Comment.Math.Univ.Carolin. 32,4 (1991)743–748

Extremal solutions of a general marginal problem

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Abstract. The characterization of extremal points of the set of probability measures with given marginals is given in the general context of a marginal system. The sets of marginal uniqueness are studied and an example is added to illustrate the theory.

Keywords: marginal problem, marginal system, simplicial measure, set of marginal uniqueness

Classification: Primary 60B05; Secondary 52A05

1. Introduction.

We shall say that $\mathcal{L} = \{X \xrightarrow{q_j} X_j |\, j \in J\}$ is a marginal system if X, X_j are Polish spaces, $q_j : X \to X_j$ Borel measurable maps for $j \in J$ (called projections here) and where J is a nonempty index set. Denote by $M(X)\,(M_1(X))$ a set of bounded Borel signed (probability) measures defined on X and define a map $\mathrm{MARG}(P)$: $M(X) \to \bigotimes_{j\in J} M(X_j)$ by $\mathrm{MARG}(P) = (P_j |\, j \in J)$, where $P_j = q_j \circ P$ are the image measures that will be called marginals (or projections) of P. Hoffmann–Jørgensen [7] considers a marginal system of probability measures, i.e. the system

$$\{X \xrightarrow{q_j} (X_j,\, Q_j) |\, j \in J\}, \quad \text{where } Q_j \in M_1(X_j) \text{ are fixed,}$$

and presents necessary and sufficient conditions for the existence of a $P \in M_1(X)$, such that $\mathrm{MARG}(P) = (Q_j, |\, j \in J)$. (See also [6].) Our problem is to characterize extremal solutions of the above equation.

We shall say, that $P \in M_1(X)$ is a simplicial measure w.r.t. a marginal system $\mathcal{L}$ if it is an extremal point of the (nonempty) set

$$\mathcal{L}(P) = \{Q \in M_1(X) |\; \mathrm{MARG}(Q) = \mathrm{MARG}(P)\}.$$

We shall say, that a Borel set $B \subset X$ is a set of marginal uniqueness (w.r.t. a marginal system $\mathcal{L}$) (or shortly a MU-set) if

$$Q(B) = R(B) = 1,\; \mathrm{MARG}(Q) = \mathrm{MARG}(R) \Rightarrow R \equiv Q$$

holds for every R, $Q \in M_1(X)$.

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*Presented by Prof. Josef Štěpán. We regret to have to say that Dr. Petra Linhartová, née Beránková, died in an accident on August 20, 1991.

It is easy to see that each set $\mathcal{L}(P)$ $(P \in M_1(X))$ is a nonempty convex set and contains a simplicial measure only if the projections q_j are continuous mappings, as in this case the set $\mathcal{L}(P)$ is weakly closed. In addition, the boundary of the set, $\operatorname{ex}\mathcal{L}(P)$, is rich enough to make valid the Choquet theorem for any $P \in M_1(X)$. The same conclusion is true in the case when q_j are continuous for $j \in J \setminus S$, where S is at most countable subset of J. The argument for this is as follows:

For $i \in S$ there is a uniformity of X_i which makes the set $U(X_i)$ of bounded uniformly continuous functions on X_i separable. Denote U_i a countable dense subset of $U(X_i)$, put $D = \bigcup_{i \in S} \{f \circ g \mid f \in U_i\}$ and observe that each $\mathcal{L}(P)$ is a nonempty convex set closed w.r.t. the coarsest topology of $M_1(X)$ for which the maps $Q \to \int_X h\, dQ$ are continuous for any $h \in C(X) \cup D$. Using [14] or [12], we get the desired conclusion.

The problem of characterization of simplicial measures has a remarkable history (see [3]). In the case of

$$\mathcal{L} = \{X = X_1 \times X_2 \xrightarrow{q_j} X_j,\ j = 1, 2\},$$

where q_j are continuous projections, Štěpán [13] has proved that $P \in M_1(X)$ is a simplicial measure if and only if $\operatorname{ess\,inf} \frac{dP'}{d|n|} = 0$ for any $n \in M(X)$, $\operatorname{MARG}(n) = 0$, $n \neq 0$, where P' is the absolutely continuous part of P w.r.t. $|n|$.

Our aim is to extend this result to general marginal systems $\mathcal{L}$. For this purpose we specify the Douglas density theorem [4] to our situation. Fix a marginal system $\mathcal{L}$ and denote

$$(1)\qquad \begin{aligned} D = \{f : X \to \mathbb{R} \mid f(x) = \sum_{j \in \alpha} f_j(q_j(x)),\ \alpha \subset J \text{ a finite set },\\ f_j \in C(X_j) \text{ for } j \in \alpha\}. \end{aligned}$$

Observe that D is a linear set of bounded Borel measurable functions defined on X, containing all constant functions, with the property

$$(2)\qquad \begin{aligned} &\operatorname{MARG}(P) = \operatorname{MARG}(Q) \text{ iff } \int_X f\, dP = \int_X f\, dQ\\ &\text{for any } f \in D,\ P, Q \in M(X). \end{aligned}$$

Hence, according to Douglas (1964), we have

Lemma. *P is a simplicial measure if and only if D is dense in $L_1(P)$.*

In connection with Lemma, let us observe that Hahn–Banach Theorem and Riesz Representation Theorem yield the following characterization of compact MU-sets.

Theorem 1. *Consider a marginal system $\mathcal{L}$ with all the projections q_j continuous and $K \subset X$ a compact set. Then K is a MU-set if and only if $D \restriction_K$ is a dense set in $C(K)$ (w.r.t. the supremum norm).*

In 1957, Arnol'd and Kolmogorov proved that for any $n \in \mathbb{N}$ there exists a set $S \subset \mathbb{R}^{2n+1}$ homeomorphic to $< 0,\, 1 >^n$, such that

$$C(S) = \{f : S \to \mathbb{R},\ f(x_1, \dots, x_{2n+1}) = \sum_{j=1}^{2n+1} f_j(x_j)$$
$$\text{for some } f_j \in C(\mathbb{R}),\ 1 \leq j \leq 2n+1\},$$

and provided thus very nontrivial examples of sets of marginal uniqueness. Indeed, according to Theorem 1 the set S is a MU-set when considering the marginal system $\{\mathbb{R}^{2n+1} \xrightarrow{\pi_j} \mathbb{R},\ j = 1, 2, \dots, 2n+1\}$ with the canonical projections π_j. From Theorem 1 we can also see that $< 0,\, 1 >^n$ is a MU-set w.r.t. the marginal system $\{< 0,\, 1 >^n \xrightarrow{q_j} \mathbb{R},\ j = 1, 2, \dots, 2n+1\}$, where $q_j = \pi_j(h)$ and h is a homeomorphism of $< 0,\, 1 >^n$ and S.

2. A characterization of simplicial measures.

Consider a marginal system $\mathcal{L} = \{X \xrightarrow{q_j} X_j |\, j \in J\}$, a $P \in M_1(X)$ and a Borel set $B \subset X$. Denote

$$\begin{aligned}
M_0(B) =& \{n \in M(X) |\ \mathrm{MARG}(n) = 0,\ |n|(\complement B) = 0\},\\
M(P, B) =& \{n \in M(X) |\ |n| \restriction_B \leq b \cdot P \text{ for a } b \in \mathbb{R}^+\},\\
M_1(P, B) =& M_1(X) \cap M(P, B),\\
\mathcal{K}_0 =& \{K \subset X \text{ a compact set } |\, n = 0 \text{ for every } n \in M_0(X) \cap M(P, \complement K)\},\\
\mathcal{K}_1 =& \{K \subset X \text{ a compact set } |\, n \restriction_K = 0 \text{ for any } n \in M_0(X) \cap M(P, \complement K)\}.
\end{aligned}$$

Now, we are prepared to generalize Theorem 1 of Štěpán [13].

Theorem 2. *Let $\mathcal{L} = \{X \xrightarrow{q_j} X_j |\, j \in J\}$ be a marginal system. The following statements are equivalent:*

(a) *P is a simplicial probability measure on X,*
(b) $\sup\{P(K) |\, K \in \mathcal{K}_0\} = 1$,
(c) $\sup\{P(K) |\, K \in \mathcal{K}_1\} = 1$,
(d) $\operatorname{ess\,inf} (\frac{dP'}{d|n|}) = 0$ *for any* $n \in M_0(X),\ n \neq 0$,
(e) $\operatorname{ess\,inf} (\frac{dP'}{d|n|}) = 0$ *for any* $n \in M_0(X),\ 0 \neq n \ll P$,
(f) $\operatorname{ess\,sup} |\frac{dn}{dP}| = +\infty$ *for any* $n \in M_0(X),\ 0 \neq n \ll P$,
(g) $g \in L_\infty(P),\ E_P[g |\, q_j] = 0,\ j \in J$ *implies that* $g = 0$ *a.s.* $[P]$,

where the essential infima and suprema are defined w.r.t. the dominating measures and P' denotes an absolutely continuous part of P w.r.t. the $|n|$. In (g) by $E_P[g |\, q_j]$ we have denoted the conditional expectation of g w.r.t. P relative to the σ-algebra

$$\sigma(q_j) = \{[q_j \in B_j],\ B_j \text{ Borel set in } X_j\}.$$

Corollary. *If P is a simplicial measure then*

$$\sup\{P(K),\ K \textit{ is a compact MU–set }\} = 1.$$

The assertion follows easily from (c) as each $K \in \mathcal{K}_1$ is easily seen to be a compact MU-set. Let us also observe that any of the conditions (a)–(g) implies that

$$\begin{aligned}&P \text{ is completely determined by its restriction to the}\\ &\sigma\text{–algebra } \sigma(q_j,\ j \in J) = \sigma\Big(\bigcup_{j\in J} \sigma(q_j)\Big).\end{aligned} \tag{3}$$

Proof: (a) $\Rightarrow$ (b) X is a separable metric space, so there exists an equivalent metric d, such that the space $U(X)$ of bounded functions on X uniformly continuous w.r.t. d is separable w.r.t. the usual supremum norm. Denote $\{f_1, f_2, ...\}$ a countable dense subset of $U(X)$.

According to Lemma there exist functions $a_n^i \in D$ (the set defined by (1)) for $i,\ n \in \mathbb{N}$, such that

$$\begin{aligned}&a_n^i \to f_i, \text{ as } n \to \infty\ a.s. \text{ w.r.t. } P\\ &\text{and in } L_1(P) \text{ for } i \in \mathbb{N}.\end{aligned}$$

Take $\varepsilon > 0$. The Jegoroff's theorem implies the existence of compact sets $K_i \subset X$, such that

$$\begin{aligned}&P(K_i) > 1 - \varepsilon 2^{-i},\\ &a_n^i \to f_i, \text{ uniformly on } K_i,\ n \to \infty,\ i \in \mathbb{N}.\end{aligned}$$

Denote $K = \bigcap_{i=1}^{\infty} K_i$. Then $P(K) > 1 - \varepsilon$ and $a_n^i \to f_i$ uniformly on K, for $n \to \infty$, $i \in \mathbb{N}$. Now we only need to show that the compact set K, we have just constructed, is an element of $\mathcal{K}_0$. So, let $n \in M(P, \complement K) \cap M_0(X)$, it follows from (2) that $n(a) = 0$ for $a \in D$. We may write that

$$\begin{aligned}|n(f_i)| = |n(f_i) - n(a_k^i)| &\le |n|(\mathbf{1}_K |a_k^i - f|) + |n|(\mathbf{1}_{\complement K} |a_k^i - f_i|) \le\\ &\le |n|(\mathbf{1}_K |a_k^i - f_i|) + b \cdot P(|a_k^i - f_i|)\end{aligned}$$

holds for $i,\ k \in \mathbb{N}$ and some $b \in \mathbb{R}$. The limit of the first term as $k \to \infty$ is zero, because a_k^i converge to f uniformly on K, the limit of the second one is zero too, as a_k^i converge to f in $L_1(P)$. Thus we have proved that $n(f_i) = 0$ for all $i \in \mathbb{N}$, hence $n = 0$.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$(d) Suppose that (c) holds for a $P \in M_1$, assume that there are $n \in M_0(X), n \neq 0$, and $\delta > 0$, such that $\operatorname{ess\,inf} h_n \ge \delta$, where $h_n \in [\frac{dP'}{d|n|}]$. Take $K \in \mathcal{K}_1$ an arbitrary set. It is easy to see that

$$|n| \restriction_{\complement K} \le \delta^{-1} P' \le \delta^{-1} P,$$

hence $|n|$ is dominated by P on $\complement K$, which means that $n \in M(P, \complement K)$. As $K \in \mathcal{K}_1$, we have $n \restriction_K = 0$ and therefore $P'(K) = 0$. But it is in contradiction with (c).

(d) $\Rightarrow$ (e) Obvious.

(e) $\Rightarrow$ (f) Consider $n \in M_0(X)$, $0 \neq n \ll P$ and observe that

$$\left|\frac{dn}{dP}\right| = \frac{d|n|}{dP} = \frac{d|n|}{dP'} \text{ a.s. } [P]$$

holds as $|n|$ and $(P - P')$ are singular measures. Hence, $|\frac{dn}{dP}| \cdot \frac{dP'}{d|n|} = 1$ holds almost everywhere w.r.t. both P' and $|n|$ and thus it follows from (e) that $\operatorname{ess\,sup} |\frac{dn}{dP}| = +\infty$, when the essential supremum is defined w.r.t. P'. This, of course, implies (f).

(f) $\Rightarrow$ (g) Consider $g \in L_\infty(P)$ such that $E[g|\, q_j] = 0$ for each $j \in J$. Define $n \in M(X)$ by $dn = g \cdot dP$. It is easy to see that the signed measure n vanishes at each set in $\bigcup_{j \in J} \sigma(q_j)$, hence $n \in M_0(X)$. According to (f) we get $n = 0$ and the validity of implication (g).

(g) $\Rightarrow$ (a) Assume that P is not a simplicial measure. By Hahn–Banach Theorem and Lemma above there is $g \in L_\infty$, $P[g \neq 0] > 0$, such that

$$\int_X g \cdot f \, dP = 0 \text{ holds for any } f \in D. \tag{4}$$

As $C(X_j)$ is a dense set in $L_1(q_j \circ P)$ for any $j \in J$, we may see that (4) is equivalent to $E[g|\, q_j] = 0$ for $j \in J$ which contradicts the implication (g). □

To illustrate the theory, we have presented, let us consider a marginal system $\mathcal{L} = \{X \xrightarrow{p} Y,\, X \xrightarrow{q} Z\}$ and a measure $P \in M_1(X)$, such that

$$P[(p,\, q) \in S] = 1 \text{ and } P[p = y,\, q = z] > 0 \text{ for } (y,\, z) \in S$$

holds for a finite set $S \subset Y \times Z$. Using (g) we are able to prove that P is a simplicial measure if and only if (see [9])

$$\begin{gathered} P = \sum_{j=1}^{h} \alpha_j \varepsilon_{x_j} \text{ for some } x_j \in X \\ \text{and } \alpha_j > 0 \text{ with } h = \operatorname{card} S \end{gathered} \tag{5}$$

and

$$\begin{gathered} \text{there is no finite sequence } (y_1,\, z_1), \ldots, (y_{2n},\, z_{2n}) \text{ of distinct points} \\ \text{in } S \text{ such that } y_1 = y_2,\, z_2 = z_3, \ldots, y_{2n-1} = y_{2n},\, z_{2n} = z_1 \text{ – a cycle .} \end{gathered} \tag{6}$$

Indeed, if P is a simplicial measure then according to (3) P is completely determined by its values in the sets $[p = y,\, q = z]$, $(y,\, z) \in S$. Hence, these sets are atoms of P, which implies that P has a form of (5). Now, assume that there is a cycle $(y_1, z_1), \dots, (y_{2n}, z_{2n})$ in S. Without loss of generality, assume that $\mathrm{card}\{y_1, \dots, y_{2n}\} = \mathrm{card}\{z_1, \dots, z_{2n}\} = n$. Define $g \in L_\infty(P)$ by

$$g = \sum_{i=1}^{2n} (-1)^{i+1} P[p = y_i,\, q = z_i] \cdot I_{[p=y_i,\, q=z_i]}$$

and observe that $E[g|\,p] = E[g|\,q] = 0$. Indeed, if, for example, $1 \le i \le 2n$ is odd, then $P[p = y_i] = P[p = y_i, q = z_i] + P[p = y_i, q = z_{2i+1}]$ implies that $E[g|\,p = y_i] = 0$. Using (g) we arrive to contradiction.

To finish our reasoning, assume that a measure P defined by (5) is not simplicial. According to (g) there is a $g \in L_\infty$, $P[g \neq 0] > 0$ such that $E[g|\,p] = E[g|\,q] = 0$. Now, it is easy to construct a cycle in S by induction:

We start with a $(y_1, z_1) \in S$, such that $E[y|\,p = y_1, q = z_1] > 0$. As $E[g|\,p] = 0$, we may find $(y_1, z_2) \in S$, such that $E[g|\,p = y_1, q = z_2] < 0$. Now, $E[g|\,q] = 0$ implies the existence of $(y_3, z_2) \in S$ with $E[g|\,p = y_3, q = z_2] > 0 \dots$ etc. Continuing this procedure we construct a sequence $(y_i, z_i) \in S$ which necessarily contains a cycle segment $(y_j, z_j), (y_{j+1}, z_{j+1}), \cdots, (y_{j+l}, z_{j+l})$.

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(*Received* September 23, 1991)