Extremal and optimal solutions in the transshipment problem

Viktor Beneš

Abstract. The paper yields an investigation of the set of all finite measures on the product space with given difference of marginals. Extremal points of this set are characterized and constructed. Sets of uniqueness are studied in the relation to marginal problem. In the optimization problem the support of the optimal measure is described for a class of cost functions. In an example the optimal value is reached by an unbounded sequence of measures.

Keywords: transshipment problem, set of uniqueness, simplicial measure, optimal solution

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1. Introduction.

Let $(Y,\mathcal{B}_Y)$ be a Polish space with the Borel σ-algebra and $(X,\mathcal{B})=(Y\times Y,\mathcal{B}_Y\otimes\mathcal{B}_Y)$ the product space. Given two probability measures P_1 and P_2 on $\mathcal{B}_Y$, in the transshipment problem (Kemperman, 1983) we shall study the set $T(P_1,P_2)$ of finite nonnegative measures on $\mathcal{B}$ which have the difference of marginals equal to P_1-P_2:

$$T(P_1,P_2)=\{Q\in\mathcal{M}(X),\ Q^1-Q^2=P_1-P_2\},\tag{1}$$

where $Q^1(B)=Q(B\times Y)$, $Q^2(B)=Q(Y\times B)$, $B\in\mathcal{B}_Y$, $\mathcal{M}(X)$ denotes the space of finite nonnegative measures on $\mathcal{B}$.

If $Q_1,Q_2\in T(P_1,P_2)$ and $Q=\alpha Q_1+(1-\alpha)Q_2$, $0<\alpha<1$, then $Q^1-Q^2=\alpha(Q^1-Q^2)+(1-\alpha)(Q_2^1-Q_2^2)=P_1-P_2$, therefore $Q\in T(P_1,P_2)$, which yields that $T(P_1,P_2)$ is a convex set of measures.

Considering the product measure $P=P_1\times P_2$ on $\mathcal{B}$, we can alternatively define $T(P_1,P_2)$ by

$$T(P_1,P_2)=\{Q\in\mathcal{M}(X),\ Q\sim P\}\tag{2}$$

using the equivalence relation $\sim$ on $\mathcal{M}(X)$

$$P\sim Q\iff\int_X a\,dP=\int_X a\,dQ\ \text{ for all }\ a\in A,\tag{3}$$

$$A=\{a:X\to R,\ a(x,y)=f(x)-f(y),\ f\in C(Y)\},\tag{4}$$

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$C(Y)$ being the space of bounded continuous functions on Y.

Indeed, if $Q \in T(P_1, P_2)$ in (1) then $Q^1 - Q^2 = P_1 - P_2$ and $\int a\,dQ = \int f\,dQ^1 - \int f\,dQ^2 = \int f\,d(Q^1 - Q^2) = \int f\,d(P_1 - P_2) = \int f\,dP$ for all $a \in A$. Conversely $\int a\,dP = \int a\,dQ$, $a \in A$, means $\int f\,d(Q^1 - Q^2) = \int f\,d(P_1 - P_2)$ for all $f \in C(Y)$ and $Q^1 - Q^2 = P_1 - P_2$.

It is easy to see now that $T(P_1, P_2)$ is a closed set with respect to standard weak topology. If $Q_n \to Q$, $Q_n \in T(P_1, P_2)$, then $Q \sim Q_n$ for all n as $\{\int a\,dQ_n\}$ is a constant sequence converging to $\int a\,dQ$ for each $a \in A$.

In Štěpán (1979), the properties of sets of the type (2) called solutions of a moment problem were studied for general linear set of functions A containing all constant functions. Then equivalent measures have the same norm and the problem can be reduced to probability measures. An example is the marginal problem, see Beneš, Štěpán (1991), Linhartová (1991), in which

$$\begin{aligned} M(P_1, P_2) =& \{Q \in \mathcal{M}(X),\ Q^1 = P_1,\ Q^2 = P_2\} = \\ =& \{Q \in \mathcal{M}(X),\ Q \sim P\}, \end{aligned} \tag{5}$$

where the equivalence relation corresponds to (3) with $A = A_M$,

$$A_M = \{a : X \to R,\ a(x,y) = f(x) + g(y),\ f, g \in C(Y)\}. \tag{6}$$

The set A in (4) is linear, however, it does not contain constant functions with the exception of the function identically equal to zero. Therefore $T(P_1, P_2)$ is generally an unbounded set and our problem is over the scope of probability measures.

Nevertheless, in Section 2 the technique of Štěpán (1991) is used for the characterization of extremal points of $T(P_1, P_2)$ called T-simplicial measures. Another characterization is given in Section 3 in the special case of discrete finite space Y. An important notion is that of the set of uniqueness (Letac, 1966). $D \in \mathcal{B}$ is the set of uniqueness if (D^c being the complement of D in X)

$$Q_1(D^c) = Q_2(D^c) = 0,\ Q_1 \sim Q_2 \Longrightarrow Q_1 = Q_2 \text{ whenever } \ Q_1, Q_2 \in \mathcal{M}(X). \tag{7}$$

The set of uniqueness in the transshipment problem (A in (4)) and in the marginal problem (A_M in (6)) will be denoted here a TU-set and an MU-set, respectively. In Beneš and Štěpán (1987), MU-sets were studied and using the transfinite construction it was proved that each MU-set can be decomposed in a union of two graphs of functions $f, g : Y \to Y$. Even if problems with measurability appear and counterexamples were constructed (Losert, 1982) of M-simplicial measures (extremal points of $M(P_1, P_2)$) which have zero measure on any measurable graph, for practical purposes it is desirable to look for simplicial measures supported by graphs. Trivial relations $M(P_1, P_2) \subset T(P_1, P_2)$ and D is a TU-set $\Rightarrow$ D is an MU-set, lead to the same intentions in the transshipment problem.

In Section 4 two examples are given. First a generalized hairpin set (Kaminski et al., 1987) is studied which is not a TU-set and secondly a T-simplicial measure with given support is constructed using Theorem 2 from Section 2.

The last Section 5 is devoted to the optimization transshipment problem, to find $\inf \int c\,dQ$, where $Q \in T(P_1, P_2)$ and c is a given cost function. In Rachev, Shortt (1990), the optimal value was obtained for a class of cost functions using duality relations. Theorem 5 yields a necessary condition for the support of the optimal measure, i.e. the unknown measure for which the extreme is realized. Using this result we show in an example that the optimal value may be reached asymptotically by an unbounded sequence of measures from $T(P_1, P_2)$.

2. Characterization of extremal solutions.

When investigating extremal points of a convex weakly closed set H of measures, the general approach is first to verify the validity of Choquet type integral representation property from which it follows that H has enough extremal points. However, the theorems on integral representation property for sets of solutions of a moment problem (Winkler, Weizsacker, 1980; Bican, Štěpán, 1985) depend on the assumption of boundedness of H which is not valid for $T(P_1, P_2)$ in (1). Therefore we proceed to another way here: first a direct characterization of T-simplicial measures is given and an example how to use Theorem 2 to the construction of a T-simplicial measure is presented in Section 4. We start by proving the characterization of Douglas (1964) in the transshipment problem using a standard technique.

Theorem 1. *Let $Q \in \mathcal{M}(X)$. Q is a T-simplicial measure if and only if the set A in* (4) *is dense in $L_1(Q)$* (*the space of Q-integrable functions*).

PROOF: If A is not dense in $L_1(Q)$, we can find a function $f \in L_\infty(Q)$, $f \neq 0$, such that $\int af\,dQ = 0$ for all $a \in A$. Assuming that $0 < \operatorname{ess\,sup}|f| < 1$ we define a measure $S \in \mathcal{M}(X)$ by $dS = f\,dQ$. Then $Q_1 = Q + S$, $Q_2 = Q - S$ are different nonnegative measures equivalent to Q such that $Q = \frac{1}{2}(Q^1 + Q^2)$ hence Q is not T-simplicial. Conversely, if Q is not T-simplicial, there exists a measure $Q_1 \sim Q$, $Q_1 \neq Q$ and $Q_1 \leq 2Q$. Hence $0 \neq 1 - f \in L_\infty(Q)$, where $f = \frac{dQ_1}{dQ}$ and $\int (1-f)a\,dQ = 0$ for all $a \in A$, i.e. A is not dense in $L_1(Q)$. □

For the main characterization theorem we need the following notation (cf. Štěpán, 1991). Let $\mathcal{M}_0(X)$ be the space of bounded signed measures on X and for $B \in \mathcal{B}$ and $Q \in \mathcal{M}(X)$

$$
\begin{aligned}
\mathcal{N}_0(B) =& \{N \in \mathcal{M}_0(X);\ \ N^1 = N^2,\ |N|(B^c) = 0\},\\
\mathcal{N}(Q,B) =& \{N \in \mathcal{M}_0(X);\ \ |N|(B) \leq bQ \ \text{ for a constant } \ b \geq 0\},\\
\mathcal{K} =& \{K \subset X \ \text{ compact};\ \ N \,|\, K = 0 \ \text{ for any } \ N \in \mathcal{N}_0(X) \cap \mathcal{N}(Q, K^c)\},
\end{aligned}
\tag{8}
$$

where $|N| = |N^+| + |N^-|$ is the total variation of a signed measure N. Having finite measures N, Q on X and writing $\frac{dQ'}{dN}$ we mean by Q' the absolutely continuous part of Q with respect to the measure N.

Theorem 2. *The following statements are equivalent:*

(a) *Q is a T-simplicial measure,*

(b) $\sup\{Q(K),\ K \in \mathcal{K}\} = |Q|$,

(c) $\operatorname{ess\,inf} \frac{dQ'}{d|N|} = 0$ *for each* $N \in \mathcal{N}_0(X),\ 0 \neq N \ll Q$. $\quad$ (9)

PROOF: (a) $\Rightarrow$ (b): X is a Polish space, we may choose a uniformity of the space which makes the set $U(X)$ of bounded uniformly continuous functions on X separable. Assume that $\{g_i\}_{i=1}^{\infty}$ is a dense set in $U(X)$. If Q is T-simplicial, Theorem 1 yields a sequence $a_n^i \in A$ such that $\int |a_n^i - g_i|\, dQ \to 0$ as $n \to \infty$ for i integer. Then there is a sequence of integers $n_1 < n_2 < \dots$ for which $Q(\{x;\ a_{n_k}^i(x)$ does not converge to $g_i(x)$ for $k \to \infty\}) = 0$. For given $\varepsilon > 0$ by Jegoroff's theorem there are compact sets $K_i \subset X$ with $Q(K_i) = |Q| - \varepsilon 2^{-i}$ such that $a_n^i \to g_i$, $n \to \infty$, uniformly on K_i for all i. Hence $K = \bigcap_i K_i$ is a compact set with $Q(K) \geq |Q| - \varepsilon$ for which $a_n^i \to g_i$, $n \to \infty$, uniformly on K for each i. The constructed set K is an element of $\mathcal{K}$ as for $N \in \mathcal{N}_0(X) \cap \mathcal{N}(Q, K^c)$ and corresponding b in (8) it holds $|N(g_i)| = |N(g_i) - N(a_n^i)| \leq |N|(I_K|a_n^i - g_i|) + bQ(|a_n^i - g_i|)$, denoting $N(g) = \int g\, dN$ and I_K the indicator function of K. Hence $N(g_i) = 0$ for each i, which implies $N = 0$.

(b) $\Rightarrow$ (c): Suppose that (b) holds for a $Q \in \mathcal{M}(X)$ and that there are $N \in \mathcal{N}_0(X)$, $\delta > 0$ such that $\operatorname{ess\,inf} \frac{dQ'}{d|N|} \geq \delta$. As $|N| \mid K^c \leq \delta^{-1} Q' \leq \delta^{-1} Q$, we get $Q'(K) = 0$ for any $K \in \mathcal{K}$. Hence $Q' = 0$, a contradiction.

(c) $\Rightarrow$ (a): Assume that Q is not T-simplicial, $Q = \frac{Q_1+Q_2}{2}$, $Q_1 \neq Q_2$, $Q_1 \sim Q_2 \sim Q$. Put $N = Q_1 - Q_2$, $|N| \leq 4Q$, $N \in \mathcal{N}_0(X)$. From (c) it holds $\operatorname{ess\,inf} \frac{dQ'}{d|N|} = 0$. As Q' and N are equivalent measures, we may choose $h_N = \frac{dQ'}{d|N|}$ such that $Q' = Q \,|\, [h_N > 0]$. Then it follows $Q[0 < h_N < \frac{1}{5}] = Q'[0 < h_N < \frac{1}{5}] \leq \frac{1}{5}|N|[0 < h_N < \frac{1}{5}] < \frac{1}{4}|N|[0 < h_N < \frac{1}{5}]$, a contradiction. $\square$

If $P_1 = P_2$, it holds $T(P_1, P_2) \subset \mathcal{N}_0(X)$ and the condition (c) in (9) cannot be fulfilled with the exception that Q is the zero measure, the only extremal point of $T(P_1, P_2)$ in this trivial case.

A consequence of Theorem 2 is that the support of T-simplicial measures is built of TU-sets in the following sense:

Corollary 1. *If Q is T-simplicial, then*

$$\sup\{Q(K);\ K \ \textit{ compact a } TU\textit{-set}\} = |Q|, \tag{10}$$

PROOF: The set K constructed in the first part of the proof of Theorem 2 is a TU-set: If $Q_1(K^c) = Q_2(K^c) = 0$, $Q_1 \sim Q_2$, $Q_1, Q_2 \in \mathcal{M}(X)$ then $Q_1(g_i) = Q_2(g_i)$ for all i, therefore $Q_1(f) = Q_2(f)$ for $f \in U(X)$, which implies $Q_1 = Q_2$. $\square$

Further properties of TU-sets can be obtained from its relation to the marginal problem. Let $\mathcal{D} = \{(y, y), y \in Y\}$ be the diagonal set.

Proposition 1. *$D \subset X$ is a TU-set if and only if $D \cap \mathcal{D} = \emptyset$ and $D \cup \mathcal{D}$ is an MU-set.*

PROOF: Obviously the singleton $\{(y, y)\}$, $y \in Y$ is not a TU-set. The nontrivial implication is proved as follows: Let $D \cap \mathcal{D} = \emptyset$. If D is not a TU-set, there exist $P, Q \in \mathcal{M}(X)$, $P \neq Q$, $(P - Q)^1 = (P - Q)^2$. Define a measure ν on $\mathcal{D}$ by

$\nu^1 = (Q-P)^1$, $\nu = \nu_+ - \nu_-$ let be its Jordan decomposition. Then $\lambda_1 = \nu_+ + P$, $\lambda_2 = \nu_- + Q$ are nonnegative measures on $D \cup \mathcal{D}$ such that $(\lambda_1 - \lambda_2)^1 = (\lambda_1 - \lambda_2)^2 = 0$, i.e. $D \cup \mathcal{D}$ is not an MU-set. $\square$

The following necessary and sufficient condition will be of use in Section 4.

Proposition 2. *D is a TU-set if and only if $\mathcal{N}_0(D)$ is a singleton containing the zero measure.*

PROOF: It holds $0 \in \mathcal{N}_0(D)$. If $0 \neq Q \in \mathcal{N}_0(D)$, $Q = Q_+ - Q_-$, then $(Q_+ - Q_-)^1 = (Q_+ - Q_-)^2$, which implies $Q_+ \sim Q_-$ and D is not a TU-set. $\square$

3. The discrete case.

Consider that $Y = \{1, 2, \dots, n\}$ is a finite set, $X = Y \times Y$. For $D \subset X$, $\operatorname{card} D = k$, we construct the matrix H_D of the type $n \times k$ in the following way: to each point $(i,j) \in D$, $i \neq j$, there corresponds a column $(h_1, \dots, h_n)$ of H_D for which $h_i = -1$, $h_j = 1$, $h_l = 0$, $l \neq i$, $l \neq j$. Diagonal points (i,i) contribute a zero column to H_D. The range $\operatorname{ran} H_D$ is of interest only, therefore the columns may be arbitrarily ordered.

Theorem 3. *$D \subset X$ is a TU-set if and only if $\operatorname{ran} H_D \geq k$.*

PROOF: Let P, Q be finite nonnegative measures on D, $P = \{p_{ij}\}$, $Q = \{q_{ij}\}$, $1 \leq i \leq n$, $1 \leq j \leq n$, where $p_{ij} = P(\{(i,j)\})$, $q_{ij} = Q(\{(i,j)\})$. Denote $r_{ij} = p_{ij} - q_{ij}$. Then $P \sim Q$ if and only if $P^1 - P^2 = Q^1 - Q^2 \Leftrightarrow (P-Q)^1 = (P-Q)^2$, which is equivalent to

$$\sum_{i=1}^{n} r_{il} = \sum_{j=1}^{n} r_{lj}, \quad 1 \leq l \leq n.$$

This is a system of n equations for k unknown r_{ij} with indices corresponding to the points (i,j) of D. The matrix of this system is H_D and a nontrivial solution exists if and only if $\operatorname{ran} H_D < k$, in which case D is not a TU-set. $\square$

Corollary 2. *If D is a TU-set then $\operatorname{card} D \leq n$.*

It follows immediately from Theorem 3. As an example take $D = \{(i,j), (j,i)\}$. Then

$$H_D = \begin{array}{cc} \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & -1 \\ \cdot & \cdot \\ 0 & 0 \end{pmatrix} & \begin{array}{c} \\ i \\ \\ \\ j \\ \\ \\ \end{array} \end{array}$$

and $\operatorname{ran} H_D = 1$ which implies that D is not a TU-set.

The characterization given by Theorem 3 lacks a geometrical interpretation. Using Proposition 1 we obtain another characterization. Further let $Y = \{1, 2, \dots\}$ be a countable set, $X = Y \times Y$ endowed with discrete topology. A finite sequence $(x_i, y_i)_{i=1}^{2n}$ of points in X is called a cycle if it can be rearranged in such an order that $x_1 = x_2$, $y_2 = y_3$, $x_3 = x_4$, $\dots$, $y_{2n} = y_1$.

Proposition 3. *$D \subset X$ is a TU-set if and only if $D \cap \mathcal{D} = \emptyset$ and there is no cycle in $D \cup \mathcal{D}$.*

PROOF: Follows from Letac (1966) and Proposition 1. □

Concerning the characterization of T-simplicial measures we obtain a similar result as in the marginal problem (Letac, 1966), valid in the discrete case only.

Corollary 3. *Q is a T-simplicial measure if and only if its support* $\operatorname{supp} Q$ *is a TU-set.*

PROOF: If $\operatorname{supp} Q$ is a TU-set, then obviously Q is a simplicial measure. The converse implication follows from Corollary 1, there exists a nondecreasing sequence $\{K_n\}$ of compact TU-sets such that $Q(K) = |Q|$, $K = \bigcup_{n=1}^{\infty} K_n$. It holds $K \cap \mathcal{D} = \emptyset$ and if K contains a cycle C then for some n it holds $C \subset K_n$, a contradiction. By Proposition 3, K is a TU-set. □

4. Special results.

In this section, we consider $X = Y_1 \times Y_2$, $Y_1 = Y_2 = Y = \langle 0, 1 \rangle$ is the closed unit interval. At first we answer the question whether the set $D = \operatorname{graph}(f)$, where $f : Y \to Y$ is a measurable function and $\operatorname{graph}(f) = \{(y, f(y)), y \in Y\}$ its graph in X, is a TU-set. $Q \in \mathcal{M}(Y)$ is an f-invariant measure if $Q(f^{-1}(B)) = Q(B)$ for all $B \in \mathcal{B}_Y$.

Proposition 4. *$D = \operatorname{graph}(f)$ is a TU-set if and only if $D \cap \mathcal{D} = \emptyset$ and there does not exist an f-invariant measure on Y.*

PROOF: If there exists an f-invariant measure $Q \in \mathcal{M}(Y)$, then define a measure P on D by $P^1 \in \mathcal{N}_0(D)$ and D is not a TU-set. Conversely, let $D \cap \mathcal{D} = \emptyset$ and D be not a TU-set. Then $D \cup \mathcal{D}$ is not an MU-set and according to Beneš, Štěpán (1987), there exists a nontrivial $(g \circ f)$-invariant measure on Y, where $\operatorname{graph}(g) = \mathcal{D}$. This measure is f-invariant. □

Further a union of two graphs is studied,

$$D = \operatorname{graph}(f) \cup \operatorname{graph}(g), \tag{11}$$

where $f : Y_1 \to Y_2$, $g : Y_2 \to Y_1$ are measurable functions. When investigating if D is a TU-set, we always suppose that D does not contain the trivial points $(0, 0)$, $(1, 1)$, these points are released from the graphs. The work by Seethoff, Shiflett (1978) and Sherwood, Taylor (1988) yields special negative results here, because if there exists a doubly stochastic measure on D then according to Proposition 2, D is not a TU-set. D will be called a generalized hairpin set if

(a) f, g are increasing homeomorphisms onto Y, (12)
(b) $(f \circ g)(x) < x$, $(g \circ f)(x) < x$ for all $0 < x < 1$.

The condition (b) says that the graph(f) (and graph(g)) is always below and to the right (above and to the left) of the diagonal, see Figure 1.

According to Seethoff, Shiflett (1978), a generalized hairpin such that f, g are differentiable and there is a one-sided neighborhood of 0 in which $\frac{df}{dy} < 1$, $\frac{dg}{dy} < 1$, supports a signed measure with uniform marginals and again by Proposition 2 is not a TU-set.

We shall not proceed in characterizing TU-sets, the attention will be paid now to T-simplicial measures. From Proposition 2 and Theorem 2, the key role of the set $\mathcal{N}_0(D)$ in (8) becomes evident:

(a) if $\mathcal{N}_0(D)$ is a singleton, then D is a TU-set and each measure supported by D is T-simplicial;

(b) if $\mathcal{N}_0(D)$ is not a singleton, we must verify (9) for each $N \in \mathcal{N}_0(D)$ to decide whether the measure supported by D is T-simplicial or not.

Therefore a full description of $\mathcal{N}_0(D)$ is desirable. Theorem 4 yields conditions for $Q \in \mathcal{N}_0(D)$ on a generalized hairpin set D.

For a measure $N \in \mathcal{M}_0(Y)$ its mass-spreader h_N is a function defined on Y by

$$h_N(x) = N(\langle 0, x\rangle). \tag{13}$$

The mass-spreader of a probability measure coincides with the distribution function. Let D be a generalized hairpin and $Q \in \mathcal{M}_0(D)$. We denote $r_f(x) = Q \,|\, f(\langle 0, x\rangle \times Y)$ and $r_g(x) = Q \,|\, g(Y \times \langle 0, x\rangle)$ the mass-spreaders corresponding to the marginals of Q restricted to graph(f) and graph(g), respectively. Further $f^i(x)$ denotes the i-th composition $f \circ f \circ \ldots \circ f(x)$.

Theorem 4. *Let D be a generalized hairpin and $r_g(x)$ a mass-spreader on* graph(g). *Put*

$$r_f(x) = \sum_{i=1}^{\infty} \big(r_g(g^{-1} \circ f^i(x)) - r_g(f^i(x))\big). \tag{14}$$

Then r_f, r_g define a measure $Q \in \mathcal{N}_0(D)$ if and only if

$$m(x) = \sum_{i=-\infty}^{\infty} \big(r_g(g^{-1} \circ f^i(x)) - r_g(f^i(x))\big) \tag{15}$$

is constant for $0 < x < 1$.

PROOF: Let r_f, r_g define a measure $Q \in \mathcal{N}_0(D)$, then from the properties of a generalized hairpin it is $r_f(1) = \lim_{y \to 1_-} r_f(y) = \lim_{k \to -\infty} \sum_{i=k}^{\infty} (r_g(g^{-1} \circ f^i(x)) - r_g(f^i(x))) = m(x)$ constant for $0 < x < 1$. Conversely, if $m(x)$ is constant, then r_f, r_g define a measure $Q \in \mathcal{M}_0(D)$. According to (14) it is $r_f(x) - r_f(f(x)) = \sum_{i=1}^{\infty} (r_g(g^{-1} \circ f^i(x)) - r_g(f^i(x))) - \sum_{i=2}^{\infty} (r_g(g^{-1} \circ f^i(x)) - r_g(f^i(x))) = r_g(g^{-1}(f(x))) - r_g(f(x))$ which is a necessary and sufficient condition for $h_1(f(x)) = h_2(f(x))$, $x \in Y$, h_i being the mass-spreader of Q^i, $i = 1, 2$, see Figure 1. Therefore $Q^1 = Q^2$. $\square$

In Example 1 below, Theorem 4 is used for the description of the set $\mathcal{N}_0(D)$. This knowledge is used in Example 2 for a construction of a T-simplicial measure with a prescribed support.

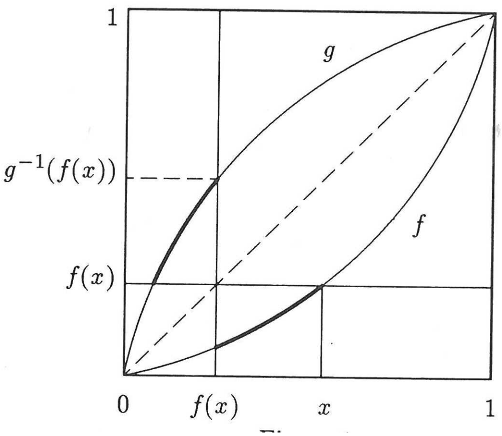

Figure 1

Example 1. Let for $0 < k < l < 1$, such that

$$\frac{\ln l}{\ln k} \text{ is an irrational number,} \tag{16}$$

$$\begin{aligned} f(x) &= kx,\ x \le \frac{1}{k+1}, \quad f(x) = \frac{x+k-1}{k},\ x > \frac{1}{k+1}, \\ g(y) &= ly,\ y \le \frac{1}{l+1}, \quad g(y) = \frac{y+l-1}{l},\ y > \frac{1}{l+1}, \end{aligned} \tag{17}$$

see Figure 2.

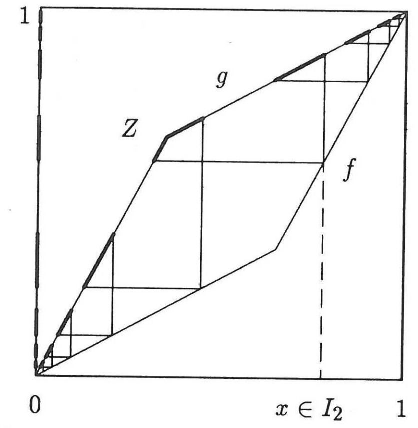

Figure 2

We look for r_g satisfying (15). Due to periodicity it is enough to study $x \in I_1 \cup I_2 \cup I_3$, where

$$I_1 = \langle \frac{1}{k+1}, \frac{l-k+1}{l+1} \rangle,$$
$$I_2 = \langle \frac{l-k+1}{l+1}, \frac{l-lk+1}{l+1} \rangle,$$
$$I_3 = \langle \frac{l-lk+1}{l+1}, \frac{1+k-k^2}{k+1} \rangle.$$

Transforming graph (g) linearly to a segment $S = \langle -\frac{1}{l+1}, \frac{1}{l+1} \rangle$ such that $0 \in S$ corresponds to the point Z in Figure 2, our problem is reformulated as follows: Does there exist a measure μ on S such that $\mu(S_x) = $ const., where

$$S_x = \bigcup_{i=0}^{\infty} \langle k^i(x+k-1) - \frac{1}{l+1}, \frac{k^i}{l}(x+k-1) - \frac{1}{l+1} \rangle \cup$$
$$\cup \bigcup_{i=0}^{\infty} \langle \frac{1}{l+1} - (1-x)\frac{k^i}{l}, \frac{1}{l+1} - (1-x)k^i \rangle \cup T_x,$$
$$T_x = \langle \frac{x+k-1}{k} - \frac{1}{l+1}, \frac{x+k-1}{lk} - \frac{1}{l+1} \rangle \quad \text{for} \quad x \in I_1,$$
$$T_x = \langle \frac{x+k-1}{k} - \frac{1}{l+1}, \frac{1}{l+1} - \frac{1-x}{k} \rangle \quad \text{for} \quad x \in I_2,$$
$$T_x = \langle \frac{1}{l+1} - \frac{1-x}{lk}, \frac{1}{l+1} - \frac{1-x}{k} \rangle \quad \text{for} \quad x \in I_3,$$

see Figure 3.

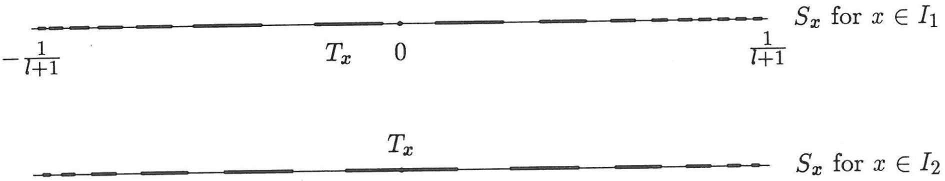

Figure 3

Notice that the Lebesgue measure λ on S is not a solution. It is $\lambda(S_x) = \frac{k(1-l)}{l(1-k)} + \lambda(T_x)$, but $\lambda(T_x) = (x+k-1)\frac{1-l}{kl}$, $x \in I_1$, $\lambda(T_x) = \frac{(1-l)(1-x)}{lk}$, $x \in I_3$, and $\lambda(S_x) = $ const. for $x \in I_2$ only, as $\lambda(T_x) = \frac{1-l}{1+l}$ for $x \in I_2$.

A lot of solutions yields the logarithmic transformation

$$L : \langle -\frac{1}{l+1}, \frac{1}{l+1} \rangle \to \langle -\infty, \infty \rangle,$$
$$L(x) = \ln(lx+x+1), \quad x \in \langle -\frac{1}{l+1}, 0 \rangle, \tag{18}$$
$$L(x) = -\ln(-lx-x+1), \quad x \in \langle 0, \frac{1}{l+1} \rangle,$$

which transforms subintervals of S_x to intervals of constant length $-\ln l$ (with the exception of T_x for $x \in I_2$) and constant period $-\ln k$, i.e. of type

$$J_{i,x} = \langle i \ln k + \ln(x+k-1) - \ln \frac{1}{l+1}\,,\ i \ln k + \ln(x+k-1) - \ln \frac{l}{l+1}\rangle\,. \tag{19}$$

Now choosing the Lebesgue measure on the interval

$$J_k = \langle 0, -\ln k\rangle \subset L(S), \tag{20}$$

we obtain after the inverse transformation to (18) finally the desired mass-spreader

$$\begin{aligned} r_g(x) &= 0,\ x < \frac{1}{l+1}, \quad r_g(x) = 1,\ x > 1 - \frac{lk}{l+1}, \\ r_g(x) &= \frac{\ln((1-x)(l+1))}{\ln k} - \frac{\ln l}{\ln k}\,,\ x \in \langle \frac{1}{l+1}, 1 - \frac{lk}{l+1}\rangle, \end{aligned}$$

and using (14),

$$\begin{aligned} r_f(x) &= 0,\ x < 1 - \frac{k}{l+1}, \quad r_f(x) = \frac{\ln l}{\ln k},\ x > 1 - \frac{lk}{l+1}, \\ r_f(x) &= \frac{\ln((1-x)(l+1))}{\ln k} - 1,\ x \in \langle 1 - \frac{k}{l+1}, 1 - \frac{lk}{l+1}\rangle. \end{aligned}$$

The corresponding measure Q is supported by $D' = (\mathrm{graph}(f) \cup \mathrm{graph}(g)) \cap X'$, where X' is the square $\langle \frac{l}{l+1}, 1 - \frac{lk}{l+1}\rangle^2$, see Figure 4.

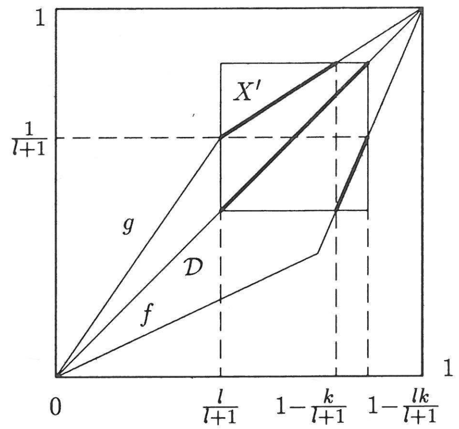

Figure 4

By Theorem 4, it is $Q^1 = Q^2$, the mass spreader h_i of Q^i being $h_i(x) = \frac{\ln((l+1)(1-x))}{\ln k}$, $\frac{l}{l+1} \le x \le 1 - \frac{lk}{l+1}$, $i = 1, 2$. Moreover, from the construction it follows that up to a scaling factor Q is the only signed measure on D' satisfying $Q^1 = Q^2$, i.e.

$$\mathcal{N}_0(D') = \{bQ,\, b \in R\}, \tag{21}$$

see (8). Indeed, under the condition (16), the Lebesgue measure chosen on J_k in (20) was the only measure invariant with respect to the sets $\bigcup_{i=-\infty}^{\infty} J_{i,x} \cap J_k$, $x \in I_1 \cup I_2 \cup I_3$.

Example 1 is related to the marginal problem. As $D = \mathrm{graph}(f) \cup \mathrm{graph}(g)$, see (17), is not a TU-set, according to Proposition 1 the union of three graphs $D \cup \mathcal{D}$ is not an MU-set. Moreover, $B' = D' \cup \mathcal{D}$ is not an MU-set. However, B' can be expressed as the union of two graphs only, see Figure 4. Our situation confirms to the conjecture, still not proved in the marginal problem: If B is not an MU-set, then there exists $B' \subset B$, B' not being an MU-set, B' being the union of two measurable graphs.

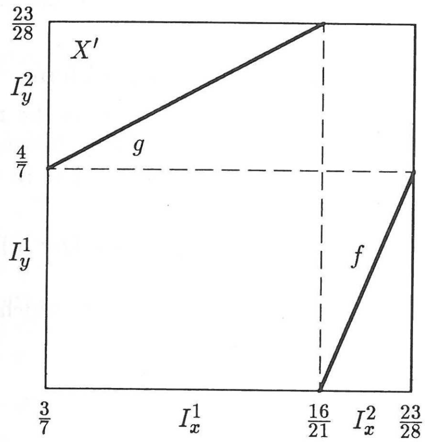

Figure 5

Example 2. In the situation of Example 1 put $l = \frac{3}{4}$, $k = \frac{5}{12}$. Then $f(x) = \frac{12x-7}{5}$, $g(y) = \frac{4y-1}{3}$, X' is the square in Figure 5, we denote $I_x^1 = \langle \frac{3}{7}, \frac{16}{21} \rangle$, $I_x^2 = (\frac{16}{21}, \frac{23}{28} \rangle$, $I_y^1 = \langle \frac{3}{7}, \frac{4}{7} \rangle$, $I_y^2 = (\frac{4}{7}, \frac{23}{28} \rangle$. We shall study measures supported by $D' = (\mathrm{graph}(f) \cup \mathrm{graph}(g)) \cap X'$. Consider the probability measure P uniformly distributed on D'. It is described by the density $\frac{dP^1}{d\lambda}$ of its marginal P^1 with respect to the Lebesgue measure λ : $\frac{dP^1}{d\lambda}(x) = \frac{35}{16}$ for $x \in I_x^1$, $\frac{dP^1}{d\lambda}(x) = \frac{91}{20}$ for $x \in I_x^2$. Let $\kappa = P^1 - P^2$ be the difference of marginals of P, then $\frac{d\kappa}{d\lambda}(x) = \frac{7}{24}$, $x \in I_y^1$, $\frac{d\kappa}{d\lambda}(x) = -\frac{35}{48}$, $x \in I_x^1 \cap I_y^2$, $\frac{d\kappa}{d\lambda}(x) = \frac{49}{30}$, $x \in I_x^2$. Having described the set $\mathcal{N}_0(D')$ in (21), we can construct a T-simplicial measure (extremal point of $T(P^1, P^2)$) supported by D'. According to the Theorem 2 such measure Q is given by the condition $\operatorname{ess\,inf} \frac{dQ'}{d|N|} = 0$ for all $N \in \mathcal{N}_0(X)$, $0 \neq N \ll Q$, i.e. $N \in \mathcal{N}_0(D')$. It follows from Example 1 that $\frac{dN^1}{d\lambda}(x) = \frac{c}{1-x}$ for any $N \in \mathcal{N}_0(D')$ and some constant c. Especially $\operatorname{ess\,inf} \frac{d|N^1|}{d\lambda} > 0$ on $I_x^1 \cup I_x^2$. Therefore a sufficient condition for T-simpliciality of Q

is

$$\operatorname{ess\,inf} \frac{dQ^1}{d\lambda} = 0. \tag{22}$$

We construct Q by subtracting suitable $N \in \mathcal{N}_0(D')$ from P to obtain a nonnegative measure satisfying (22). As $\frac{d|N^1|}{d\lambda}$ is increasing, it may hold $\frac{dQ^1}{d\lambda}(x) = 0$ in the right endpoint of either I_x^1 or I_x^2 only. The result is

$$\begin{aligned} \frac{dQ^1}{d\lambda}(x) &= \frac{5}{16}\Big(7 - \frac{5}{3(1-x)}\Big), \quad x \in I_x^1, \\ &= \frac{1}{4}\Big(\frac{91}{5} - \frac{25}{12(1-x)}\Big), \quad x \in I_x^2. \end{aligned}$$

This marginal density defines a T-simplicial measure Q supported by D' having the same difference of marginals as P.

5. Optimal solutions in the transshipment problem.

Throughout this section, we assume that $X = R^2$ is the real plane and $c(x,y)$ is a measurable cost function on X. Given probability measures P_1, P_2 on $R = Y$, the Kantorovich and Rubinstein (1958) problem consists in the evaluation of the extreme

$$e(P_1,P_2) = \inf_{Q \in T(P_1,P_2)} \int c(x,y)\, dQ(x,y). \tag{23}$$

Under suitable assumptions, the duality theorem holds which says that

$$e(P_1,P_2) = \sup \int f\, d(P_1 - P_2),$$

where the supremum is taken over the set $\{f : Y \to R;\ f(x) - f(y) \le c(x,y)$ for all $x, y \in R\}$. Remember that in the case when $c(x,y)$ is a metric on X, it holds

$$e(P_1,P_2) = \inf_{Q \in M(P_1,P_2)} \int c(x,y)\, dQ(x,y),$$

i.e. the problem coincides with the optimization marginal problem.

The question is when we can write min instead of inf in (23) and if so, what is the optimal measure Q' for which

$$\int c\, d(Q') = \min_{Q \in T(P_1,P_2)} \int c\, dQ. \tag{24}$$

We shall call c a quasi-monotone function if for all $x \le x'$, $y \le y'$,

$$\mu_c\{(x,x'\rangle \times (y,y'\rangle\} = c(x,y) + c(x',y') - c(x,y') - c(x',y) \ge 0; \tag{25}$$

c is quasi-antitone if $-c$ is quasi-monotone.

Theorem 5. *Let c be a quasi-antitone right continuous function, $c(x,y)=c(y,x)$ for all x,y and there exists an optimal measure Q in (23), the extreme being finite. Then the support of Q is a graph of an increasing function $f: Y \to Y$ satisfying the functional equation*

$$h_Q^1(x) - h_Q^1(f^{-1}(x)) = F_1(x) - F_2(x), \tag{26}$$

where $h_Q^1(x)$ is the mass-spreader of Q^1 and F_1, F_2 given distribution functions of P_1, P_2, respectively.

Proof: From (25), $-c$ may be viewed as a "distribution function" corresponding to a nonnegative measure μ_c on X. It is $T(P_1,P_2) = \{\alpha P,\ P$ probability measure; $\alpha(P^1 - P^2) = P_1 - P_2$ for $\alpha \geq 0\}$. For $Q \in T(P_1,P_2)$, $Q = \alpha P$, let U, V be random variables with joint distribution P and distribution function $F(x,y)$. Then it holds

$$\begin{aligned} 2\int c\,dQ = -\alpha[Ec(U,U) + Ec(V,V) - \qquad\qquad\qquad\qquad\qquad\qquad & (27)\\ - \int_X (H(x\wedge y) + G(x\wedge y) - F(x\vee y, x\wedge y) - F(x\wedge y, x\vee y))\,d\mu_c(x,y)], & \end{aligned}$$

see Cambanis et al. (1976), where $x \vee y = \max\{x,y\}$, $x \wedge y = \min\{x,y\}$ and H, G are marginal distribution functions of U, V, respectively. Let Q be an optimal measure. As for given marginals of Q (27) is a monotone functional of the joint distribution function F, P must correspond to the upper Fréchet bound with $F(x,y) = \min\{H(x), G(y)\}$. The support of this measure is a graph of an increasing function $f: Y \to Y$. Therefore the mass-spreader h_Q^2 of Q^2 is equal to $h_Q^2(x) = h_Q^1(f^{-1}(x))$, putting this into (1) we obtain the condition (26). □

The condition (26) is not sufficient. In fact, any $Q \in T(P_1,P_2)$ supported by a graph of an increasing function f satisfies it.

The explicit solution of the problem (23) was obtained by Rachev and Shortt (1990) for cost functions of type

$$c_p(x,y) = |x-y| \max\{1, |x-b|^{p-1},\ |y-b|^{p-1}\}, \tag{28}$$

$p \geq 1$, $x, y, b \in R$, and it was evaluated as

$$e(P_1,P_2) = \int_Y \max(1, |x-b|^{p-1}) |F_1(x) - F_2(x)|\,dx. \tag{29}$$

Notice that even this result does not guarantee the existence of an optimal measure.

Example 3. Let $Y = \langle 0,1\rangle$, $X = Y \times Y$ the unit square, $F_1(x) = x$, $x \in Y$, $F_2(x) = ax$, $x \in \langle 0,c\rangle$, $F_2(x) = \frac{1-ac}{1-c}x + \frac{c(a-1)}{1-c}$, $x \in \langle c,1\rangle$, $c \in Y$, $a < 1$. Consider the function c_p in (28) with $p=2$, $b=2$, then

$$c_p(x,y) = |x-y|(2 - \min(x,y)), \quad (x,y) \in X, \tag{30}$$

c_p is not a metric but it is quasi-antitone. From (29), it holds

$$e(P_1, P_2) = (1-a)\big(c^2 - \frac{c^3}{3} + \frac{c}{1-c}(\frac{5}{6} - 2c + \frac{3c^2}{2} - \frac{c^3}{3})\big). \tag{31}$$

By differentiation of (26) we obtain a necessary condition for an optimal measure

$$g(x) - \frac{g(f^{-1}(x))}{f'(f^{-1}(x))} = \kappa(x), \tag{32}$$

where $f' = \frac{df}{dx}$, g is the density of h^1_Q and $\kappa(x) = 1-a$, $x \in \langle 0, c\rangle$, $\kappa(x) = \frac{c(a-1)}{1-c}$, $x \in (c, 1\rangle$ is given. Intuitively, we look for the support of the optimal measure Q in a parametric form $f(x) = bx$, $x \in \langle 0, \frac{c}{b}\rangle$, $f(x) = \frac{b(1-c)}{b-c}x + \frac{c(b-1)}{b-c}$, $x \in (\frac{c}{b}, 1\rangle$, $b > 1$, see Figure 6, with the marginal density $g(x) = d$, $x \in Y$.

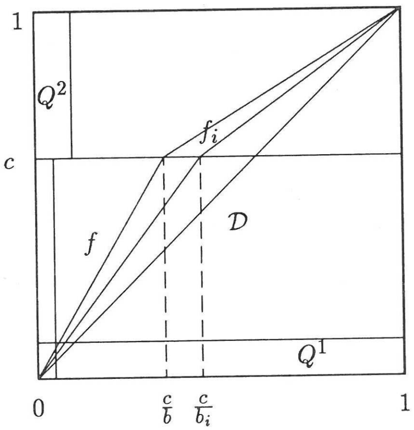

Figure 6

From (32) we obtain two conditions for the unknown parameters b, d:

$$d - \frac{d}{b} = 1-a, \quad d - \frac{d(b-c)}{b(1-c)} = \frac{c(a-1)}{1-c} \tag{33}$$

which are linearly dependent. Putting the first condition $d = \frac{b(1-a)}{b-1}$ into the evaluation of $\int c_p\, dQ$ we obtain

$$\begin{aligned}\int c_p(x,y)\, dQ(x,y) &= \int |x - f(x)|(2 - \min(x, f(x)))g(x)\, dx = \\ &= b(1-a)\Big[\frac{c^2}{b^2} - \frac{c^3}{3b^3} + \frac{c}{b-c}(\frac{5}{6} - \frac{2c}{b} + \frac{3c^2}{2b^2} - \frac{c^3}{3b^3})\Big].\end{aligned} \tag{34}$$

For fixed a, c, (34) is a decreasing function of b for $b \to 1_+$ ($b \le 1$ are excluded due to (33)) which tends to the optimal value (31). Taking a sequence $b_n \to 1_+$ the corresponding $d \to \infty$ (33) form an unbounded sequence of measures Q_n such that $\lim_{n\to\infty} \int c_p\, dQ_n = e(P_1, P_2)$. The supports of Q_n converge to the diagonal.

The presented results may be generalized in the following way.

Theorem 6. *Let the support of $P_1 - P_2$ be a compact interval $I \subset R$ and for the corresponding distribution functions F_1, F_2, $\kappa = F_1 - F_2$, there is a $K \in R$ such that for all $x \in I$, $|\kappa'(x)| < K$, $\kappa' = \frac{d\kappa}{dx}$. Let $c(x,y)$ be a function of the type $c(x,y) = |x-y|\xi(x,y)$, where for any $x < t < y$, $\xi(t,t) \leq \xi(x,y)$, $\xi(x,y) = \xi(y,x)$, $\xi(x,y)$ is continuous in x and $t \to \xi(t,t)$ is locally bounded. Then*

$$e(P_1, P_2) = \int \xi(t,t)|\kappa(t)|\, dt \tag{35}$$

and there exists an unbounded sequence $Q_n \in T(P_1, P_2)$ such that $e(P_1, P_2) = \lim_{n\to\infty} \int c\, dQ_n$.

Proof: For the first part, i.e. the validity of (35), see Rachev, Ruschendorf (1991). Further for an integer $n > K$ put $f^{-1}(x) = x - \frac{\kappa(x)}{n}$ and $h_n^1(x) = nx + b$, $x \in I$, $h_n^1(x) = 0$ elsewhere, where b is a constant such that h_n^1 is nonnegative. Then f is increasing and (26) is satisfied, it follows that $Q_n \in T(P_1, P_2)$ for the measure Q_n supported by f and defined by h_n^1 being the mass-spreader of its first marginal. It is $c(f^{-1}(y), y) = \frac{1}{n}|\kappa(y)|\xi(y - \frac{\kappa(y)}{n}, y)$ and

$$\int c\, dQ_n = \int c(x, f(x)) dh_n^1(x) = n \int c(f^{-1}(y), y)(1 - \frac{\kappa'(y)}{n})\, dy =$$
$$= \int_I |\kappa(y)|\xi(y - \frac{\kappa(y)}{n}, y)\, dy - \frac{1}{n}\int_I \kappa'(y)|\kappa(y)|\xi(y - \frac{\kappa(y)}{n}, y)\, dy.$$

Obviously, for $n \to \infty$, $\int c\, dQ_n \to e(P_1, P_2)$ in (35) while $|Q_n| \to \infty$. □

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Department of Mathematics, Faculty of Mechanical Engineering, Czech Technical University, Karlovo nám. 13, 121 35 Prague, Czechoslovakia

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