varphi_1}\right)^* = \Psi_{2i}^{\varphi_2} \quad \textit{and} \quad \left(\Psi_{2i}^{\varphi_2}\right)^* = \Psi_{1i}^{\varphi_1} .$$

3. Inductive limit topologies on E^φ.

Let $\varphi \in \Phi_{1i}$ $(i = 1, 2, 3, 4)$. Then in view of Theorem 2.1, one can consider on E^φ the inductive limit topologies $\mathcal{T}_{I_1}^\varphi$ and $\mathcal{T}_{I_2}^\varphi$ with respect to the families $\{(E^\psi, \mathcal{T}_\psi \,|_{E^\psi}) : \psi \in \Psi_{1i}^\varphi\}$ and $\{(L^\psi, \mathcal{T}_\psi) : \psi \in \Psi_{1i}^\varphi\}$, respectively (see [10, Chapter V, § 2]). Thus $\mathcal{T}_{I_1}^\varphi$ (resp. $\mathcal{T}_{I_2}^\varphi$) is the finest of all locally convex topologies ξ on E^φ that satisfy, for each $\psi \in \Psi_{1i}^\varphi$, the condition $\xi \,|_{E^\psi} \subset \mathcal{T}_\psi \,|_{E^\psi}$ (resp. $\xi \,|_{L^\psi} \subset \mathcal{T}_\psi$). It is seen that

$$\mathcal{T}_\varphi \,|_{E^\varphi} \subset \mathcal{T}_{I_2}^\varphi \subset \mathcal{T}_{I_1}^\varphi . \tag{3.1}$$

Our aim is to show that the topology $\mathcal{T}_\varphi \,|_{E^\varphi}$ coincides with $\mathcal{T}_{I_1}^\varphi$ and $\mathcal{T}_{I_2}^\varphi$. For this purpose, the following theorem will be of importance.

Theorem 3.1. *Let $\varphi \in \Phi_1$ and let μ be a σ-finite measure. Then for a linear functional f on E^φ the following statements are equivalent:*

(a) *f is $\mathcal{T}_{I_1}^\varphi$-continuous.*

(b) *There exists a unique $y \in L^{\varphi^*}$ such that*

$$f(x) = f_y(x) = \int_\Omega x(t)y(t)\, d\mu \quad \textit{for all} \quad x \in E^\varphi.$$

Proof: (a) $\Rightarrow$ (b). Let $\varphi \in \Phi_{1i}$ $(i = 1, 2, 3, 4)$. Then for each $\psi \in \Psi_{1i}^\varphi$, the functional $f \,|_{E^\psi}$ is continuous for $\mathcal{T}_\psi \,|_{E^\psi}$, so according to [5, Chapter II, § 3, Theorem 2] there exists a unique function $y_\psi \in L^{\psi^*}$ such that

$$f(x) = \int_\Omega x(t)y_\psi(t)\, d\mu \quad \text{for all} \quad x \in E^\psi . \tag{+}$$

Assume that there exist $\psi_1, \psi_2 \in \Psi_{1i}^\varphi$ such that $y_{\psi_1} \neq y_{\psi_2}$, and $f(x) = \int_\Omega x(t)y_{\psi_k}(t)\, d\mu$ for $x \in E^{\psi_k}$, where $k = 1, 2$. Let us assume, for example, that $\mu(\{t \in \Omega : y_{\psi_1}(t) > y_{\psi_2}(t)\}) > 0$, and let $A \subset \{t \in \Omega : y_{\psi_1}(t) > y_{\psi_2}(t)\}$ be a measurable set with $0 < \mu(A) < \infty$. Denoting by χ_A the characteristic function of A, we have $\chi_A \in E^{\psi_1} \cap E^{\psi_2}$, so by (+) we get

$$\int_\Omega \chi_A(t)\,(y_{\psi_1}(t) - y_{\psi_2}(t))\, d\mu = \int_A (y_{\psi_1}(t) - y_{\psi_2}(t))\, d\mu = 0.$$

This contradiction establishes that there exists a unique

$$y \in \bigcap_{\psi \in \Psi_{1i}^{\varphi}} L^{\psi^*} \quad \text{such that} \quad f(x) = \int_{\Omega} x(t)y(t)\, d\mu \quad \text{for all} \quad x \in E^{\varphi}\,.$$

On the other hand, since $\varphi^* \in \Phi_{2i}$ and $(\Psi_{1i}^{\varphi})^* = \Psi_{2i}^{\varphi^*}$ (see Theorem 2.3), according to Theorem 2.2,

$$\bigcap_{\psi \in \Psi_{1i}^{\varphi}} L^{\psi^*} = \bigcap_{\psi \in (\Psi_{1i}^{\varphi})^*} L^{\psi} = \bigcap_{\psi \in \Psi_{2i}^{\varphi^*}} L^{\psi} = L^{\varphi^*}\,.$$

(b) $\Rightarrow$ (a). Let $\varphi \in \Phi_{1i}$ (i=1,2,3,4). Then for each $\psi \in \Psi_{1i}^{\varphi}$, by Theorem 2.3, $\psi^* \in \Psi_{2i}^{\varphi^*}$. Hence $L^{\varphi^*} \subset L^{\psi^*}$, and the functional $f\,|_{E^{\psi}}$ is continuous for $\mathcal{T}_{\psi}\,|_{E^{\psi}}$ (see [5, Chapter 2, § 3, Theorem 2]). Therefore, in view of [10, Chapter V, Proposition 5], the functional f is continuous for $\mathcal{T}_{I_1}^{\varphi}$.

Thus the proof is completed. □

Now we are in a position to prove our main theorem.

Theorem 3.2. *Let $\varphi \in \Phi_1$ and μ be a σ-finite measure. Then the norm topology $\mathcal{T}_{\varphi}$ restricted to E^{φ} coincides with the inductive limit topologies $\mathcal{T}_{I_1}^{\varphi}$ and $\mathcal{T}_{I_2}^{\varphi}$, that is*

$$\mathcal{T}_{\varphi}\,|_{E^{\varphi}} = \mathcal{T}_{I_1}^{\varphi} = \mathcal{T}_{I_2}^{\varphi}\,.$$

Proof: Since the space $(E^{\varphi}, \mathcal{T}_{\varphi}\,|_{E^{\varphi}})$ is barrelled and $(E^{\varphi}, \mathcal{T}_{\varphi}\,|_{E^{\varphi}})^* = \{f_y : y \in L^{\varphi^*}\}$ (see [5, Chapter II, § 3, Theorem 2]), the equality $\mathcal{T}_{\varphi}\,|_{E^{\varphi}} = \beta(E^{\varphi}, L^{\varphi^*})$ holds (see [10, Chapter IV, § 1, Corollary 1]).

On the other hand, the space $(E^{\varphi}, \mathcal{T}_{I_1}^{\varphi})$ is barrelled, because an inductive limit of barrelled spaces is barrelled (see [10, Chapter 2, Proposition 6]). Hence, in view of Theorem 3.1, the equality $\mathcal{T}_{I_1}^{\varphi} = \beta(E^{\varphi}, L^{\varphi^*})$ holds. Thus $\mathcal{T}_{\varphi}\,|_{E^{\varphi}} = \mathcal{T}_{I_1}^{\varphi}$, and by (3.1) our proof is completed. □

4. A characterization of continuity of linear operators on E^{φ}.

As an application of Theorem 3.2, in view of the general property of inductive limit topologies (see [10, Chapter V, 2, Proposition 5]), we obtain a characterization of linear operators of E^{φ} into a locally convex space X. The details follow.

Theorem 4.1. *Let $\varphi \in \Phi_{1i}$ $(i = 1,2,3,4)$ and let (X, ξ) be a locally convex space. For a linear operator $A : E^{\varphi} \to X$, the following statements are equivalent:*

(a) *A is $(\mathcal{T}_{\varphi}\,|_{E^{\varphi}}, \xi)$-continuous.*
(b) *$A\,|_{E^{\psi}}$ is $(\mathcal{T}_{\psi}\,|_{E^{\psi}}, \xi)$-continuous for every $\psi \in \Psi_{1i}^{\varphi}$.*
(c) *$A\,|_{E^{\psi}}$ is $(\mathcal{T}_{\psi}, \xi)$-continuous for every $\psi \in \Psi_{1i}^{\varphi}$.*

We close this section with an application of Theorem 2.1 and Theorem 4.1 to the spaces: $L^p, L^1 + L^p$ $(p > 1)$ and c_0.

Examples.

A. Let $p > 1$. Then $\varphi_p \in \Phi_{11}$ and in view of Theorem 2.1 and Theorem 4.1 we get the following

Corollary 4.2. *Let $p > 1$. Then the following equalities hold:*

$$L^p = \bigcup_{\psi} E^{\psi} = \bigcup_{\psi} L^{\psi},$$

where the unions are taken over all N-functions ψ such that $\psi(u)/u^p \to \infty$ as $u \to 0$ and $u \to \infty$.

Moreover, if the measure μ is σ-finite, then for a locally convex space (X, ξ) and a linear operator $A : L^p \to X$, the following statements are equivalent:

(a) A is $(\mathcal{T}_{L^p}, \xi)$-continuous.
(b) $A\,|_{E^\psi}$ is $(\mathcal{T}_\psi\,|_{E^\psi}, \xi)$-continuous for every N-function ψ such that $\psi(u)/u^p \to \infty$ as $u \to 0$ and $u \to \infty$.
(c) $A\,|_{L^\psi}$ is $(\mathcal{T}_\psi, \xi)$-continuous for every N-function ψ such that $\psi(u)/u^p \to \infty$ as $u \to 0$ and $u \to \infty$.

B. For $p > 1$ let us put

$$\varphi(u) = \begin{cases} u^p & \text{for } 0 \le u \le 1, \\ pu + 1 - p & \text{for } u > 1, \end{cases}$$

and let $\varphi'(u) = \min(\varphi_1(u), \varphi_p(u))$. Then φ is a convex Orlicz function and $\varphi \overset{a}{\sim} \varphi'$, so $E^\varphi = L^\varphi = L^{\varphi'} = L^1 + L^p$ and $\mathcal{T}_\varphi = \mathcal{T}_{\varphi'}$, where the topology $\mathcal{T}_{\varphi'}$ is generated by the norm:

$$\|x\|_{L^1+L^p} = \inf\{\|x_1\|_{L^1} + \|x_2\|_{L^p} : x = x_1 + x_2, \quad x_1 \in L^1, \quad x_2 \in L^p\}.$$

Since $\varphi \in \Phi_{12}$, according to Theorem 2.1 and Theorem 4.1 we have

Corollary 4.3. *Let $p > 1$. Then the following equalities hold:*

$$L^1 + L^p = \bigcup_{\psi} E^{\psi} = \bigcup_{\psi} L^{\psi},$$

where the unions are taken over the set of all N-functions ψ such that $\psi(u)/u^p \to \infty$ as $u \to 0$.

Moreover, if the measure μ is σ-finite, then for a locally convex space (X, ξ) and a linear operator $A : L^1 + L^p \to X$, the following statements are equivalent:

(a) A is $(\mathcal{T}_{L^1+L^p}, \xi)$-continuous.
(b) $A\,|_{E^\psi}$ is $(\mathcal{T}_\psi\,|_{E^\psi}, \xi)$-continuous for every N-function ψ such that $\psi(u)/u^p \to \infty$ as $u \to 0$.
(c) $A\,|_{L^\psi}$ is $(\mathcal{T}_\psi, \xi)$-continuous for every N-function ψ such that $\psi(u)/u^p \to \infty$ as $u \to 0$.

In particular, if the measure μ is finite, then

$$L^1 = \bigcup_{\psi} E^{\psi} = \bigcup_{\psi} L^{\psi},$$

where the unions are taken over the set of all N-functions ψ.

Moreover, for a linear operator $A : L^1 \to X$, the following statements are equivalent:

(a) A is $(\mathcal{T}_{L^1}, \xi)$-continuous.
(b) $A|_{E^\psi}$ is $(\mathcal{T}_\psi|_{E^\psi}, \xi)$-continuous for every N-function ψ.
(c) $A|_{L^\psi}$ is $(\mathcal{T}_\psi, \xi)$-continuous for every N-function ψ.

C. Let

$$\varphi(u) = \begin{cases} 0 & \text{for } 0 \le u \le 1, \\ u-1 & \text{for } u > 1. \end{cases}$$

Then φ is a convex Orlicz function and $\varphi \overset{s}{\sim} \varphi_0$. Hence $l^\varphi = l^{\varphi_0} = l^\infty$ and $h^\varphi = h^{\varphi_0} = c_0$, and the topology $\mathcal{T}_\varphi$ on l^φ agrees with the topology $\mathcal{T}_\infty$ of the norm $\|x\|_\infty = \sup_i |x(i)|$ on l^∞. Since $\varphi \in \Phi_{14}$, in view of Theorem 2.1 and Theorem 4.1, we have

Corollary 4.4. *The following equalities hold:*

$$c_0 = \bigcup_{\psi} h^{\psi} = \bigcup_{\psi} l^{\psi},$$

where the unions are taken over the set of all N-functions.

Moreover, for a locally convex space (X, ξ) and a linear operator $A : c_0 \to X$, the following statements are equivalent:

(a) A is $(\mathcal{T}_\infty|_{c_0}, \xi)$-continuous.
(b) $A|_{h^\psi}$ is $(\mathcal{T}_\psi|_{h^\psi}, \xi)$-continuous for every N-function ψ.
(c) $A|_{l^\psi}$ is $(\mathcal{T}_\psi, \xi)$-continuous for every N-function ψ.

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