On the behaviour of solutions to the nonlinear elliptic Neumann problem in unbounded domains

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Abstract. The asymptotic behaviour is studied for minima of regular variational problems with Neumann boundary conditions on noncompact part of boundary.

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Let F be a Caratheodory function such that

$$C_1|\xi_m|^p - g(x) \leqq F(x,\xi_0,\dots,\xi_m) \leqq C_2|\xi_m|^p + f(x), \tag{1}$$

where $p>1, C_1, C_2$ are positive constants, $x=(x_1,\dots,x_n)\in G\subset R^n$, $\xi_i\in R^{N_i}$, $N_i = C^{n-1}_{n+i-1}$, $0\leqq i\leqq m$; $g(x)\geqq 0$, $f(x)\geqq 0$ are given functions, G is an arbitrary domain that may be unbounded and on which some natural conditions will be imposed below. We shall denote $C^\infty_0(G,\Gamma)$ the space of functions vanishing in a neigbourhood of Γ and having a finite norm

$$\|f\| = \left(\int_G \sum_{|\alpha|=m} |D^\alpha f(x)|^p\,dx\right)^{\frac{1}{p}},$$

where Γ is a part of the boundary ∂G of the domain G with $(n-1)$ dimensional finite and positive measure. Then $W^{p,m}_0(G,\Gamma)$ is a completion of $C^\infty_0(G,\Gamma)$ according to the above mentioned norm.

Let ψ be a given function having the generalized derivatives $D^\alpha\psi\in L_p(G)$ for all α with $|\alpha|=m$. We denote by $\mathcal{M}_\psi(G,\Gamma)$ the set of all functions v having the generalized derivatives $D^\alpha v\in L_p(G)$ for all α with $|\alpha|=m$, such that $v-\psi\in W^{p,m}_0(G,\Gamma)$.

The function $u\in\mathcal{M}_\psi(G,\Gamma)$ is said to be a solution of the variational problem for a functional

$$\Phi(v,G) = \int_G F(x,v(x),Dv(x),\dots,D^m v(x))\,dx \tag{2}$$

satisfying the Dirichlet data on Γ and Neumann zero conditions on $\tilde\Gamma = \partial G\setminus\bar\Gamma$ if

$$\Phi(u,G) = min\{\Phi(v,G); v\in\mathcal{M}_{\psi(G,\Gamma)}\}.$$

The aim of this paper is to get estimates of the rate of decrease in the infinity for solutions of the variational problem satisfying Neumann zero conditions on the non-compact part of the boundary. The analogous questions for solutions of higher order elliptic equations satisfying the Dirichlet zero conditions on the lateral surface of a cylinder and having the bounded Dirichlet integral was first investigated by P.D. Lax ([1]). Recently, these problems and in particular the asymptotics of solutions have been exploited in a significant number of articles (i.e. [2], [3], [4], [5], [6]). We would like to remark that the problem being considered is not covered by the above mentioned papers that prevalently study the Dirichlet boundary value problem or use much stronger conditions on the functional. Moreover, as there are no requirements as to the smoothness of F, in general the Euler equation does not exist for the functional $\Phi(u,G)$.

If Γ is sufficiently smooth part of ∂G then for any function ψ having the generalized derivatives $D^\alpha\psi \in L_p(G)$ for all α with $|\alpha| = m$ there are the traces $D^\beta\psi\Big|_\Gamma \in L_p(\Gamma)$ for β such that $|\beta| \le m-1$. Then the condition $v-\psi \in W_0^{p,m}(G,\Gamma)$ means that the traces of v and ψ on Γ coincide.

Let $H = \{Q^{n-1}\times]-\infty;\infty[\}$, where Q^{n-1} is an $(n-1)$ dimensional ball and Q_r is an n-dimensional ball with radius r.

We shall denote by $u\Big|_{G_T}$ restriction of the function u on the domain $G_T = G \cap \{x_n > T\}$, as well as $\Gamma_T = G \cap \{x_n = T\}$.

Now we introduce a definition of the type of domains we shall deal with in the theorem.

Definition. *Let Ω be a domain in R^n, Ω_1 its convex subdomain. We call Ω starshaped with respect to Ω_1 if for every point $s \in \Omega$ all intervals connecting point s with any point $t \in \Omega_1$ are contained in Ω.*

Theorem. *Let a domain $G \subset H$ have the following structure: there are such numbers T_0 and positive r, a that*

(i) $\partial G_{T_0} \cap \Gamma = \emptyset$,

(ii) *for every number $A \geqq T_0$ a domain $G \cap \{A-a < x_n < A+a\}$ is starshaped with respect to the ball Q_r contained in $G \cap \{A - \frac{a}{4} < x_n < A + \frac{a}{4}\}$. Let functions f, g from (1) decrease exponentially at infinity, i.e. $f \cdot exp(\frac{\eta}{q}x_n) \in L_q(G)$, $g \cdot exp(\frac{\eta_1}{q_1}x_n) \in L_{q_1}(G)$ with q, $q_1 > 1$ and η, η_1 positive. Let $\mathcal{M}_\psi(G,\Gamma) \neq \emptyset$ and u be a solution of the variational problem (2). Then there is a number $\kappa > 0$ such that*

$$\int_G e^{\kappa\cdot x_n} \sum_{|\alpha|=m} |D^\alpha u|^p \, dx < \infty.$$

Before starting the proof of the theorem we need the following

Lemma. *Let the domain G and the function u be the same as in the theorem.*

Then for every number T such that $T \geqq T_0$ we have

$$\Phi(u, G_T) = min\{\Phi(v, G_T); v \in \mathcal{M}_{u|_{G_T}}(G_T, \Gamma_T)\}.$$

PROOF: Let there exist a function $\hat{u}$ different from u such that $\hat{u} \in \mathcal{M}_{u|_{G_T}}(G_T, \Gamma_T)$ and $\Phi(\hat{u}, G_T) = min\{\Phi(v, G_T); v \in \mathcal{M}_{u|_{G_T}}(G_T, \Gamma_T)\} < \Phi(u, G_T)$. Denote now

$$\tilde{u} = \begin{cases} \hat{u} - u, & \text{for } x \in G_T \\ 0, & \text{for } x \in G \setminus G_T \end{cases}$$

and consider a new function $w = u + \tilde{u}$. It follows from the definition of generalized derivatives that there exist $D^\alpha w \in L_p(G)$ for all $\alpha, |\alpha| = m$. In addition, $w \in \mathcal{M}_\Psi(G, \Gamma)$. Let us evaluate

$$\begin{aligned}
\Phi(w, G) &= \int_{G \cap \{x_n < T\}} F(x, u(x), Du(x), \dots, D^m u(x))\, dx \\
&\quad + \int_{G_T} F(x, \hat{u}(x), D\hat{u}(x), \dots, D^m \hat{u}(x))\, dx \\
&= \int_{G \cap \{x_n < T\}} F(x, u(x), Du(x), \dots, D^m u(x))\, dx + \Phi(\hat{u}, G_T) \\
&< \int_{G \cap \{x_n < T\}} F(x, u(x), Du(x), \dots, D^m u(x))\, dx + \Phi(u, G_T) < \Phi(u, G).
\end{aligned}$$

The last inequality gives the contradiction. □

PROOF OF THE THEOREM: Let us introduce the domains

$$\Omega_i = G \cap \{A + (2i-3)a < x_n < A + (2i-1)a\}, i \geqq 1.$$

Taking into account the structure of the domain G it can be shown that in every domain Ω_i there exist balls Q_r^i of fixed radius r such that

$$Q_r^i \subset G \cap \{A + (2i - \frac{9}{4})a < x_n < A + (2i - \frac{7}{4})a\}$$

and the domains Ω_i are starshaped with respect to Q_r^i.

Let us consider a function

$$z_i(x) = u(x) - \bar{u}_i - \sum_{i_1=1}^{n} a_{i_1}^i x_{i_1} - \dots - \sum_{i_1, \dots, i_{m-1}=1}^{n} a_{i_1 \dots i_{m-1}}^i x_{i_1} \dots x_{i_{m-1}}$$

where the numbers $\bar{u}_i, a_{i_1}^i, \dots, a_{i_1 \dots i_{m-1}}^i$ are chosen so that all the derivatives $D^\alpha z_i$ with $|\alpha| < m$ are orthogonal to constants in a domain Ω_i.

Let $\sigma_i(x_n) \in C^\infty(R)$, $\sigma_i \le 1$,

$$\sigma_i(x_n) = \begin{cases} 1, & \text{for } x_n \geqq A + (2i-1)a, \\ 0, & \text{for } x_n \le A + (2i-3)a. \end{cases}$$

Denote for the simplicity $u\big|_{G_{A+(2i-3)a}} = \psi_i$. Taking into account the Lemma, we can easily conclude that for every $i \geqq 1$

$$\text{(3)} \quad \Phi(u, G_{A+(2i-3)a}) = \\ = min\{\Phi(v, G_{A+(2i-3)a}); v \in \mathcal{M}_{\Psi_i}(G_{A+(2i-3)a}, \Gamma_{A+(2i-3)a})\}.$$

Let us consider a function $u - \sigma_i z_i$ in a domain $G_{A+(2i-3)a}$. It is clear that $u - \sigma_i z_i \in \mathcal{M}_{\Psi_i}(G_{A+(2i-3)a}, \Gamma_{A+(2i-3)a})$. By using the assumption of the theorem and the inequality (1) and (3) we obtain

$$\begin{aligned} &\int_{G_{A+(2i-3)a}} \sum_{|\alpha|=m} \left|D^\alpha u(x)\right|^p dx \\ &\qquad \le C \int_{G_{A+(2i-3)a}} g(x)\,dx + \Phi(u, G_{A+(2i-3)a}) \\ \text{(4)} \qquad &\qquad \le C \int_{G_{A+(2i-3)a}} g(x)\,dx + \Phi(u - \sigma_i z_i, G_{A+(2i-3)a}) \\ &\qquad \le C\Big[e^{-\frac{\eta}{q}[A+(2i-3)a]} + e^{-\frac{\eta_1}{q_1}[A+(2i-3)a]} \\ &\qquad + \int_{G_{A+(2i-3)a}} \sum_{|\alpha|=m} \left|D^\alpha (u - \sigma_i z_i)\right|^p dx\Big], \end{aligned}$$

where the constant C depends on $C_1, C_2, H, A, a, m, n, \eta, \eta_1, q, q_1, f$ and g but does not depend on i.

Taking into account the choice of σ_i, z_i we get the following estimate for the last integral in (4)

$$\begin{aligned} \int_{G_{A+(2i-3)a}} \sum_{|\alpha|=m} \left|D^\alpha (u - \sigma_i z_i)\right|^p dx &= \int_{\Omega_i} \sum_{|\alpha|=m} \left|D^\alpha (u - \sigma_i z_i)\right|^p dx \\ &\le C\Big[\int_{\Omega_i} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx + \int_{\Omega_i} \sum_{|s_1|+|s_2|=m, |s_2| \le m-1} \left|D^{s_1}\sigma_i D^{s_2} z_i\right|^p dx\Big] \\ &\le C\Big[\int_{\Omega_i} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx + \int_{\Omega_i} \sum_{|s_2| \le m-1} \left|D^{s_2} z_i\right|^p dx\Big], \end{aligned}$$

where the constant C does not depend on i. (We shall denote by C different constants specifying, if necessary, what they depend on. Generally, they do not depend on i.)

Applying the Steklov inequality ([7]) for domains Ω_i starshaped with respect to Q_r^i and taking into account the orthogonality of functions z_i to constants, we obtain for all i and s_2 such that $|s_2| \le m-1$

$$\int_{\Omega_i} |D^{s_2} z_i|^p\,dx \le C\Big[\Big|\int_{\Omega_i} D^{s_2} z_i\,dx\Big| + \Big(\int_{\Omega_i} \sum_{|s_3|=|s_2|+1} |D^{s_3} z_i|^p\,dx\Big)^{\frac{1}{p}}\Big]^p. \tag{5}$$

Since the first integral on the right hand side of this inequality is equal to zero for all s_2 such that $|s_2| \le m-1$ then

$$\int_{\Omega_i} |D^{s_2} z_i|^p\,dx \le C\int_{\Omega_i} \sum_{|s_3|=|s_2|+1} |D^{s_3} z_i|^p\,dx.$$

If $|s_3| < m$ we apply the Steklov inequality once more. Finally we get

$$\int_{\Omega_i} |D^{s_2} z_i|^p\,dx \le C\int_{\Omega_i} \sum_{|\alpha|=m} |D^{\alpha} u|^p\,dx \tag{6}$$

for every s_2 such that $|s_2| \le m-1$. Note that in the inequality (6) the constant C depends on meas Q_r^i, meas Ω_i, m, p but does not depend on i. Because of (5) and (6) we have from (4)

$$\int_{G_{A+(2i-3)a}} \sum_{|\alpha|=m} |D^{\alpha} u|^p\,dx \le$$
$$\le C\Big[e^{-\frac{\eta}{q}[A+(2i-3)a]} + e^{-\frac{\eta_1}{q_1}[A+(2i-3)a]} + \int_{\Omega_i} \sum_{|\alpha|=m} |D^{\alpha} u|^p\,dx\Big].$$

It follows from the last inequality

$$\int_{\Omega_i} \sum_{|\alpha|=m} |D^{\alpha} u|^p\,dx \geqq \tag{7}$$
$$\geqq \frac{1}{C}\int_{G_{A+(2i-3)a}} \sum_{|\alpha|=m} |D^{\alpha} u|^p\,dx - \big(e^{-\frac{\eta}{q}[A+(2i-3)a]} + e^{-\frac{\eta_1}{q_1}[A+(2i-3)a]}\big).$$

By using (7) we can estimate

$$\int_{G_{A+(2i-1)a}} \sum_{|\alpha|=m} |D^{\alpha} u|^p\,dx = \int_{G_{A+(2i-3)a}} \sum_{|\alpha|=m} |D^{\alpha} u|^p dx - \int_{\Omega_i} \sum_{|\alpha|=m} |D^{\alpha} u|^p\,dx$$
$$\le (1-\frac{1}{C})\int_{G_{A+(2i-3)a}} \sum_{|\alpha|=m} |D^{\alpha} u|^p\,dx + e^{-\frac{\eta}{q}[A+(2i-3)a]} + e^{-\frac{\eta_1}{q_1}[A+(2i-3)a]}$$
$$\le e^{-\kappa_1}\int_{G_{A+(2i-3)a}} \sum_{|\alpha|=m} |D^{\alpha} u|^p\,dx + e^{-\frac{\eta}{q}[A+(2i-3)a]} + e^{-\frac{\eta_1}{q_1}[A+(2i-3)a]},$$

where $e^{-\kappa_1} = 1 - \frac{1}{C}$ and does not depend on i.

Let $\kappa_2 = min\{\frac{\eta}{q}, \frac{\eta_1}{q_1}, \kappa_1\}$.It follows from the last inequality that

$$\int_{G_{A+(2i-1)a}} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx \leq \tag{8}$$

$$\leq e^{-\kappa_2} \int_{G_{A+(2i-3)a}} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx + 2e^{-\kappa_2[A+(2i-3)a]}.$$

Let us denote $A - a = T$, thus (8) can be rewritten as follows

$$\int_{G_{T+2ia}} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx \leq \tag{9}$$

$$\leq e^{-\kappa_2} \int_{G_{T+(2i-2)a}} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx + 2e^{-\kappa_2[T+2(i-1)a]}.$$

By induction we can get from (9) the inequalities
(10)

$$\int_{G_{T+2ia}} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx \leq \begin{cases} e^{-i\kappa_2} \int_{G_T} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx + 2ie^{-\kappa_2[T+2(i-1)a]}, \\ \text{if } 2a < 1, \\ e^{-i\kappa_2} \int_{G_T} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx + 2ie^{-\kappa_2[T+i-1]}, \\ \text{if } 2a \geqq 1. \end{cases}$$

Let $2a < 1$. Then by using (10) we can get the estimate

$$\begin{aligned} &\int_{G_T} e^{\kappa x_n} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx \\ &\quad = \int_{\Omega_1} e^{\kappa x_n} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx + \int_{\Omega_2} e^{\kappa x_n} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx + \dots \\ &\quad + \int_{\Omega_i} e^{\kappa x_n} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx + \dots \\ &\quad \leq e^{\kappa(T+2a)} \int_{G_T} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx \\ &\quad + e^{\kappa(T+4a)} \left[e^{-\kappa_2} \int_{G_T} \sum_{|\alpha|=m} \left|D^\alpha u\right|^p dx + 2e^{-\kappa_2 T}\right] + \dots \end{aligned}$$

$$+ e^{\kappa(T+2ia)}\left[e^{-(i-1)\kappa_2}\int_{G_T}\sum_{|\alpha|=m}\left|D^\alpha u\right|^p dx + 2(i-1)e^{-\kappa_2[T+2(i-2)a]}\right] + \dots$$

$$= e^{\kappa(T+2a)}\int_{G_T}\sum_{|\alpha|=m}\left|D^\alpha u\right|^p dx \cdot \sum_{i=1}^{\infty} e^{(2a\kappa-\kappa_2)(i-1)}$$

$$+ 2e^{(\kappa-\kappa_2)T+4a\kappa_2}\sum_{i=2}^{\infty}(i-1)e^{2ai(\kappa-\kappa_2)} < \infty,$$

if $\kappa < \kappa_2$. By the same way we get the result for $2a \geqq 1$ and $\kappa < \frac{\kappa_2}{2a}$. □

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