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Smoothing effect and discretization in time to semilinear parabolic equations with nonsmooth data

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Abstract. The purpose of this paper is to derive the error estimates for discretization in time of a semilinear parabolic equation in a Banach space. The estimates are given in the norm of the space $\mathbb{X}_\alpha$ for $0 < \alpha < 1$ when the initial condition is not regular.

Keywords: error estimates, parabolic equation, backward Euler method, nonsmooth initial data

Classification: 65M15, 35K22, 65M20

1. Introduction.

The aim of this paper is to study the error estimates for discretization in time (backward Euler method, Rothe method) applied to the abstract semilinear evolution equation ($t \in \langle 0, T\rangle$)

$$u'(t) + Au(t) = f(t, u(t)) \qquad (1.1)$$
$$u(0) = v \in \mathbb{X}$$

in a Banach space $\mathbb{X}$ with the norm $\| \;\|$. The operator A is assumed to be sectorial in $\mathbb{X}$ with the domain $D(A)$, where $\operatorname{Re}\sigma(A) > \delta_0 > 0$ and $\sigma(A)$ is the spectrum of A.

Definition 1 (cf. [2, D.1.3.1]). A linear operator A in a Banach space $\mathbb{X}$ is said to be sectorial in $\mathbb{X}$ iff:

(i) A is closed, $\overline{D(A)} = \mathbb{X}$,

(ii) there exist $\xi \in (0, \pi/2)$, $a \in \mathbb{R}$ such that

$$S_{a,\xi} = \{\lambda;\ \xi \le |\arg(\lambda - a)| \le \pi,\ \lambda \ne a\} \subset \rho(A)$$

where $\rho(A)$ denotes the resolvent set of A and

$$\|(\lambda - A)^{-1}\| \le C|\lambda - a|^{-1} \quad \forall\, \lambda \in S_{a,\xi}.$$

The function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is global Hölder continuous (with the Hölder coefficient $0 < \theta \le 1$) in the first variable and global Lipschitz continuous in the second variable. We are interested here in the case when the initial element v is rough, i.e. the only assumption is $v \in \mathbb{X}$.

It is well known that there exists a unique solution of (1.1) and it can be described in the following way

$$u(t) = T(t)v + \int_0^t T(t-s) f(s, u(s))\, ds, \tag{1.2}$$

where

$$T(t) = (2\pi\,\mathrm{i})^{-1} \int_\Gamma e^{\lambda t} (\lambda + A)^{-1}\, d\lambda$$

and Γ is a curve in $\rho(-A)$ (the resolvent set of $-A$) such that $\arg\lambda \to \pm\phi$ as $|\lambda| \to \infty$ for any fixed $\phi \in (\pi/2, \pi)$.

Without loss of generality we can suppose that Γ is described as follows

$$\lambda \in \Gamma \Leftrightarrow \lambda = -\delta - s\cos\varphi \pm \mathrm{i}\; s\,\sin\varphi, \tag{1.3}$$

where $s \in \langle 0, \infty)$, $\varphi \in (0, \pi/2)$, $\delta = \delta(\delta_0) > 0$.

During the past ten years many authors have been studying the error estimates for discretization in space or in time applied to (1.1), cf. Helfrich [1], Johnson–Larsson–Thomee–Wahlbin [3], Le Roux [4], Le Roux–Thomee [5], Luskin–Rannacher [6], Mingyou–Thomee [7], Sammon [8], Slodička [9], [10], Thomee [12], [13], Thomee–Zhang [14]. The most of the works mentioned above are written in Hilbert spaces and the operator A is assumed to be selfadjoint and positive definite.

Using backward Euler method for discretization in time we get

$$\begin{gathered} (u_i - u_{i-1})\tau^{-1} + Au_i = f(t_i, u_{i-1}) \\ u_0 = v, \end{gathered} \tag{1.4}$$

for $i = 1, 2, \ldots$; τ is a time step; $t_i = i\tau$.

The following error estimate is known (see [10, Th. 1]) for $0 < \tau < \tau_0 < 1$

$$\|u(i\tau) - u_i\| \le C\left(i^{-1} + \tau^\theta + \tau\ln\tau^{-1}\right), \tag{1.5}$$

where $i = 1, 2, \ldots$; u is the exact solution of (1.1) and u_i is the solution of (1.4). The formula (1.5) was obtained without any regularity assumptions of the initial element $v \in \mathbb{X}$.

The smoothing property for parabolic equations is familiarly known. We show that this property takes place for discretization in time, too. Using this we are able to establish the error estimate for backward Euler method in the norm of the space $\mathbb{X}_\alpha$, $0 < \alpha < 1$ (the definition of $\mathbb{X}_\alpha$ can be found in [2, Def. 1.4.7]). Our main results are formulated in Theorems 1–3 without any regularity assumptions of the initial element $v \in \mathbb{X}$.

Remark. C denotes a generic positive constant independent of τ but it may depend on $\delta_0, \phi, v, T, \alpha$.

2. Homogeneous problem.

In this section we suppose $f \equiv 0$. Solving (1.1) by backward Euler method we get elliptic problems

$$(u_i - u_{i-1})\tau^{-1} + Au_i = 0$$
$$u_0 = v,$$

where τ is a time step; u_i is the approximate solution of (1.1) at the time $t_i = i\tau$; $i = 1, 2, \dots$. This system can be solved successively for $i = 1, 2, \dots$ and it is easy to find that

$$u_i = (I + \tau A)^{-i} v.$$

Let us denote $g(\lambda) = (1 - \tau\lambda)^{-t/\tau}$ for arbitrary positive fixed t, τ. Let the range of definition of $g(\lambda)$ be

$$D = \mathbb{S} - \{\lambda \in \mathbb{S};\ |\lambda - \tau^{-1}| \le \varepsilon\}$$

for sufficiently small $\varepsilon > 0$; $\mathbb{S}$ denotes the closed complex plane.

One can see that D is an open set in $\mathbb{S}$ which contains $\sigma(-A)$ because of $\operatorname{Re}\sigma(A) > \delta_0 > 0$ and A is sectorial. The complement of D is compact. Further, g is differentiable in D and $g(\lambda)$ is bounded as $|\lambda| \to \infty$, because of

$$g(\infty) = \lim_{|\lambda| \to \infty} g(\lambda) = 0.$$

So, $g(-A)$ can be described in the following way (see [11, § 5.6])

$$T_\tau(t) = (I + \tau A)^{-t/\tau} = (2\pi\,\mathrm{i})^{-1} \int_\Gamma (1 - \tau\lambda)^{-t/\tau} (\lambda + A)^{-1}\, d\lambda \tag{2.1}$$

where Γ is taken from (1.3).

Let us note that the integral in (2.1) is absolutely convergent for every positive t, τ. On the other hand, we can say that $T_\tau(t)$ is a fractional power of $(I + \tau A)^{-1}$.

It is well known that for $\alpha \ge 0$ we have (see [2, Th. 1.4.3])

$$T(t)v \in D(A^\alpha) \qquad \forall\, v \in \mathbb{X},\ \forall\, t > 0.$$

The definition of $D(A^\alpha)$ can be found in [2, Def. 1.4.1].

This fact is known as smoothing effect. Let us remark that $T(t)$, $t \ge 0$, is an analytic semigroup. We know (see [9, Th. 1]) that $T_\tau(t)$, $t \ge 0$ is a semigroup, too. We shall prove that the smoothing effect takes place for $T_\tau(t)$. More exactly, the following lemma holds.

Lemma 1. *Let $\alpha \ge 0$; $t, \tau > 0$ such that $t > \tau\alpha$. Then $T_\tau(t)x \in D(A^\alpha)$ for every $x \in \mathbb{X}$.*

Proof: We consider the case when $0 \le \alpha \le 1$ first. Using [2, Th. 1.4.4] for $\lambda \in \Gamma$ we have

$$\|A^\alpha (\lambda + A)^{-1}\| \le C|\lambda|^{\alpha - 1}. \tag{2.2}$$

From this we obtain

$$\begin{aligned}\|A^\alpha T_\tau(t)\| &= \Big\|(2\pi\,\mathrm{i})^{-1}\int_\Gamma (1-\tau\lambda)^{-t/\tau} A^\alpha(\lambda+A)^{-1}\,d\lambda\Big\| \le \\ &\le C\int_\Gamma \Big|(1-\tau\lambda)^{-t/\tau}\Big|\ |\lambda|^{\alpha-1}\,|d\lambda| \le \\ &\le C\int_\Gamma (1-\tau\operatorname{Re}\lambda)^{-t/\tau}|\operatorname{Re}\lambda|^{\alpha-1}\,|d\lambda| \le \\ &\le C\tau^{-t/\tau}\int_\Gamma |\operatorname{Re}\lambda|^{\alpha-t/\tau-1}\,|d\lambda|.\end{aligned}$$

The last integral is convergent if $t > \alpha\tau$.

Let us consider $\alpha > 1$. Then we can put $\alpha = n+\beta$ where n is an integer and $\beta \in \langle 0,1)$. So we deduce

$$\begin{aligned} A^\alpha T_\tau(t) &= A^{n+\beta} T_\tau(tn\alpha^{-1}) T_\tau(t\beta\alpha^{-1}) = \\ = A^n T_\tau(tn\alpha^{-1}) &= A^\beta T_\tau(t\beta\alpha^{-1}) = \Big(AT_\tau(t\alpha^{-1})\Big)^n A^\beta T_\tau(t\beta\alpha^{-1})\end{aligned}$$

because of $t > \alpha\tau$. □

$T_\tau(t)v$, as an approximate solution of (1.1) for $f \equiv 0$, was introduced in [9]. It was proved there that

$$\|T(t) - T_\tau(t)\| \le C\,\tau t^{-1}. \tag{2.3}$$

In virtue of Lemma 1 we know that the both solutions (exact and approximate) become smoother with increasing time. So, there arises such a question: "How does the estimate of $(T(t) - T_\tau(t))$ look like in the norm of the space $\mathbb{X}_\alpha$?". The answer to this question (in the case when $\alpha = 0$) is given by (2.3). In order to establish such an estimate, when $0 < \alpha \le 1$, we need the following lemmas.

Lemma 2. *If* $\lambda \in \mathbb{C}$ (*complex plane*), $\operatorname{Re}\lambda < 0$ *and* $t, \tau > 0$, *then*

$$\Big|(1-\tau\lambda)^{-t/\tau} - e^{\lambda t}\Big| \le |\lambda|^2 |\operatorname{Re}\lambda|^{-2}\Big|(1-\tau\operatorname{Re}\lambda)^{-t/\tau} - e^{\operatorname{Re}\lambda t}\Big|.$$

Proof: See [9]. □

Lemma 3. *If* $\min\{1,\beta\} > \alpha > 0$, *then*

$$\int_0^\infty z^{\alpha-1}\Big[\big(1+\beta^{-1}z\big)^{-\beta} - e^{-z}\Big]\,dz \le \beta^\alpha\,(\beta-\alpha)^{-1}.$$

Proof: Let us fix α, β and for arbitrary $N > 0$ we define

$$I_N = \int_0^N z^{\alpha-1}\Big[\big(1+\beta^{-1}z\big)^{-\beta} - e^{-z}\Big]\,dz.$$

It is easy to see that ($\forall\, z > 0$)

$$\partial_z\Big[e^z\big(1+\beta^{-1}z\big)^{-\beta}-1\Big]=\beta^{-1}z\;e^z\big(1+\beta^{-1}z\big)^{-\beta-1},$$
$$\partial_z\Big[\int_N^z e^{-s}s^{\alpha-1}\,ds\Big]=e^{-z}\;z^{\alpha-1}.$$

Using integration by parts one can find

$$\begin{aligned}
I_N&=\int_0^N z^{\alpha-1}\Big[\big(1+\beta^{-1}z\big)^{-\beta}-e^{-z}\Big]\,dz=\\
&=\Big[\int_N^z e^{-s}s^{\alpha-1}\,ds\int_0^z\beta^{-1}s\;e^s\big(1+\beta^{-1}s\big)^{-\beta-1}\,ds\Big]_0^N+\\
&+\int_0^N\int_z^N e^{-s}s^{\alpha-1}\,ds\;\beta^{-1}z\;e^z\big(1+\beta^{-1}z\big)^{-\beta-1}\,dz\le\\
&\le\int_0^\infty\int_z^\infty e^{-s}s^{\alpha-1}\,ds\;\beta^{-1}z\;e^z\big(1+\beta^{-1}z\big)^{-\beta-1}\,dz.
\end{aligned}$$

One can prove that ($\forall\, z > 0$)

$$z\;e^z\int_z^\infty e^{-s}s^{\alpha-1}\,ds\le z^\alpha.$$

Because of this we obtain

$$\begin{aligned}
I_N&\le\int_0^\infty\beta^{-1}z^\alpha\big(1+\beta^{-1}z\big)^{-\beta-1}\,dz=\beta^\alpha\int_0^\infty w^\alpha\big(1+w\big)^{-\beta-1}\,dw\le\\
&\le\beta^\alpha\int_0^\infty\big(1+w\big)^{\alpha-\beta-1}\,d(1+w)=\beta^\alpha\,(\beta-\alpha)^{-1}.
\end{aligned}$$

The assertion of the lemma follows from the last estimate taking the limit as $N\to\infty$. □

Now, we are able to derive the estimate of $(T(t)-T_\tau(t))$ in the norm of the space $\mathbb{X}_\alpha$ for $0<\alpha\le 1$ without any regularity assumption of the initial element $v\in\mathbb{X}$. We do it for $t>\tau$ first.

Theorem 1. *Let A be a sectorial operator in a Banach space $\mathbb{X}$ where $\operatorname{Re}\sigma(A)>\delta_0>0$. Then for $t>\tau$, $\tau<\tau_0$ we have*

(i) $$\|T(t)-T_\tau(t)\|_1\le C\;\tau t^{-1}(t-\tau)^{-1},$$

(ii) $$\|T(t)-T_\tau(t)\|_\alpha\le C\;\tau t^{-1}(t-\tau)^{-\alpha},\quad 0\le\alpha\le 1.$$

($\|\ \|_\alpha$ denotes the norm in $\mathbb{X}_\alpha$, $\|w\|_\alpha=\|A^\alpha w\|$.)

Proof: (i) In fact, using (2.2) we find

$$\|T(t)-T_\tau(t)\|_1=\|A[T(t)-T_\tau(t)]\|\le C\int_\Gamma\big|(1-\tau\lambda)^{-t/\tau}-e^{\lambda t}\big|\;|d\lambda|.$$

In virtue of Lemma 2 we get

$$\begin{aligned}\|T(t)-T_\tau(t)\|_1 &\le C\int_\Gamma \left|(1-\tau\operatorname{Re}\lambda)^{-t/\tau}-e^{\operatorname{Re}\lambda t}\right|\,|d\lambda| \le\\ &\le C\int_0^\infty \left[(1+\tau y)^{-t/\tau}-e^{-yt}\right]dy = C\,\tau t^{-1}(t-\tau)^{-1}.\end{aligned}$$

(ii) For $t>\tau$ we have $(T(t)-T_\tau(t))v\in D(A)$. So applying [2, Th. 1.4.4] one can prove $(0\le\alpha\le 1)$

$$\begin{aligned}\|T(t)-T_\tau(t)\|_\alpha &= \|A^\alpha(T(t)-T_\tau(t))\| \le\\ &\le C\|A(T(t)-T_\tau(t))\|^\alpha\,\|T(t)-T_\tau(t)\|^{1-\alpha}.\end{aligned}$$

The rest of the proof follows from this fact, (2.3) and (i). □

By now, we have established the error estimate in the norm of $\mathbb{X}_\alpha$ in the case when $t>\tau$. But, for the discretization in time, it is necessary to derive this error in all time steps $t_i=i\tau$; $i=1,2,\dots$. So we must still do it for $t=\tau$.

Theorem 2. *Let A be a sectorial operator in a Banach space $\mathbb{X}$ where $\operatorname{Re}\sigma(A)>\delta_0>0$. Then for $0<\alpha<1$, $\tau<\tau_0$ and $t>\alpha\tau>0$ we have*

$$\|T(t)-T_\tau(t)\|_\alpha \le C\,\tau^{1-\alpha}(t-\alpha\tau)^{-1}.$$

Proof: We know that $(T(t)-T_\tau(t))v\in D(A^\alpha)$ because of $t>\alpha\tau$. Further, applying (2.2) we can write

$$\begin{aligned}\|T(t)-T_\tau(t)\|_\alpha &= \|A^\alpha(T(t)-T_\tau(t))\| \le\\ &\le C\int_\Gamma \left|(1-\tau\lambda)^{-t/\tau}-e^{\lambda t}\right|\,\|A^\alpha(\lambda+A)^{-1}\|\,|d\lambda| \le\\ &\le C\int_\Gamma \left|(1-\tau\lambda)^{-t/\tau}-e^{\lambda t}\right|\,|\lambda|^{\alpha-1}\,|d\lambda|.\end{aligned}$$

Using Lemma 2 we estimate

$$\begin{aligned}\|T(t)-T_\tau(t)\|_\alpha &\le C\int_\Gamma \left|(1-\tau\operatorname{Re}\lambda)^{-t/\tau}-e^{\operatorname{Re}\lambda t}\right|\,|\operatorname{Re}\lambda|^{\alpha-1}\,|d\lambda| \le\\ &\le C\int_\delta^\infty \left[(1+\tau y)^{-t/\tau}-e^{-yt}\right]y^{\alpha-1}\,dy \le\\ &\le C\,t^{-\alpha}\int_0^\infty z^{\alpha-1}\left[\left(1+\tau t^{-1}z\right)^{-t/\tau}-e^{-z}\right]dz.\end{aligned}$$

The rest of the proof follows from this applying Lemma 3 for $\beta=t\tau^{-1}>\alpha>0$.

□

3. Nonhomogeneous problem.

In this section we suppose that the function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ satisfies

$$\begin{gathered} \|f(t,x) - f(s,y)\| \le C\big(|t-s|^\theta + \|x-y\|\big) \\ \forall\, x,y \in \mathbb{X};\ \forall\, t,s \in \mathbb{R};\ 0 < \theta \le 1. \end{gathered} \tag{3.1}$$

Considering the discretization scheme (1.4) with the time step τ ($0 < \tau < \tau_0 < 1$) one can prove

$$u_i = T_\tau(t_i)v + \sum_{k=0}^{i-1} T_\tau(t_i - t_k)\, f(t_{k+1}, u_k)\,\tau, \tag{3.2}$$

where $T_\tau(t)$ is defined by (2.1).

In the following we shall need the following estimates.

Lemma 4. *If* $0 < \alpha < 1$, *then for all* $n \in \mathbb{N}$ *we have*

(i) $$\sum_{k=1}^{n} (k-\alpha)^{-1} \le 2(1-\alpha)^{-1} \ln\big(1 + n(1-\alpha)^{-1}\big)$$

(ii) $$\sum_{k=1}^{n} k^{-\alpha} \le (1-\alpha)^{-1} n^{1-\alpha}.$$

Proof: The proof is straightforward and so it is left to the reader. □

Lemma 5. *Suppose* $0 < \alpha < 1$.

(i) *Let* u *be the solution of* (1.1) *defined by* (1.2). *Then*

$$\|u(t)\|_\alpha \le C\ t^{-\alpha} \qquad \forall\, t \le T.$$

(ii) *Let* u_i *be the solution of* (1.4) *defined by* (3.2). *Then*

$$\|u_i\|_\alpha \le C\ t_i^{-\alpha} \qquad \forall\, i = 1, 2, \dots .$$

Proof: (i) This assertion follows immediately from (3.1) applying the semigroup theory.

(ii) Using Theorem 2, [2, Th. 1.4.3] we get

$$\begin{aligned} &\|T_\tau(t_i)\|_\alpha \le \|T(t_i)\|_\alpha + \|T_\tau(t_i) - T(t_i)\|_\alpha \le \\ &\le C\big(t_i^{-\alpha} + \tau^{-\alpha}(i-\alpha)^{-1}\big) = C\ t_i^{-\alpha}\big(1 + i^\alpha (i-\alpha)^{-1}\big) \le C\ t_i^{-\alpha}. \end{aligned}$$

In virtue of [10, Lemma 1] and (3.1) one can write

$$\|f(t_i, u_j)\| \le C$$

for $i, j = 1, 2, \ldots$.

So we have

$$\|u_i\|_\alpha \leq \|T_\tau(t_i)\|_\alpha \|v\| + \sum_{k=0}^{i-1} \|T_\tau(t_i - t_k)\|_\alpha \|f(t_{k+1}, u_k)\| \tau \leq$$
$$\leq C\Big[t_i^{-\alpha} + \sum_{k=0}^{i-1} (i-k)^{-\alpha} \tau^{1-\alpha}\Big] = C\Big[t_i^{-\alpha} + \sum_{k=1}^{i} k^{-\alpha}\tau^{1-\alpha}\Big] \leq C\ t_i^{-\alpha}.$$

□

Theorem 3. *Let A be a sectorial operator in a Banach space $\mathbb{X}$ where* $\operatorname{Re}\sigma(A) > \delta_0 > 0$. *Suppose* (3.1), $0 < \alpha < 1$. *Then*

$$\|u(t_i) - u_i\|_\alpha \leq C\big(\tau^{-\alpha}(i-\alpha)^{-1} + \tau^{\theta-\alpha} + \tau^{1-\alpha}\ln \tau^{-1}\big)$$

for all $i = 1, 2, \ldots$.

Proof: We can write

$$u(t_i) - u_i = I_1 + I_2 + I_3 + I_4 + I_5, \tag{3.2}$$

where

$$I_1 = (T(t_i) - T_\tau(t_i))v,$$
$$I_2 = \sum_{k=0}^{i-1} T(t_i - t_k)\big[f(t_{k+1}, u(t_k)) - f(t_{k+1}, u_k)\big]\,\tau,$$
$$I_3 = \sum_{k=0}^{i-1} \big[T(t_i - t_k) - T_\tau(t_i - t_k)\big]\, f(t_{k+1}, u_k)\,\tau,$$
$$I_4 = \int_\tau^{t_{i-1}} T(t_i - s)\, f(s, u(s))\, ds - \sum_{k=1}^{i-2} T(t_i - t_k)\, f(t_{k+1}, u(t_k))\,\tau,$$
$$I_5 = \int_0^\tau T(t_i - s)\, f(s, u(s))\, ds + \int_{t_{i-1}}^{t_i} T(t_i - s)\, f(s, u(s))\, ds -$$
$$- \big[T(t_i)\, f(\tau, v) + T(\tau)\, f(t_i, u(t_{i-1}))\big]\,\tau.$$

Let us estimate $I_1, \ldots, I_5$. Using Theorem 2 we have

$$\|I_1\|_\alpha \leq C\ \tau^{-\alpha}(i-\alpha)^{-1}. \tag{3.3}$$

It is easy to see that

$$\|I_5\|_\alpha \leq C\,\Big[\int_0^\tau \|T(t_i - s)\|_\alpha\, ds + \int_{t_{i-1}}^{t_i} \|T(t_i - s)\|_\alpha\, ds +$$
$$+ \|T(t_i)\|_\alpha \tau + \|T(\tau)\|_\alpha \tau\Big] \leq$$
$$\leq C\,\Big[\int_0^\tau (t_i - s)^{-\alpha}\, ds + \int_{t_{i-1}}^{t_i} (t_i - s)^{-\alpha}\, ds + \tau t_i^{-\alpha} + \tau^{1-\alpha}\Big] \leq$$
$$\leq C\ \tau^{1-\alpha}\Big[i^{1-\alpha} - (i-1)^{1-\alpha} + 1\Big].$$

So we can write

$$\|I_5\|_\alpha \le C\,\tau^{1-\alpha}. \tag{3.4}$$

The second term can be estimated in the following way

$$\begin{aligned}\|I_2\|_\alpha &\le C\sum_{k=0}^{i-1}\|T(t_i-t_k)\|_\alpha\,\|u(t_k)-u_k\|\,\tau \le\\ &\le C\sum_{k=0}^{i-1}\tau^{1-\alpha}(i-k)^{-\alpha}\,\|u(t_k)-u_k\|.\end{aligned}$$

In virtue of [10, Th. 1] we get

$$\|u(t_k)-u_k\| \le C\left(\tau^\theta + k^{-1} + \tau\ln\tau^{-1}\right).$$

Hence

$$\begin{aligned}\|I_2\|_\alpha \le& C\sum_{k=1}^{i-1}\tau^{1-\alpha}(i-k)^{-\alpha}\left(\tau^\theta+k^{-1}+\tau\ln\tau^{-1}\right) =\\ =& C\,\tau^{1-\alpha}\left(\tau^\theta+\tau\ln\tau^{-1}\right)\sum_{k=1}^{i-1}(i-k)^{-\alpha} + C\,\tau^{1-\alpha}\sum_{k=1}^{i-1}(i-k)^{-\alpha}k^{-1} \le\\ \le& C\,\tau^{1-\alpha}\left(\tau^\theta+\tau\ln\tau^{-1}\right)\sum_{k=1}^{i-1}k^{-\alpha} + C\,\tau^{1-\alpha}\sum_{k=1}^{i-1}k^{-1}.\end{aligned}$$

From this we deduce

$$\|I_2\|_\alpha \le C\left(\tau^\theta + \tau^{1-\alpha}\ln\tau^{-1}\right). \tag{3.5}$$

For the third term we get (using Theorem 2)

$$\begin{aligned}\|I_3\|_\alpha \le& \sum_{k=0}^{i-1}\|T(t_i-t_k)-T_\tau(t_i-t_k)\|_\alpha\,\|f(t_{k+1},u_k)\|\,\tau \le\\ \le& C\sum_{k=0}^{i-1}(i-k-\alpha)^{-1}\tau^{1-\alpha} = C\,\tau^{1-\alpha}\sum_{k=1}^{i-1}(k-\alpha)^{-1}.\end{aligned}$$

So

$$\|I_3\|_\alpha \le C\,\tau^{1-\alpha}\left(1+\ln\tau^{-1}\right). \tag{3.6}$$

Let us rewrite the fourth term into the following form

$$I_4 = S_1 + S_2, \tag{3.7}$$

where

$$S_1 = \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} T(t_i - s) \, \Big[f(s, u(s)) - f(t_{k+1}, u(t_k)) \Big] \, ds,$$

$$S_2 = \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} \Big[T(t_i - s) - T(t_i - t_k) \Big] \, f(t_{k+1}, u(t_k)) \, ds.$$

One can see that

$$\|S_1\|_\alpha \le C \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} \|T(t_i - s)\|_\alpha \big(\tau^\theta + \|u(s) - u(t_k)\| \big) \, ds.$$

Applying [10, Lemma 2] we get

$$\|u(s) - u(t_k)\| \le C \left(k^{-1} + \tau + \tau \ln k \right).$$

Hence

$$\|S_1\|_\alpha \le C \sum_{k=1}^{i-2} \big(k^{-1} + \tau^\theta + \tau \ln k \big) \int_{t_k}^{t_{k+1}} (t_i - s)^{-\alpha} \, ds.$$

Using

$$\int_{t_k}^{t_{k+1}} (t_i - s)^{-\alpha} \, ds \le (1 - \alpha)^{-1} \tau^{1-\alpha}$$

one can find

$$\|S_1\|_\alpha \le C \left[\tau^{\theta - \alpha} + \tau^{1-\alpha} \ln \tau^{-1} \right]. \tag{3.8}$$

At the end we estimate S_2. Applying [2, Th. 1.4.3] we have

$$\begin{aligned}
\|S_2\|_\alpha =& \left\| \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} \Big[T(s - t_k) - I \Big] A^\alpha \, T(t_i - s) \, f(t_{k+1}, u(t_k)) \, ds \right\| \le \\
\le& C \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} (s - t_k)^{1-\alpha} \|A \, T(t_i - s) \, f(t_{k+1}, u(t_k))\| \, ds \le \\
\le& C \sum_{k=1}^{i-2} \int_{t_k}^{t_{k+1}} (s - t_k)^{1-\alpha} (t_i - s)^{-1} \, ds \le \\
\le& C \sum_{k=1}^{i-2} \tau^{1-\alpha} \int_{t_k}^{t_{k+1}} (t_i - t_{k+1})^{-1} \, ds \le C \, \tau^{1-\alpha} \sum_{k=1}^{i} k^{-1}.
\end{aligned}$$

From this we get

$$\|S_2\|_\alpha \le C \, \tau^{1-\alpha} \Big[1 + \ln \tau^{-1} \Big]. \tag{3.9}$$

Using (3.2)–(3.9) we conclude the proof. □

Consequence. (i) If $0 \leq \alpha < \theta < 1$, then

$$(t_i - \alpha\tau)\,\|u(t_i) - u_i\|_\alpha = O(\tau^{\theta-\alpha}).$$

(ii) If $0 \leq \alpha < 1 = \theta$, then

$$(t_i - \alpha\tau)\,\|u(t_i) - u_i\|_\alpha = O(\tau^{1-\alpha}\ln\tau^{-1}).$$

PROOF: If $\alpha > 0$ the assertion follows from Theorem 3. If $\alpha = 0$ we use [10, Th. 1]. $\square$

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