QTAG torsionfree modules

Ladislav Bican, Blas Torrecillas

Abstract. The structure theory of abelian p-groups does not depend on the properties of the ring of integers, in general. The substantial portion of this theory is based on the fact that a finitely generated p-group is a direct sum of cyclics. Given a hereditary torsion theory on the category R-**Mod** of unitary left R-modules we can investigate torsionfree modules having the corresponding property for all torsionfree factor-modules (and a natural requirement concerning extensions of some homomorphisms). This paper continues in our previous investigations of the structural properties of such modules.

Keywords: torsion theory, torsionfree module, σ-QTAG-module, kernel of purity, center of purity

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Introduction.

In the study of the structure of abelian p-groups the decomposability theorems play a very important role. This theory has been extended to modules over different types of rings (not necessarily commutative), see for example [S_1], [S_2], [B_1] and [B_2]. Singh observed in [S_1] that a large number of results on decomposability does not depend upon the nature of the ring involved, but of the properties of the module. Following this idea, many authors have been interested in the class of modules (TAG-modules) where results on decomposibility can be obtained.

In [T_1], [T_2], [T_3], [BT_1] and [BT_2] similar properties are assumed for a torsionfree module and some results about decomposability are extended to this class of modules. Recently Singh [S_3] weakened the condition on the modules in the class (he called them QTAG-modules) and he also showed that many results can be extended to these modules (cf. [S_3]). The aim of this paper is to study the class of torsionfree modules with similar properties, then we extend our preceding results to this new class. We also investigated the kernels and centers of the modules in this class. Both concepts are established using the h-purity or the more general α-purity and they are modelled after the usual notions in primary abelian groups.

We will establish our results in the general setting of an arbitrary hereditary torsion theory σ. The paper is organized as follows. After fixing notation in first section, we will establish the properties on σ-QTAG-modules that will be used throughout the paper. In Section 2 we extend the results about decomposability of [BT_1]. The third section is dedicated to the study of the h-kernels of purity. We obtain some properties of these submodules of a σ-QTAG-module. In the last section we study the centers of α-purity, α any ordinal, and we give some characterizations of such submodules.

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1. General properties.

All the rings will be unitary and the modules will be left unitary modules. We denote by R-**Mod** the category of left R-modules. Let σ be a hereditary torsion theory on R-**Mod** and $\mathcal{L}_\sigma$ the Gabriel filter associated to σ. $\mathcal{T}_\sigma$ (resp. $\mathcal{F}_\sigma$) will denote the torsion (resp. torsionfree) class. See [AN], [BN], [G] and [S] for basic results on torsion theories.

Let M be an R-module and N be a submodule of M. The submodule

$$\mathrm{Cl}_\sigma^M(N) = \{x \in M \mid (N : x) \in \mathcal{L}_\sigma\}$$

of M is called the σ-closure of N in M. It is clear that $t_\sigma(M/N) = \mathrm{Cl}_\sigma^M(N)/N$, where t_σ is the radical functor associated to σ. Since we use always the same torsion theory σ, the subscript σ will be usually omitted. We say that N is σ-closed (resp. σ-dense) in M if $\mathrm{Cl}^M(N) = N$ (resp. M).

A left R-module M is called σ-finitely generated (resp. σ-cyclic) if there exists a finitely generated (resp. cyclic) submodule of M that is σ-dense in M.

A nonzero left R-module M is said to be σ-cocritical if and only if M is σ-torsionfree and for any nonzero submodule N of M, M/N is σ-torsion. Let M be a σ-torsionfree left module. We say that M has a σ-composition series if there exists a chain of submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

having the property that M_{i+1}/M_i is a σ-cocritical module for $0 \le i \le n-1$. If such σ-composition series exists, we shall say that M has finite σ-length and write $\ell(M) = n$.

We recall the following definitions from [T_1]. A σ-torsionfree module M is called σ-uniserial if it has exactly one σ-composition series. An element $x \in M$ is called σ-uniserial (resp. uniform) if the cyclic module Rx is σ-uniserial (resp. uniform). A left R- module M is called a σ-strongly uniserial module if it is σ-uniserial and for any Rx, $Ry \subseteq M$ such that $\mathrm{Cl}_\sigma^M(Rx) \subseteq \mathrm{Cl}_\sigma^M(Ry)$, Rx is isomorphic to a submodule of Ry.

We say that a σ-torsionfree module M is a σ-QTAG-module if it satisfies the following two conditions:

(I) Every σ-finitely generated submodule of a σ-torsionfree homomorphic image of M is a direct sum of σ-uniserial submodules;

($\mathrm{I_e}$) For every σ-uniserial submodule A of a σ-torsionfree homomorphic image N of M every homomorphism $f : A \to K$ into a σ-closed σ-uniserial submodule K of N can be extended to $g : \mathrm{Cl}^N(A) \to K$.

For the undefined notions we refer to [BT_1].

Now we extend Lemma 2.14 of [BT_1] to σ-QTAG-modules.

Lemma 1.1. *Let U be a σ-closed submodule of a σ-QTAG-module M. If $\varphi : U \to M$ is a monomorphism, then $\varphi(U)$ is σ-closed in M.*

Proof: Denote $V = \mathrm{Cl}^M(\varphi(U))$. Proving indirectly, let $v \in V \setminus \varphi(U)$ be a uniform element. Then $Rv \cap \varphi(U)$ is σ-dense in Rv and denoting $Y = \mathrm{Cl}^M(\varphi^{-1}(Rv \cap \varphi(U)))$ the condition ($\mathrm{I_e}$) gives the following commutative diagram

$$\begin{array}{ccc} Rv \cap \varphi(U) & \longrightarrow & Rv \\ \psi\Big\downarrow & & \Big\downarrow \\ Y & = & Y \end{array}$$

where ψ is φ^{-1} composed with the corresponding embedding.

The rest of the proof is the same as [BT$_1$, Lemma 2.14]. □

We will denote by $\mathrm{J}_\sigma(M)$ the intersection of all submodules K of M such that M/K is a σ-cocritical module.

Proposition 1.2. *Let $A_1, \dots, A_n$, $U_1, \dots, U_m$ be σ-uniserial modules such that $\Sigma_{i=1}^n A_i = \oplus_{j=1}^m U_j$. Then $m \le n$.*

Proof: Let A be the external direct sum of A_i's, $U = \Sigma_{i=1}^n A_i = \oplus_{j=1}^m U_j$ and $f : A \to U$ be the natural epimorphism. It is easy to see that $U/\mathrm{J}_\sigma(U)$ is a direct sum of m σ-cocritical modules and it is an epimorphic image of $A/\mathrm{J}_\sigma(A)$ which is sum of n σ-cocritical modules. By [BT$_1$, Remark 1.1] we get the inequality $m \le n$. □

Proposition 1.3. *Let A, B be σ-uniserial submodules of a σ-QTAG-module M such that $\ell(A) \le \ell(B)$ and B is a σ-closed in M. Then B is a direct summand of $A + B$.*

Proof: Without loss of generality we can assume that $A \cap B \neq 0$. If $A + B$ is σ-uniserial, then $A \subseteq B$ by [BT$_1$, Proposition 2.6]. Therefore we can with respect to the condition (I) and Proposition 1.2 suppose that $A + B = Y \oplus C$ with Y, C σ-uniserial and $\ell(C) \le \ell(Y)$.

Composing canonical embeddings of A and B with the canonical projections onto Y and C, we get the following four homomorphism:

$$\varphi_{AY} : A \to Y, \;\; \varphi_{AC} : A \to C, \;\; \varphi_{BY} : B \to Y \text{ and } \;\; \varphi_{BC} : B \to C.$$

Claim: $\varphi = \varphi_{BY}$ is an isomorphism.

Suppose that φ_{BY} is not monic. Since Ker $\varphi_{BY} \cap$ Ker $\varphi_{BC} = 0$ and B is σ-uniserial, we necessarily have Ker $\varphi_{BC} = 0$. Hence $\ell(B) \le \ell(C)$, which yields a contradiction. Thus φ_{BY} is monic.

On the other hand, since Y is σ-uniserial and $Y = \varphi_{AY}(A) + \varphi_{BY}(B)$, [BT$_1$, Proposition 2.6] implies that $Y = \mathrm{Cl}^Y(\varphi_{BY}(B))$ and Lemma 1.1 yields the claim.

Now for $b \in B \cap C$ it is $\varphi(b) = 0$, hence $b = 0$. It remains to show that $A + B = B + C$. For $a \in A$ we have $a = y + c, y \in Y, c \in C$ and $y = \varphi(b)$ for some $b = y + c' \in Y \oplus$ C. Thus $a = y + c = b + c - c' \in B \oplus C$ and we are through. □

Proposition 1.4. *Let A, B be any two σ-uniserial submodules of a σ-QTAG-module M such that $A \cap B \neq 0, B$ is σ-closed in M and $\ell(A) \le \ell(B)$. Then there exists a monomorphism $\varphi : A \to B$ extending the identity on $A \cap B$.*

Proof: By Proposition 1.3 we have $A + B = B \oplus C$ where, obviously, $\ell(A) > \ell(C)$. Therefore, the mapping $A \hookrightarrow B \oplus C \to C$ cannot be monic and consequently $A \hookrightarrow B \oplus C \to B$ must be monic, A being σ-uniserial. The rest is clear. □

Proposition 1.5. *Let A, B be two σ-uniserial submodules of a σ-QTAG-module M such that $A \cap B = 0$ and B is σ-closed in M. If $\varphi : W \to B$ is a homomorphism of a submodule W of A into B such that $\ell(A/\operatorname{Cl}^A(W)) \le \ell(B/\operatorname{Cl}^B(\varphi(W)))$, then φ can be extended to a homomorphism $\psi : A \to B$.*

Proof: Consider the pushout diagram

$$\begin{array}{ccc} W & \xrightarrow{\iota} & A \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ B & \xrightarrow{j} & K \end{array}$$

where $K = (A \oplus B)/L$, $L = \{(w, -\varphi(w)) \mid w \in W\}$, $i(a) = (a, 0) + L$, $j(b) = (0, b) + L$.

The homomorphism j is obviously monic, hence $j(B)$ is σ-closed by Lemma 1.1. Since $\operatorname{Ker} i = \operatorname{Ker} \varphi$, by the hypothesis we have that $\ell(i(A)) \le \ell(j(B))$. By Proposition 1.3, $j(B)$ is a direct summand of K and we denote by p the corresponding canonical projection of K into $j(B)$. Now $\psi = j^{-1}pi : A \to B$ is a homomorphism and for $w \in W$ we have $\psi(w) = j^{-1}pi(w) = j^{-1}p((w, 0) + L) = j^{-1}p((0, \varphi(w)) + L) = j^{-1}pj\varphi(w) = \varphi(w)$ and the proof is complete. □

A uniform element x in M is said to be of σ-exponent n (denoted $\mathrm{e}(x) = n$) if $\ell(Rx) = n$. Let $x \neq 0$ be uniform in M; the supremum of the $\ell(U/\operatorname{Cl}^U(Rx))$, where U is a σ-uniserial module containing x, is called the σ-height of x in M. It will be denoted by $\mathrm{H}^M(x)$ (or simply $\mathrm{H}(x)$ if there is no confusion). We put $\mathrm{H}(0) = \infty$ and $\mathrm{H}(x) \le \mathrm{H}(0)$ for any nonzero uniform element x from M.

Lemma 1.6. *Let A, B be σ-uniserial submodules of a σ-QTAG-module M with $A \cap B = 0$, W be a σ-closed submodule of B and $\varphi : A \to B/W$ be any homomorphism with $\ell(W) \le \ell(\operatorname{Ker} \varphi)$. Then there is $\psi : A \to B$ lifting φ.*

Proof: The direct sum $A \oplus B$ as a submodule of M is a σ-QTAG-module and we can work with $M = A \oplus B$. Consider the pullback diagram

$$\begin{array}{ccc} K & \xrightarrow{q} & B \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\varphi} & B/W \end{array}$$

with $K = \{(a, b) \in A \oplus B \mid \varphi(a) = \pi(b)\}$, p, q projections. It is easy to see that p is onto and $q(K) = \pi^{-1}(\varphi(A))$.

Now

$$\ell(\varphi(A)) = \ell(A) - \ell(\operatorname{Ker} \varphi) \le \ell(A) - \ell(W)$$

gives

$$\ell(q(K)) = \ell(\pi^{-1}(\varphi(A))) = \ell(\varphi(A)) + \ell(W) \le \ell(A).$$

It follows from Proposition 1.3 that $K = A' \oplus C$ where $\ell(C) \le \ell(A')$ (the case $C = 0$ is also possible) and A', C are σ-uniserial.

Consider the map $\rho : B \to (A \oplus B)/K$ given by $\rho(b) = (0, b) + K$. If $(a, b) \in A \oplus B$ is arbitrary, then $\varphi(a) = \pi(b')$ for some $b' \in B$. Hence $(a, b') \in K$. Now $(a, b) + K = (a, b) - (a, b') + K = \rho(b - b')$ and ρ is surjective. However, Ker $\rho = W$, hence $B/W \cong (A \oplus B)/K$ and K is σ-closed in $A \oplus B$.

Since A' is σ-uniserial then either $p|_{A'}$ is monic and then $\ell(A') \le \ell(A)$ or $q|_{A'}$ is monic and also $\ell(A') = \ell(q(A')) \le \ell(q(K)) \le \ell(A)$, in any case

$$\ell(C) \le \ell(A') \le \ell(A). \tag{1}$$

If $p|_{A'}$ is monic then $p(A')$ is σ-closed in A by Lemma 1.1. By (1) we have $\mathrm{Cl}^A(p(C)) \subseteq p(A') = \mathrm{Cl}^A(p(A'))$, hence $p(A') = A$ since $p(K) = p(A') + p(C) = A$. Therefore $p|_{A'}$ is an isomorphism.

If $p|_{A'}$ is not monic then $\mathrm{Cl}^A(p(A')) \ne A$, for otherwise

$$\ell(A) = \ell(p(A')) < \ell(A')$$

contradicts (1). Therefore $\mathrm{Cl}^A(p(C)) = A$ and consequently

$$\ell(A) = \ell(p(C)) = \ell(C) - \ell(\mathrm{Ker}(p|_C)) \le \ell(C) \le \ell(A)$$

implies that $p|_C$ is monic. By Lemma 1.1 $p(C)$ is σ-closed in A. Hence $p(C) = A$ and $p|_C$ is an isomorphism.

Thus, without loss of generality we can assume that $p : A' \to A$ is an isomorphism. Denoting $\eta = ip^{-1} : A \to A' \to K$ we set $\psi = q\eta : A \to B$ and for $a \in A$ we have $\pi\psi(a) = \pi q i p^{-1}(a) = \pi q(a, b) = \pi(b) = \varphi(a)$ by the definition of K. □

Lemma 1.7. *Let M be a σ-QTAG- module. If $U_1 \oplus \ldots \oplus U_m \subseteq M$ is a direct sum of σ-closed σ-uniserial submodules of M and $0 \ne u_i \in U_i$ are such that $\mathrm{H}^{U_i}(u_i) = k$ and $x = u_1 + \ldots + u_m$ is σ-uniserial, then $\mathrm{H}^M(x) \ge k$.*

Proof: For each $j \in \{1, \ldots, m\}$ the composed map

$$Rx \subseteq Ru_1 \oplus \ldots \oplus Ru_m \to Ru_j$$

is the natural epimorphism $\varphi_j : Rx \to Ru_j$. Since Rx is σ-uniserial, $\bigcap_{j=1}^m \mathrm{Ker}\,\varphi_j = \mathrm{Ker}\,\varphi_t$ for some $t \in \{1, \ldots, m\}$ and so we have $Rx \cong Ru_t$ and $f_j : Ru_t \to Ru_j \to 0$ naturally.

Take $v_t \in U_t$ with $\mathrm{Cl}^{U_t}(Rv_t) = U_t$. Then $Ru_t \cap Rv_t$ is σ-dense in Ru_t and we can choose an element $rv_t = su_t$ with $\mathrm{Cl}^{Ru_t}(Rsu_t) = Ru_t$. For each $j \ne t$, Proposition 1.5 gives the commutative diagram

$$\begin{tikzcd} Ru_t \arrow[r] \arrow[d, "f_j"'] & U_t \arrow[d, "h_j"] \\ Ru_j \arrow[r] & U_j \end{tikzcd}$$

Set $h_t = 1_{U_t}$, $h_j(v_t) = w_j$ and $v = w_1 + \ldots + w_m$. Now it is easy to see that Rv_t and Rv are naturally isomorphic, $V = \mathrm{Cl}^M(Rv)$ is σ-uniserial and σ-closed and therefore the condition ($\mathrm{I_e}$) and Lemma 1.1 give the existence of the isomorphism ψ making the diagram

$$\begin{array}{ccc} Rv & \longrightarrow & V \\ \bar{\psi}\downarrow & & \downarrow\psi \\ Rv_t & \longrightarrow & V_t \end{array}$$

commutative.

For $\alpha \in (Rsu_t : u_t) \in \mathcal{L}$ we have $\alpha u_t = r_\alpha s u_t$ for some $r_\alpha \in R$, and $\alpha x = \Sigma\alpha f_j(u_t) = \Sigma f_j(r_\alpha s u_t) = r_\alpha s x$ showing that $\mathrm{Cl}^{Rx}(Rsx) = Rx$. Moreover, $sx = s\Sigma u_j = \Sigma h_j(su_t) = \Sigma h_j(rv_t) = rv$ and so $Rx = \mathrm{Cl}^{Rx}(Rsx) = \mathrm{Cl}^{Rx}(Rrv) \subseteq \mathrm{Cl}^M(Rv) = V$. Consequently $\ell(V/\mathrm{Cl}^V(Rx)) = \ell(V) - \ell(Rsx) = \ell(U_t) - \ell(Rsu_t) = \ell(U_t/\mathrm{Cl}^{U_t}(Ru_t)) = k$ in view of the fact that $Rsx = Rrv \cong Rrv_t = Rsu_t$. Thus $\mathrm{H}^M(x) \geq k$, as desired. $\square$

Using this result, we are able to extend the Lemma 2.4 of [$\mathrm{T_1}$] for σ-QTAG-modules. The proof follows the same argument, but we include it for completeness.

Lemma 1.8. *Let $x_1, \ldots x_n$ be uniform elements of a σ-QTAG-module M. If $\mathrm{H}^M(x_i) \geq k$ for some $k \in \mathcal{N}$ and all $i = 1, \ldots, n$, then $\mathrm{H}^M(x) \geq k$ for every uniform element $x \in \sum_{i=1}^n Rx_i$.*

Proof: By hypothesis there exists a σ-closed σ-uniserial submodule T_i of M containing x_i such that $\ell(T_i/\mathrm{Cl}^{T_i}(Rx_i)) \geq k$. There is $z_i \in T_i$ with $\mathrm{Cl}^M(Rz_i) = T_i$ and by (I) we have

$$\sum_{i=1}^n T_i = \sum_i \mathrm{Cl}^M(Rz_i) \subseteq \mathrm{Cl}^M(\sum_i Rz_i) = \bigoplus_{j=1}^m U_j,$$

where U_j are σ-closed σ-uniserial submodules of M.

We have $x = y_1 + \cdots + y_n = u_1 + \cdots + u_m$, $y_i \in Rx_i$ and $u_j \in U_j$. Consider an arbitrary $t \in \{1, \ldots, m\}$ with $u_t \neq 0$. Denoting by p_{it} the composed map $T_i \hookrightarrow \oplus_{j=1}^m U_j \to U_t$ we see that $u_t = \sum_{i=1}^n p_t(y_i) = \sum_{i=1}^n p_{it}(y_i)$.

For $p_t(y_i) \neq 0$ we have $y_i \notin \operatorname{Ker} p_{it}$ and therefore $\operatorname{Ker} p_{it} \subset \mathrm{Cl}^{T_i}(Ry_i)$, as T_i is σ-uniserial. Moreover, it is not difficult to see that

$$p_t(\mathrm{Cl}^{T_i}(Ry_i)) = \mathrm{Cl}^{p_t(T_i)}(Rp_t(y_i)).$$

Hence

$$T_i/\mathrm{Cl}^{T_i}(Ry_i) \cong p_t(T_i)/p_t(\mathrm{Cl}^{T_i}(Ry_i)) = \\ p_t(T_i)/\mathrm{Cl}^{p_t(T_i)}(Rp_t(y_i)) \subseteq U_t/\mathrm{Cl}^{p_t(T_i)}(Rp_t(y_i))$$

and consequently $\ell[U_t/\operatorname{Cl}^{p_t(T_i)}(Rp_t(y_i))] \geq k$ for every $i \in \{1, \dots, n\}$ with $p_t(y_i) \neq 0$. Now

$$R_t = R\sum_i p_t(y_i) \subseteq \sum_i Rp_t(y_i) \subseteq \sum_i \operatorname{Cl}^{U_t}(Rp_t(y_i)).$$

The last sum equals its greatest summand, U being σ-uniserial, and by the preceding we obtain $\ell(U_t/\operatorname{Cl}^{U_t}(Ru_t)) \geq k$. Now it is sufficient to use Lemma 1.7. □

As in [T$_1$], we define the following submodule:

$$\mathrm{H}_k(M) = \operatorname{Cl}^M(\sum\{Rx \mid x \text{ is uniform element of } M \text{ with } \mathrm{H}^M(x) \geq k\}).$$

From Lemma 1.8 it follows that all uniform elements in $\mathrm{H}_k(M)$ are of σ-height at least k.

Lemma 1.9. *Let $M = A + B$ be a σ-QTAG-module. Then:*

(i) *For each $k \in \mathcal{N}$, $\mathrm{H}_k(M) = \operatorname{Cl}^M(\mathrm{H}_k(A) + \mathrm{H}_k(B))$;*
(ii) *If $M = A \oplus B$, then $\mathrm{H}_k(M) = \mathrm{H}_k(A) \oplus \mathrm{H}_k(B)$;*
(iii) *If $M = A \oplus B$ and $0 \neq x \in A$ is uniform, then $\mathrm{H}^M(x) = \mathrm{H}^A(x)$.* □

2. Decomposability results.

Lemma 2.1. *Let U, V be σ-closed σ-uniserial submodules of a σ-QTAG-module M. Then:*

(i) *If $U \subseteq V$, then $\ell(V/U) = m$ if and only if $\mathrm{H}_m(V) = U$;*
(ii) *If $\mathrm{H}_1(U) = \mathrm{H}_1(V) = X \neq 0$, then there exists a monomorphism $f : U \to V$ extending the identity on X;*
(iii) *If $\mathrm{H}_1(U) = \mathrm{H}_1(V) = X \neq 0$, there are $u \in U\backslash X$ and $v \in V\backslash X$ such that $u - v \in \operatorname{Soc}(M)$ is uniform;*
(iv) *If $u \in U\backslash \mathrm{H}_1(U)$ and $v \in V\backslash \mathrm{H}_1(V)$ are elements such that $u - v \in \operatorname{Soc}(M)$, then $\mathrm{H}_1(U) = \mathrm{H}_1(V)$.* □

Lemma 2.2. *Let U be a σ-closed σ-uniserial submodule of a σ-QTAG-module M, $z \in \operatorname{Soc}(M)$ and $u \in U\backslash \mathrm{H}_1(U)$ be arbitrary. If $V = \operatorname{Cl}^M(R(u+z))$, then $\mathrm{H}_1(U) = \mathrm{H}_1(V)$.*

Proof: We denote $W = \operatorname{Cl}^M(Rz)$. Using [BT$_1$, Lemma 3.8] we have

$$\begin{aligned}
\mathrm{H}_1(U) &= \mathrm{H}_1(\operatorname{Cl}^M(Ru)) \subseteq \mathrm{H}_1(\operatorname{Cl}^M(R(u+z)+Rz)) = \mathrm{H}_1(\operatorname{Cl}^M(V+W)) \\
&= \operatorname{Cl}^M(\mathrm{H}_1(V+W)) = \operatorname{Cl}^M(\mathrm{H}_1(V)+\mathrm{H}_1(W)) = \operatorname{Cl}^M(\mathrm{H}_1(V)) \\
&= \mathrm{H}_1(\operatorname{Cl}^M(V)) = \mathrm{H}_1(V) = \mathrm{H}_1(\operatorname{Cl}^M(R(u+z)) \subseteq \mathrm{H}_1(\operatorname{Cl}^M(Ru+Rz)) \\
&= \mathrm{H}_1(\operatorname{Cl}^M(U+W)) = \operatorname{Cl}^M(\mathrm{H}_1(U+W)) = \\
&\quad \text{(by Lemma 2.1 (iv))} = \operatorname{Cl}^M(\mathrm{H}_1(U)) = \mathrm{H}_1(U).
\end{aligned}$$

□

A submodule N of a σ-QTAG-module M is called h-neat if $\mathrm{H}_1(M) \cap N = \mathrm{H}_1(N)$. The results obtained in [T3] can be extended to σ-QTAG-modules. We will state the next one for the future references. A submodule N of a module M is called closed if N has no essential extension in M.

Theorem 2.3. *Let N be a submodule of a σ-QTAG-module M. Then N is closed in M if and only if N is σ-closed and h-neat in M.* □

We will say that a submodule N of M is h-pure if and only if

$$\mathrm{H}_k(N) = N \cap \mathrm{H}_k(M)$$

for all $k \in \mathcal{N}$.

Lemma 2.4. *Let N be a σ-closed submodule of a σ-QTAG-module M and $\bar{K} = K/N$ be a σ-closed σ-uniserial submodule of $\bar{M} = M/N$. If $U, V \subseteq K$ are σ-uniserial σ-closed submodules of K such that $U \cap V = 0$, $\ell(U) \leq \ell(V)$ and $\bar{U}, \bar{V}$ are σ-dense in $\bar{K}$, then there exists an epimorphism $\varphi : V \to U$ such that $v - \varphi(v) \in N$ for each $v \in V$.*

Proof: The intersection $\bar{W} = \bar{U} \cap \bar{V}$ is clearly σ-dense in $\bar{K}$. Let $W \subseteq U$ be the inverse image of $\bar{W}$ under $U \to \bar{U}$ and $Z \subseteq V$ that under $V \to \bar{V}$. Obviously, W is σ-dense in U and Z is σ-dense in V. Now we have $(W+N)/N = (Z+N)/N$ which gives $\ell(Z \cap N) = \ell(Z) - \ell(\bar{W}) = \ell(V) - \ell(\bar{W}) \geq \ell(U) - \ell(\bar{W}) = \ell(W) - \ell(\bar{W}) = \ell(W \cap N)$. Let $\psi : Z \to W/(W \cap N)$ be the composed map $Z \to (Z+N)/N = (W+N)/N \cong W/(W \cap N)$. We have $\operatorname{Ker} \psi = Z \cap N$ and so ψ lifts to $\rho : Z \to W$ by Lemma 1.6 and clearly, Im ρ is σ-dense in W and also in U. By the condition ($\mathrm{I_e}$) we have the following diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & V \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ W & \xrightarrow{j} & U \end{array}$$

with $\varphi(V)$ σ-dense in U. Now φ induces $\bar{\varphi} : V/\operatorname{Ker}\varphi \to U$ where Im $\varphi =$ Im $\bar{\varphi}$ is σ-dense in U. However $U \cap V = 0$ gives $U \cong (U \oplus \operatorname{Ker}\varphi)/\operatorname{Ker}\varphi$ and it follows from Lemma 1.1 that $\bar{\varphi}$ and consequently φ is onto. Finally, for $z \in Z$ we have $\psi(z) = w + W \cap N$ where $z + N = w + N$. Thus $z - \varphi(z) = z - w + w - \varphi(z) \in N$ since $\pi(w - \varphi(z)) = \psi(z) - \pi\rho(z) = 0$ where $\pi : W \to W/W \cap N$ is the canonical projection. Now for $v \in V$ there is $I \in \mathcal{L}$ with $Iv \subseteq Z$. For each $r \in I$ we have $rv - \rho(rv) \in N$, hence $I(v - \varphi(v)) \subseteq N$. Thus $v - \varphi(v) \in N$, N being σ-closed in M. □

Proposition 2.5. *Let N be a σ-closed submodule of a σ-QTAG-module M. Then N is h-pure in M if and only if for every uniform element $\bar{x} \in \bar{M} = M/N$ there exists a uniform element $y \in M$ with $R\bar{y}$ σ-dense in $R\bar{x}$ and $\mathrm{e}(\bar{x}) = \mathrm{e}(y)$.*

Proof: ($\Rightarrow$) By [BT1, Proposition 3.13] we have $Rx + N = Ry \oplus N$ and the condition obviously holds.

($\Leftarrow$) Assume that N it is not h-pure in M and let k be the smallest integer such that there exists a uniform element $x \in (\mathrm{H}_k(M) \cap N) \setminus \mathrm{H}_k(N)$. We can find a σ-closed σ-uniserial submodule $U \subseteq M$ such that $\mathrm{H}_k(U) = \mathrm{Cl}^M(Rx) = U \cap N$. Taking $y \in U$ with $\mathrm{Cl}^M(Ry) = U$ the hypothesis gives the existence of a uniform element $z \in M$ such that $R\bar{z}$ is σ-dense in $R\bar{y}$ and $\mathrm{e}(z) = k = \mathrm{e}(\bar{y})$. Since $R\bar{z} \cong Rz/(Rz \cap N)$, we have $Rz \cap N = 0$ and $V \cap N = 0$ where $V = \mathrm{Cl}^M(Rz)$. Now $U \cap V \cap \mathrm{H}_k(U) = 0$ gives $U \cap V = 0$ and by Lemma 2.4 there is an epimorphism $\varphi : U \to V$ such that $u - \varphi(u) \in N$ for each $u \in U$.

It is clear that $Rx \cap Ry$ is σ-dense in Rx. Hence there is an element $rx = sy$ such that $\mathrm{Cl}^M(Rrx) = \mathrm{H}_k(U)$. The epimorphism φ induces $\psi : Ry \to R(y - \varphi(y))$ naturally. By [BT$_1$, Lemma 3.4] it follows that $\psi(rx) \in H_k(N)$. From [BT$_1$, Lemma 4.2] we have $\mathrm{H}_k(V) = 0$, therefore $\varphi(x) = 0$, $\psi(rx) = rx \in \mathrm{H}_k(N)$ and [BT$_1$, Lemma 3.3] gives $x \in \mathrm{H}_k(N)$. This contradiction finishes the proof. $\square$

Lemma 2.6. *Let N be a σ-closed submodule of a σ-*QTAG*-module M, $0 \neq \bar{S} = S/N \subseteq M/N = \bar{M}$ be a σ-closed and σ-cocritical. Let $U, V \subseteq S$ be σ-closed σ-uniserial modules with $\bar{U}, \bar{V}$ σ-dense in $\bar{S}$ such that U has the minimal length among all such modules. Then either $U \cap V = 0$ or $\ell(U) = \ell(V)$.*

Proof: Assume $U \cap V \neq 0$ and $\ell(U) < \ell(V)$. By Proposition 1.4 there is a monomorphism $\varphi : U \to V$ extending the identity on $U \cap V$. Since $\bar{V} = V/V \cap N, V \cap N = \mathrm{H}_1(V)$ and so $\varphi(U) \subseteq V \cap N$. Take $u \in U$ such that $\mathrm{Cl}^M(Ru) = U$. Then $\varphi(u) \neq u$ since $u \notin N$ and $\varphi(u) \in N$. Thus by ($\mathrm{I_e}$) we have $\eta : U \to W = \mathrm{Cl}^M(R(u - \varphi(u)))$ induced by $Ru \to R(u - \varphi(u))$ defined naturally. Since $u - \varphi(u) \notin N$, then $\bar{W}$ is σ-dense in $\bar{S}$. But $U \cap V \subseteq \operatorname{Ker} \eta$ gives $\ell(W) < \ell(U)$, a contradiction finishing the proof. $\square$

Let N be a submodule of a module M. A submodule K of M is called a complement of N in M if K is maximal with respect to $K \cap N = 0$.

Theorem 2.7. *Let T be a σ-closed submodule of a σ-*QTAG*-module M and K be any complement of T in M. Then there exists a mapping of the set of all σ-closed σ-cocritical submodules of $M/(T \oplus K)$ onto the family of σ-closed σ-cocritical submodules of $\bar{T} = [(\mathrm{H}_1(M) + K) \cap T]/\mathrm{H}_1(T)$.*

Proof: We will follow the same arguments as in [S$_3$, Theorem 3.4]. By [BT$_1$, Proposition 4.24], $T \oplus K$ is σ-closed in M (we use [BT$_1$, Lemma 4.4] that only requires (I) and in [BT$_1$, Proposition 3.2] ($\mathrm{I_e}$) is enough).

(a) We have $T \oplus K \subseteq' M$, $(T \oplus K)/K \subseteq' M/K$ and consequently it follows from [BT$_1$, Lemma 2.7] that

$$(2) \qquad U/K \subseteq M/K \;\; \sigma\text{-cocritical implies} \quad U \subseteq T \oplus K.$$

Consider a σ-closed σ-cocritical submodule $\bar{S} = S/(T \oplus K) \subseteq M/(T \oplus K) = \bar{M}$. By [BT$_1$, Lemma 2.12] there are uniform elements $x \in S$ with $\mathrm{Cl}^M(R\bar{x}) = \bar{S}$ and we can take one of them with the smallest $\mathrm{e}(x)$ and set $U = \mathrm{Cl}^M(Rx)$. Now

$\bar{U} = (U + T \oplus K)/(T \oplus K) \cong U/U \cap (T \oplus K)$ contains $\bar{x}$ and so it is σ-dense in $\bar{S}$. Thus

$$U \cap (T \oplus K) = \mathrm{H}_1(U) = V \tag{3}$$

and we can take y with

$$\mathrm{Cl}^M(Ry) = V. \tag{4}$$

Now we can write

$$y = t + k, \quad t \in T, \quad k \in K, \quad t \neq 0. \tag{5}$$

Clearly, for $t = 0$ the natural mapping $U \to (U + K)/K$ maps y and hence V onto 0 and so $U/V \cong (U + K)/K \subseteq M/K$ is σ-cocritical and $x \in U \subseteq T \oplus K$ by (2), which contradicts the choice of x.

Obviously, $Rx \cap Ry$ is σ-dense in $Rx \cap V$ and so there is

$$rx = sy \in Ry \quad \text{with} \quad \mathrm{Cl}^{Rx \cap V}(Rsy) = Rx \cap V. \tag{6}$$

Moreover,

$$Rx/(Rx \cap V) \quad \text{is} \quad \sigma\text{-cocritical}. \tag{7}$$

(b) Now we will show that

$$t \notin \mathrm{H}_1(T). \tag{8}$$

Proving indirectly assume that $t \in \mathrm{H}_1(T)$ and denote $T_1 = \mathrm{Cl}^M(Rt)$. Then there is a σ-closed σ-uniserial submodule T_2 of T such that T_2/T_1 is σ-cocritical. Now, $k = y - t \in K \cap \mathrm{H}_1(M) = \mathrm{H}_1(K)$, K being h-neat in M by Theorem 2.3. Thus, there is a σ-closed σ-uniserial submodule K_2 of K such that K_2/K_1 is σ-cocritical and $K_1 = \mathrm{Cl}^M(Rk)$.

Assume that $U \cap T_2 = U \cap K_2 = 0$ and consider the following two commutative diagrams:

$$\begin{tikzcd} Ry \arrow[r] \arrow[d, "p"'] & V \arrow[r] \arrow[d, "\lambda"] & U \arrow[d, "\varphi"] \\ Rt \arrow[r] & T_1 \arrow[r] & T_2 \end{tikzcd}$$

$$\begin{tikzcd} Ry \arrow[r] \arrow[d, "q"'] & V \arrow[r] \arrow[d, "\mu"] & U \arrow[d, "\psi"] \\ Rk \arrow[r] & K_1 \arrow[r] & K_2 \end{tikzcd}$$

where p, q are natural projections, λ and μ exist by the condition ($\mathrm{I_e}$) and φ, ψ by Proposition 1.5.

Consider now the composition

$$\rho : Rv \to R(x - \varphi(x)) \to (R(x - \varphi(x) + K)/K$$

of natural mappings. By (5) and (6) we have $\rho(sy) = sy - \varphi(sy) + K = sy - st + K = K$, hence $Rx \cap V \subseteq \operatorname{Ker} \rho$ and $(R(x - \varphi(s)) + K)/K$ is σ-cocritical by (7). By (2) we have $x \in T \oplus K$, which is a contradiction showing that

(9) $$U \cap T_2 \neq 0 \neq V \cap T_1.$$

Moreover, $(V \cap T_1) \cap (V \cap K) \subseteq V \cap K \cap T = 0$ gives

(10) $$V \cap K = 0,$$

V being uniform.

To get a contradiction we will show that p is neither monic nor non-monic. Assume first that p is monic. Then clearly $\ell(U) = \ell(T_2)$ and so Lemma 1.1 and Proposition 1.4 give the existence of an isomorphism $\varphi : U \to T_2$ extending the identity on $U \cap T_2$. Considering the natural mapping $Rx \to R(x - \varphi(x))$ with non-trivial kernel (containing $Rx \cap U \cap T_2$) we see that $\mathrm{e}(x) > \mathrm{e}(x - \varphi(x))$, which contradicts the choice of x.

Now, assume p is non-monic. Then q is monic since $\operatorname{Ker} p \cap \operatorname{Ker} q = 0$ and Ry is σ-uniserial. By (10) we can use ψ from the second diagram and consider the natural mapping $\nu : Rx \to R(x - \psi(x))$. If ν is not monic, then $\mathrm{e}(x - \psi(x)) < \mathrm{e}(x)$ contradicts the choice of x. Thus ν is an isomorphism. By (3), (4), (6) and (7) we get $\nu(sy) = sy - sk = st$ and $\mathrm{e}(st) = \mathrm{e}(sy) = \ell(Rx \cap V) = \mathrm{e}(x) - 1 = \mathrm{e}(y)$. However, p is not monic, and so $\mathrm{e}(t) < \mathrm{e}(y) = \mathrm{e}(st) \leq \mathrm{e}(t)$, a final contradiction proving (8).

We can set

(11) $$\Phi(\bar{S}) = \mathrm{Cl}^{\tilde{T}}(R\tilde{t})$$

where $R\tilde{t} = (Rt + \mathrm{H}_1(T))/\mathrm{H}_1(T) \cong Rt/(Rt \cap \mathrm{H}_1(T)) = Rt/\mathrm{H}_1(Rt)$ by (8) and $R\tilde{t}$ is σ-cocritical.

(c) Now, we will show that $\mathrm{Cl}^{\tilde{T}}(R\tilde{t})$ does not depend on the particular choice of U and y. Assume first that U is given and let $y = t + k$, $y' = t' + k'$ both have the property (4). Clearly, $Ry \cap Ry'$ is σ-dense in Ry and so in V. Thus we can choose $ry = sy'$ with $\mathrm{Cl}^M(Rry) = V$. Now Rry is σ-dense in Ry and so using the canonical projection $Ry \to Rt$ we see that Rrt is σ-dense in Rt and similarly Rst' is σ-dense in Rt'. So, $Rr\tilde{t}$ is σ-dense in $R\tilde{t}$, $Rs\tilde{t}'$ is σ-dense in $R\tilde{t}'$, hence $Cl^{\tilde{T}}(R\tilde{t}) = \mathrm{Cl}^{\tilde{T}}(Rr\tilde{t}) = \mathrm{Cl}^{\tilde{T}}(Rs\tilde{t}') = \mathrm{Cl}^{\tilde{T}}(R\tilde{t}')$, since $ry = sy'$ obviously gives $rt = st'$.

Thus, let $U, U' \subseteq S$ be different and assume $U \cap U' \neq 0$. By Lemma 1.1 and Proposition 1.4 there is an isomorphism $\varphi : U \to U'$ extending the identity on

$U \cap U'$. Then φ induces the natural epimorphism $Rx \to R(x - \varphi(x))$ with non-zero kernel containing $Rx \cap U \cap U'$. Consequently $\mathrm{e}(x - \varphi(x)) < \mathrm{e}(x)$ which contradicts the choice of x.

(d) Assume finally that $U \cap U' = 0$ and that U' is not necessarily of minimal length, but $\Phi(\bar{S})$ is constructed in the same way once we know that the corresponding t' is not in $\mathrm{H}_1(T)$. By Lemma 2.4 there is an epimorphism $\eta : U' \to U$ with $x' - \eta(x') \in T \oplus K$. Now, by (6) and (7) we have $rx' \in \mathrm{H}_1(Rx')$, hence $y' - \eta(y') = r(x' - \eta(x')) \in \mathrm{H}_1(Rx' - \eta(x')) \subseteq \mathrm{H}_1(T \oplus K)$. However, from (6) we get that Rsy' is σ-dense in V, hence in Ry' and so [BT$_1$, Lemma 3.1] and Lemma 1.9 gives $y' - \eta(y') \in \mathrm{H}_1(T \oplus K) = \mathrm{H}_1(T) \oplus \mathrm{H}_1(K)$. Thus $t' - \eta(t') \in \mathrm{H}_1(T)$ and $R\tilde{t}' = R\eta(\tilde{t}')$. However, we already know that $\mathrm{Cl}^{\tilde{T}}(R\eta(\tilde{t})) = \mathrm{Cl}^{\tilde{T}}(R\tilde{t})$ and the proof of the independence is finished.

(e) It remains to show that the mapping Φ is onto. Let $\tilde{S} \subseteq \tilde{T}$ be a σ-closed σ-cocritical submodule. We can take t uniform with $R\tilde{t}$ σ-dense in $\tilde{T}$ (see [BT$_1$, Lemma 2.12]). Then $t \notin \mathrm{H}_1(T)$ is of the form $t = u + k$, $u \in \mathrm{H}_1(M)$, $k \in K$, and $u = t - k \neq 0$. By the condition (I) we now have $Ru = Ru_1 \oplus \ldots \oplus Ru_n$ with Ru_i σ-uniserial. Now, under the canonical projection $p : T \oplus K \to T$ we have $p(u) = t$ and $Rt = p(Ru) = \Sigma_{i=1}^{n} Rp(u_i)$ and so we can with respect to [BT$_1$, Proposition 2.6] assume that $Rt = \mathrm{Cl}^{Rt}(Rp(u_1))$. The canonical projection $Ru \to Ru_1$ shows that $u \in \mathrm{H}_1(M)$ gives $u_1 \in \mathrm{H}_1(M)$ and consequently there is a σ-closed σ-uniserial submodule U of M such that $U/\,\mathrm{Cl}^U(Ru_1)$ is σ-cocritical.

Now $p(u_1) = rt$ for some $r \in R$, where Rrt is σ-dense in Rt. By [BT$_1$, Lemma 3.3] we see that $rt \notin \mathrm{H}_1(M)$ and consequently $U \not\subseteq T \oplus K$, for otherwise $Ru_1 \subseteq \mathrm{H}_1(T \oplus K)$ would lead to $rt \in \mathrm{H}_1(T)$.

Moreover, $U \cap (T \oplus K) = \mathrm{Cl}^M(Ru_1)$ and so the construction from the part (b) would lead to $\tilde{S}$. If U is of minimal length in S, $\overline{S} = \mathrm{Cl}^{\overline{M}}(\overline{U})$ and $\Phi(\overline{S}) = \tilde{S}$ by the above construction (part (b)). If U is not of minimal length then we can take $U' \subseteq S$ of minimal length, $\ell(U') < \ell(U)$. By Lemma 2.6 we have $U \cap U' = 0$ and consequently the part (d) gives $\Phi(\overline{S}) = \tilde{S}$ (we used here freely the fact that $\mathrm{Cl}^{\tilde{T}}(Rr\tilde{t}) = \mathrm{Cl}^{\tilde{T}}(R\tilde{t}) = \tilde{S}$) and the proof is complete. □

Kulikov's theorem was obtained in [T$_1$] and extended in [BT$_1$]. We can show the same result for σ-QTAG-modules. The proof follows that one obtained there, we only have to use the results on extensions of homomorphisms in σ-QTAG-modules instead of the condition (II). We leave the proof to the reader.

Theorem 2.8. *Let M be a σ-QTAG-module. M is direct sum of σ-uniserial submodules if and only if it contains a chain of σ-closed submodules*

$$M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \ldots$$

with $\bigcup M_i$ σ-dense in M and such that for each $n \in \mathcal{N}$ there exists k_n with the property $\mathrm{H}(x) \leq k_n$ for all uniform elements of M_n. In this case the direct decomposition of M into a direct sum of σ-uniserial submodules is unique up to isomorphism.

□

3. Kernel of h-purity.

Definition 3.1. A submodule N of a module M is called a kernel of h-purity if every h-neat hull of N is h-pure in M.

Proposition 3.2. *Let N be a σ-closed submodule of a σ-QTAG-module M, $n \in \mathcal{N}$. The following two conditions are equivalent:*

(i) $\operatorname{Soc}(\mathrm{H}_n(M)) \subseteq N$ *implies* $\mathrm{H}_n(M) \subseteq N$;

(ii) *If there is a σ-closed σ-uniserial submodule U of M such that $U/(N \cap U)$ is σ-cocritical and $U \subseteq \mathrm{H}_n(M)$, then there exists a σ-closed σ-uniserial submodule W of M such that $\ell(W) = n+1$ and $W \cap N = 0$.*

Proof: (i) $\Rightarrow$ (ii): If (ii) does not hold then there exists U with the stated property and $\operatorname{Soc}(W) \subseteq N$ whenever W is σ-uniserial σ-closed and $\ell(W) = n+1$. Especially, $U \subseteq \mathrm{H}_n(M)$ means that $\mathrm{H}_n(M) \nsubseteq N$. On the other hand, if $Z \subseteq \operatorname{Soc}(\mathrm{H}_n(M))$ is σ-closed, then there is a σ-closed σ-uniserial submodule $W \subseteq M$ such that $\ell(W/Z) = n$. Then $\ell(W) = n+1$ gives $Z \subseteq N$, which contradicts (i).

(ii) $\Rightarrow$ (i): Let $\operatorname{Soc}(\mathrm{H}_n(M)) \subseteq N$ but $\mathrm{H}_n(M) \nsubseteq N$. Then there is a σ-closed σ-uniserial submodule $Z \subseteq \mathrm{H}_n(M)$ such that $Z \nsubseteq N$ and $\operatorname{Soc}(Z) \subseteq N$. Take $Z \cap N \subseteq X \subseteq Z$ with X σ-closed and $X/(Z \cap N)$ σ-cocritical. Then $X \cap N = Z \cap N$ and $X \subseteq \mathrm{H}_n(M)$. By (ii) there is a σ-closed σ-uniserial submodule W of M such that $\ell(W) = n+1$ and $W \cap N = 0$, which contradicts the hypothesis since $\operatorname{Soc}(W) \subseteq \operatorname{Soc}(\mathrm{H}_n(M))$. □

Proposition 3.3. *Let N be a σ-closed h-pure submodule of a σ-QTAG-module M, $n \in \mathcal{N}$. If $\operatorname{Soc}(\mathrm{H}_n(M)) \subseteq N$ then $\mathrm{H}_n(M) \subseteq N$.*

Proof: We shall use Proposition 3.2. Let U be a σ-closed σ-uniserial submodule of M such that $U/(U \cap N)$ is σ-cocritical and $U \subseteq \mathrm{H}_n(M)$. By h-purity we have $V = U \cap N \subseteq N \cap \mathrm{H}_{n+1}(M) = \mathrm{H}_{n+1}(N)$ and consequently there are σ-closed σ-uniserial modules $W \subseteq N$ and $Z \subseteq M$ such that $\ell(W/V) = n+1$ and $\ell(Z/U) = n$. By Proposition 1.4 there is an isomorphism $\varphi : W \to Z$ extending the identity on $W \cap Z = V$. Take $w \in W \setminus \mathrm{H}_1(W)$ and consider the natural epimorphism $\psi : Rw \to R(w - \varphi(w))$. In view of $\operatorname{Ker} \psi = Rw \cap V$ we have $\mathrm{e}(w - \varphi(w)) = n+1$. For $rw \in U \setminus V$ it is $\psi(rw) \in \operatorname{Soc}(\mathrm{H}_n(M))$, but $\psi(rw) \notin N$ since $rw \in N$ and $\varphi(rw) \notin N$. □

An element $y \in M$ is called a predecessor of $x \in M$ if $\mathrm{Cl}^M(Rx) \subseteq \mathrm{Cl}^M(Ry)$ and $\ell(Ry/\mathrm{Cl}^{Ry}(Rx)) = 1$. For uniform elements $x, y \in M$ we use the notation $x \sim y$ in case $\mathrm{Cl}^M(Rx) = \mathrm{Cl}^M(Ry)$.

Lemma 3.4. *Let V be a σ-closed σ-uniserial submodule of a σ-QTAG-module M, W be a σ-closed σ-cocritical submodule of M which is not contained in V, and $U = \mathrm{H}_1(V)$. If $X = \mathrm{Cl}^M(R(v+w))$ for arbitrary elements $v \in V \setminus U$ and $0 \neq w \in W$, then $X = V \oplus W$ if and only if $V/U \ncong W$.*

Proof: ($\Rightarrow$) Proving indirectly suppose that $\psi : V/U \to W$ is an isomorphism which naturally induces $\varphi : V \to W$ with the kernel U. Take $v \in V \setminus U$ arbitrarily, denote $w = \varphi(v)$ and consider the natural mapping $\nu : Rv \to R(v+w)$. The condition $(\mathrm{I_e})$ and Lemma 1.1 show that ν extends to an isomorphism $\mu : V \to X$.

Take $rv \in U$ with $\mathrm{Cl}^M(Rrv) = U$. We have $\mu(rv) = r(v + \varphi(v)) = rv$. Now for an arbitrary $u \in U$ we have $Iu \subseteq Rrv$ for some $I \in \mathcal{L}$. Now for each $s \in I$ we have $su = t_s rv$, so $s(\mu(u) - u) = \mu(t_s rv) - su = t_s rv - su = 0$ which yields $I(\mu(u) - u) = 0$ and consequently $\mu|_U = 1_U$. Thus μ induces the isomorphism $V/U \cong X/U$ showing that $\ell(X/U) = 1$. Hence $X \subset V \oplus W$ since $\ell((V \oplus W)/U) = 2$.

($\Leftarrow$) Assuming $(0 : w) \subseteq (U : v)$ we get the composed mapping $Rw \cong R/(0 : w) \to R/(U : v) \cong (Rv + U)/U \subseteq V/U$ which is monic, W being σ- cocritical, and so it extends to an isomorphism $W \to V/U$ by $(\mathrm{I_e})$ and Lemma 1.1. This contradiction gives the existence of an element $r \in (0 : w) \setminus (U : v)$. Then $r(v + w) = rv \in V \setminus U$ shows that $V = \mathrm{Cl}^M(Rrv) = \mathrm{Cl}^M(Rr(v+w)) \subseteq X$. Especially, $v \in X$ gives $w \in X$, hence $W \subseteq X$ and we are through. $\square$

A σ-QTAG-module M is called <u>homogeneous</u> if any two σ-cocritical submodules of any two torsionfree homomorphic images of M are isomorphic.

Theorem 3.5. *Let N be a σ-closed submodule of a σ-QTAG- module M. Then N is kernel of h-purity provided for every $n \in \mathcal{N}$ one of the following two conditions is satisfied:*

(i) $\mathrm{Soc}(M) = \mathrm{Cl}^M[\mathrm{Soc}(N) + \mathrm{Soc}(\mathrm{H}_n(M))]$;

(ii) *For every uniform element $x \in \mathrm{H}_{n+1}(M) \cap N$ and every its predecessor $y \in \mathrm{H}_n(M)$ there are $z \in \mathrm{Soc}(M)$ and $r \in R$ such that $ry \sim y$ and $ry + z \in N \cap \mathrm{H}_n(M)$.*

If M is homogeneous, then the converse is true.

Proof: Let $N \subseteq K$ be an h-neat hull of N. Then $\mathrm{H}_1(M) \cap K = \mathrm{H}_1(K)$ and we shall continue by the induction.

Let $\mathrm{H}_n(M) \cap K = \mathrm{H}_n(K)$ for some $n \geq 1$. Take $x \in \mathrm{H}_{n+1}(M) \cap K$ uniform and let $y \in \mathrm{H}_n(M) \setminus K$ be its predecessor. Since $x \in \mathrm{H}_1(M) \cap K = \mathrm{H}_1(K)$, x has also a predecessor $y' \in K$ and by Lemma 2.1 we can assume that $y - y' \in \mathrm{Soc}(M)$.

If the condition (i) is satisfied then $I(y - y') \subseteq \mathrm{Soc}(N) + \mathrm{Soc}(\mathrm{H}_n(M))$ for some $I \in \mathcal{L}$. Now, we obviously can take $s \in I \setminus (\mathrm{Cl}^M(Rx) : y)$. Then $s(y - y') = u + v$, $u \in \mathrm{Soc}(\mathrm{H}_n(M))$, $v \in \mathrm{Soc}(N)$ and $y \sim sy$. Thus $sy - u = sy' + v \in \mathrm{H}_n(M) \cap K = \mathrm{H}_n(K)$ and using Lemma 2.2 we obtain $x \in \mathrm{H}_1(\mathrm{Cl}^M(Ry)) = \mathrm{H}_1(\mathrm{Cl}^M(Rsy)) = \mathrm{H}_1(\mathrm{Cl}^M(R(sy - u))) \subseteq \mathrm{H}_1(\mathrm{H}_n(K)) \subseteq \mathrm{H}_{n+1}(K)$.

If the condition (i) is not satisfied, then it is not satisfied for all $k > n$ and so let the condition (ii) hold. If we prove that $x \in N$ then there are $z \in \mathrm{Soc}(M)$ and $r \in R$ such that $ry \sim y$ and $ry + z \in N \cap \mathrm{H}_n(M) \subseteq \mathrm{H}_n(K)$. Using Lemma 2.2 we now have $x \in \mathrm{H}_1(\mathrm{Cl}^M(Ry)) = \mathrm{H}_1(\mathrm{Cl}^M(Rry)) = \mathrm{H}_1(\mathrm{Cl}^M(R(ry + z)) \subseteq \mathrm{H}_1(\mathrm{H}_n(K)) \subseteq \mathrm{H}_{n+1}(K)$.

So, assume that $x \notin N$. By [T3, Proposition 3.5] it is $\mathrm{Soc}(N) = \mathrm{Soc}(K)$ and we can find the smallest m with $\mathrm{H}_m(\mathrm{Cl}^M(Rx)) \subseteq N$. Now take

$$v \in \mathrm{H}_{m-1}(\mathrm{Cl}^M(Rx)) \setminus \mathrm{H}_m(\mathrm{Cl}^M(Rx))$$

and

$$rv \in \mathrm{H}_m(\mathrm{Cl}^M(Rx)) \setminus \mathrm{H}_{m+1}(\mathrm{Cl}^M(Rx)).$$

Then $v \in \mathrm{H}_{n+m}(M)$, $rv \in \mathrm{H}_{n+m+1}(M) \cap N$ and by (ii), $tv + z \in \mathrm{H}_{n+m}(M) \cap N$ for some $z \in \mathrm{Soc}(M)$ and $t \in R$ with $v \sim tv$. But $tv \in \mathrm{Cl}^M(Rx) \subseteq K$ gives $z = (tv + z) - tv \in \mathrm{Soc}(M) \cap K = \mathrm{Soc}(K) = \mathrm{Soc}(N)$ and so $tv \in N$. But then $v \in \mathrm{Cl}^M(Rv) = \mathrm{Cl}^M(Rtv) \subseteq N$, which contradicts the choice of v.

Assume now M is homogeneous. Proving indirectly suppose that neither (i) nor (ii) is satisfied and find an h-neat hull K of N in M which is not h-pure in M. So let for some $n \in \mathcal{N}$

$$\mathrm{Cl}^M[\mathrm{Soc}(N) + \mathrm{Soc}(\mathrm{H}_n(M))] \subset \mathrm{Soc}(M) \tag{12}$$

and there exists $x \in \mathrm{H}_{n+1}(M) \cap N$ uniform and its predecessor

$$y \in \mathrm{H}_n(M) \tag{13}$$

such that

$$sy + z \notin \mathrm{H}_n(M) \cap N \quad \text{for each} \quad z \in \mathrm{Soc}(M) \quad \text{and each} \quad s \in R \quad \text{with} \quad sy \sim y. \tag{14}$$

Assume first, that

$$sy + z \in N \quad \text{for some} \quad z \in \mathrm{Soc}(M) \quad \text{and some} \quad s \in R \quad \text{with} \quad sy \sim y. \tag{15}$$

If $z - u \in \mathrm{H}_n(M)$ for some $u \in \mathrm{Soc}(N)$, then $sy + z - u \in N \cap \mathrm{H}_n(M)$, which contradicts (14), since $z - u \in \mathrm{Soc}(M)$. Thus

$$z - u \notin \mathrm{H}_n(M) \quad \text{for each} \quad u \in \mathrm{Soc}(N). \tag{16}$$

Now let $N \subseteq K$ be any h-neat hull of N in M. Then $x \in \mathrm{H}_{n+1}(M) \cap K = \mathrm{H}_{n+1}(K)$, since by the hypothesis all h-neat hulls of N are h-pure in M. Then, there exists a predecessor of x

$$y' \in \mathrm{H}_n(K) \tag{17}$$

which can be chosen, by Lemma 2.1, such that

$$y - y' \in \mathrm{Soc}(M) \quad \text{is uniform.} \tag{18}$$

Therefore,

$$z' = z + s(y - y') \in \mathrm{Soc}(M) \cap K = \mathrm{Soc}(K) = \mathrm{Soc}(N) \tag{19}$$

by (15), (17), (18) and [T_3, Proposition 3.5]. This gives $z - z' = s(y - y') \in \mathrm{H}_n(M)$ by (13) and (17), which contradicts (16) in view of (19).

This contradiction shows that (15) is impossible and

$$y + z \notin N \text{ for each } z \in \mathrm{Soc}(M). \tag{20}$$

Suppose that $x \in \mathrm{H}_1(N)$ and $t \in N$ be its predecessor such that $y - t \in \mathrm{Soc}(M)$, by Lemma 2.1. Then $t = y + (t - y) \in N$ contradicts (20) and

$$x \notin \mathrm{H}_1(N). \tag{21}$$

Now from (12) we get the existence of a uniform element

$$v \in \mathrm{Soc}(M) \tag{22}$$

and

$$v - u \notin \mathrm{Soc}(\mathrm{H}_n(M)) \text{ for each } u \in \mathrm{Soc}(N). \tag{23}$$

Consider $N' = \mathrm{Cl}^M(N + R(y + v))$ and

$$w = a + r(y + v),\ a \in N,\ r \in R, \tag{24}$$

be such that

$$Rw \subseteq N' \text{ is } \sigma\text{-cocritical}. \tag{25}$$

First, we are going to show that

$$rv = 0. \tag{26}$$

If not, then $w-rv = a+ry \in \mathrm{Soc}(M)$ by (22) and (25). For $Ry = Rry$ we get from (13) and Lemma 2.2 that $x \in \mathrm{H}_1(\mathrm{Cl}^M(Ry)) = \mathrm{H}_1(\mathrm{Cl}^M(Rry)) = \mathrm{H}_1(\mathrm{Cl}^M(ry - (a + ry))) = \mathrm{H}_1(\mathrm{Cl}^M(Ra)) \subseteq \mathrm{H}_1(N)$, which contradicts (21). Thus, $Rry \subset Ry$, hence $Rry \subseteq Rx$ and $w - rv \in N$. So, $rv \in N'$ and consequently $v \in N'$ by (22). By the hypothesis and Lemma 3.4, we have $\mathrm{Cl}^M(R(y+v)) \subset \mathrm{Cl}^M(Ry) \oplus \mathrm{Cl}^M(Rv)$. Hence $\mathrm{Cl}^M(R(y + v))/\mathrm{Cl}^M(Rx) = \mathrm{Cl}^M(R(y + v))/[\mathrm{Cl}^M(R(y + v)) \cap N]$ is σ-cocritical. Consequently, $(R(y+v)+N)/N \cong R(y+v)/(R(y+v)\cap N)$ is also σ-cocritical and so is N'/N, the last module being obviously σ-dense in it. Since $v \notin N$ by (23), we have $N' = N \oplus \mathrm{Cl}^M(Rv)$ and so $y = (y + v) - v \in N'$ is of the form $y = v' + u$, $v' \in \mathrm{Cl}^M(Rv)$, $u \in N$. But then $y - v' = u \in N$, which contradicts (20) owing to (22).

Thus (26) is proved and from (24) we have $w = a+ry$. As above, $\mathrm{H}_1(\mathrm{Cl}^M(Ra)) = \mathrm{H}_1(\mathrm{Cl}^M(Rry))$ and $Ry = Rry$ leads to a contradiction with (21). Thus $Rry \subseteq Rx$, $w \in N$ and $\mathrm{Soc}(N) = \mathrm{Soc}(N')$.

If K is an h-neat hull of N' in M, then by [T_3, Proposition 3.5] K is a neat hull of N in M and so it is pure in M. Then $x \in \mathrm{H}_{n+1}(M)\cap K = \mathrm{H}_{n+1}(K)$. Hence, there is a predecessor t' of x in $\mathrm{H}_n(K)$ with $y-t' \in \mathrm{Soc}(M)$ (and also $y-t' \in \mathrm{Soc}(\mathrm{H}_n(M))$). Now $v + (y - t') = (v + y) - t' \in \mathrm{Soc}(M) \cap K = \mathrm{Soc}(K) = \mathrm{Soc}(N)$ and so $t' - y = v - (v + (y - t')) \in \mathrm{Soc}(\mathrm{H}_n(M))$, which contradicts (23) and the proof is complete. □

4. Centers of α-purity.

Now we introduce the following submodules of a σ-QTAG-module M extending the definition in [BT_1].

Let M be a σ-QTAG-module and let α be any ordinal, $\mathrm{H}_\alpha(M)$ is defined by transfinite recursion as follows:
i) $\mathrm{H}_0(M) = 0$;
ii) If α is not a limit ordinal, say $\alpha = \beta + 1$, then $\mathrm{H}_\alpha(M) = \mathrm{H}_1(\mathrm{H}_\beta(M))$;
iii) If α is a limit ordinal, then

$$\mathrm{H}_\alpha(M) = \bigcap_{\beta<\alpha} \mathrm{H}_\beta(M).$$

M is called of type τ, if τ is the smallest ordinal number such that $\mathrm{H}_\tau(M) = \mathrm{H}_{\tau+1}(M)$. For any x ($\neq 0$), the generalized height $h(x)$ is defined as follows: If $x \in \mathrm{H}_\tau(M)$, put $h(x) = \infty > \alpha$, α any ordinal. Let $x \notin \mathrm{H}_\tau(M)$, then there exists an ordinal $\beta < \tau$, such that $x \in \mathrm{H}_\beta(M)$, but $x \notin \mathrm{H}_{\beta\ +1}(M)$; define $h(x) = \beta$.

Lemma 4.1. *Let M be a σ-QTAG-module. Then:*

(i) *For any $x, y \in M$, $h(x + y) \geq \min(h(x), h(y))$ and equality holds, whenever $h(x) \neq h(y)$.*
(ii) *If $M = A \oplus B$, then $\mathrm{H}_\alpha(M) = \mathrm{Cl}^M(\mathrm{H}_\alpha(A)\ \oplus \mathrm{H}_\alpha(B))$ for any ordinal α; for $x = a + b$, $a \in A$, $b \in B$, we have $h(x) = \min(h(a), h(b))$.*
(iii) *For any ordinals α, β; $\mathrm{H}_\alpha(\mathrm{H}_\beta(M)) = \mathrm{H}_{\alpha+\beta}(M)$.*
(iv) *Let K be a σ-closed submodule and $p : M \to M/N$ be the canonical projection. If $x \in \mathrm{H}_\alpha(M)$ is uniform, then $p(x) \in \mathrm{H}_\alpha(M/N)$.*
(v) *For any homomorphism $f : M \to N$, where N is also a σ-QTAG-module,*

$$f(\mathrm{H}_\alpha(M)) \subseteq \mathrm{H}_\alpha(N). \qquad \square$$

Lemma 4.2. *Let K and N be submodules of a σ-QTAG-module M, such that K is σ-closed in M. Let y be any uniform element of M such that $(Ry + K)/K$ is σ-cocritical and $(K + Ry) \cap N \neq 0$ and $K \cap N = 0$. Then there exists $r \in R$ such that $ry = x + z$, for some $x \in K$ and $z \in N$, and $ry \sim y$.*

Proof: As $(K + Ry) \cap N \neq 0$, for some $x \in K$, $u \neq 0$ in N, $r \in R$, we have $u = x + ry$. Now $K \cap N = 0$, gives $ry \notin K$. As Ry is σ-uniserial we have $Ry = \mathrm{Cl}^{Ry}(Rry)$. $\square$

A submodule N of M is said to be α-pure in M, if $\mathrm{H}_\beta(M) \cap N = \mathrm{H}_\beta(N)$ for all $\beta \leq \alpha$. Then a submodule N of M is h-pure if and only if N is $(\omega + 1)$-pure where ω is the first infinite ordinal.

Theorem 4.3. *Let M be a σ-QTAG-module, α any ordinal number and N any submodule of $\mathrm{H}_\alpha(M)$. Then any complement K of N in M is $(\alpha+1)$-pure and $\mathrm{H}_\beta(K)$ is a complement of N in $\mathrm{H}_\beta(M)$ for all $\beta \le \alpha$.*

Proof: Consider any ordinal $\beta \le \alpha+1$. To apply transfinite induction, let $\mathrm{H}_\delta(M) \cap K = \mathrm{H}_\delta(K)$ for all $\delta < \beta$. If β is a limit ordinal, then trivially $\mathrm{H}_\beta(M) \cap K = \mathrm{H}_\beta(K)$. Let $\beta = \gamma + 1$ and $\mathrm{H}_\beta(M) \cap K \ne \mathrm{H}_\beta(M)$. We can find a uniform element $x \in \mathrm{H}_\beta(M) \cap K$ such that $x \notin \mathrm{H}_\beta(K)$. As $x \in \mathrm{H}_{\gamma+1}(M) = \mathrm{H}_1(\mathrm{H}_\gamma(M))$, there exists $x \in Y \subseteq \mathrm{H}_\gamma(M)$ with $\ell(Y/\operatorname{Cl}^Y(Rx)) = 1$. Then $Y \subseteq K$. Therefore $(K+Y) \cap \operatorname{Soc}(N) \ne 0$. Then $y = k + n$ for some $k \in K$ and $n \in N$, where $Y = \operatorname{Cl}^M(Ry)$. As $n \in \mathrm{H}_\alpha(M)$, $y \in \mathrm{H}_\gamma(M)$ with $\gamma < \alpha$, $k \in \mathrm{H}_\gamma(M)$. So by the induction hypothesis $k \in \mathrm{H}_\gamma(K)$. By Lemma 2.1, $\mathrm{H}_1(Y) = \mathrm{H}_1(Ru) \subseteq \mathrm{H}_1(\mathrm{H}_\gamma(M)) = \mathrm{H}_\beta(M)$. Hence $x \in \mathrm{H}_\beta(K)$. This is a contradiction. Therefore K is $(\alpha+1)$-pure in M.

The proof of the second part is similar. □

Let N be a submodule of a σ-QTAG-module M. We will say that N is <u>center of α-purity</u> in M if every complement of N in M is α-pure submodule of M.

Theorem 4.4. *Let N be a σ-closed submodule of a σ-QTAG-module M. Then there exists a complement K of N in M which is not α-pure if and only if the following condition is satisfied:*

($*$) *There are uniform elements $u \in N$ and $v \in M$ such that $u + v$ is uniform and*

(i) $\mathrm{e}(v) > \mathrm{e}(u) = 1$,

(ii) $Rv \cap N = 0$,

(iii) $h(u) + 1 < \alpha$, $h(v) = h(u) < h(u+v)$.

Proof: ($\Rightarrow$) Let $K \cap N = 0$ be maximal and not α-pure in M and let γ be the smallest ordinal $\gamma \le \alpha$ such that $\mathrm{H}_\gamma(M) \cap K \ne \mathrm{H}_\gamma(K)$. Then γ is not a limit ordinal. Write $\gamma = \delta + 1$. We can take $x \in (\mathrm{H}_\gamma(M) \cap K) \setminus \mathrm{H}_\gamma(K)$ uniform. As K is a complement by [T3, Theorem 3.4], $\mathrm{H}_1(M) \cap K = \mathrm{H}_1(K)$. There exists $V_1 \subseteq K$ σ-closed σ-uniserial with $\ell(V_1/U) = 1$, where $U = \operatorname{Cl}^M(Rx)$. Also there is a σ-closed σ-uniserial submodule V of M such that $U \subset U_1 \subset V \subseteq K, V \subseteq \mathrm{H}_\delta(M)$. By Proposition 1.4 there is an isomorphism $\tau : U_1 \to V_1$ extending the identity on U. Since $U_1 \subseteq K$ (otherwise $U \subseteq \mathrm{H}_\alpha(K)$ by the choice of α), [BT1, Lemma 2.16] shows that $R(z - \tau(z))$ is σ-cocritical for any $z \in U_1 \setminus U$. So we can write $z - \tau(z) = u + w$, $u \in N$, $w \in K$, and set $v = z - u = \tau(z) + w \in K$ and (iii) is true. Further, v is uniform as a homomorphic image of z (since $\operatorname{Soc}(M) = \operatorname{Soc}(N) \oplus \operatorname{Soc}(K) \ni u + w$). Since $z - v = u$ is uniform, Lemma 2.1 (iv) gives $U = \mathrm{H}_1(U_1) = \mathrm{H}_1(\operatorname{Cl}^M(Rv))$, showing that $\mathrm{e}(v) > \mathrm{e}(u) = 1$.

Further, $z = u + v$ has the height $h(u+v) \ge \delta$ and to finish this part of the proof it suffices to show that $h(u) < \delta$. However, for $h(u) \ge \delta$ we have $v = z - u \in \mathrm{H}_\delta(M) \cap K = \mathrm{H}_\delta(K)$ and consequently $x \in U = \mathrm{H}_1(\operatorname{Cl}^M(Rv)) \subseteq \mathrm{H}_1(\mathrm{H}_\delta(M)) \subseteq \mathrm{H}_\alpha(M)$, which contradicts the choice o γ. Hence $h(u) < \delta$ and so $h(u) + 1 < \alpha$.

($\Leftarrow$) Let $K \subseteq Rv$ be maximal with respect to $K \cap N = 0$. Denoting $T = \mathrm{H}_1(V)$, where $V = \operatorname{Cl}^M(Rv)$, we have $T = \mathrm{H}_1(\operatorname{Cl}^M(R(u+v)))$ by Lemma 2.1 (iv). Now, $h(u) = h(v) = \beta < h(u+v)$ with $\beta + 1 < \alpha$ and $T \subseteq \mathrm{H}_1(\mathrm{H}_{\beta+1}(M)) \subseteq \mathrm{H}_{\beta+2}(M)$.

We claim that $T \nsubseteq \mathrm{H}_{\beta+2}(K)$.

So assume that $T \subseteq \mathrm{H}_{\beta+2}(K)$ and consider a σ-uniserial σ-closed submodule of K, W, with $T \subseteq W \subseteq \mathrm{H}_{\beta+1}(K)$. By Proposition 1.4 we can obtain $h : V \to W$ extending the inclusion of T in W. By [BT$_1$, Lemma 2.16] $v - h(v) \in \mathrm{Soc}(K)$. Now $u + v - h(v) \in \mathrm{H}_{\beta+1}(M)$. From [BT$_1$, Lemma 3.4] we get $\mathrm{Soc}(M) = \mathrm{Soc}(N) \oplus \mathrm{Soc}(K)$ and using the projection on $\mathrm{Soc}(N)$, Lemma 4.1 (iv) gives $u \in \mathrm{H}_{\beta+1}(M)$, which contradicts the choice of β and finishes the proof. □

Theorem 4.5. *Let N be a submodule of a σ-QTAG-module M and $T_\gamma = \mathrm{Soc}(\mathrm{H}_\gamma(M))$ for any ordinal γ. If either $\mathrm{Soc}(N) \subseteq T_\delta$, for some ordinal δ such that $\alpha \le \delta + 1$ or for some ordinal β with $\beta + 1 < \alpha$,*

$$T_{\beta+2} \subseteq \mathrm{Soc}(N) \subseteq T_\beta$$

then N is a center of α-purity. If M is homogeneous the converse is true.

Proof: For some β with $\beta+1 < \alpha$, we have $T_{\beta+2} \subseteq \mathrm{Soc}(N) \subseteq T_\beta$ and suppose that we have elements u, v from Theorem 4.4. Denote $V = \mathrm{Cl}^M(Rv)$, $W = \mathrm{Cl}^M(R(u + v)) \subseteq \mathrm{H}(M)$. By Lemma 2.1 (iv), $\mathrm{H}_1(V) = \mathrm{H}_1(W)$ and so $\mathrm{Soc}(V) \subseteq \mathrm{H}_1(V) \subseteq \mathrm{H}_1(\mathrm{H}_{\beta+1}(M)) \subseteq \mathrm{H}_{\beta+2}(M) \subseteq \mathrm{Soc}(N)$. Hence $Rv \cap N \neq 0$, which contradicts (ii).

To prove the converse assume that the condition is not satisfied. Then $\mathrm{Soc}(N)$ is not contained in T_δ for any ordinal δ such that $\delta + 1 \ge \alpha$. Let γ be the smallest ordinal such that $\mathrm{Soc}(N)$ is not contained in T_γ. Then $\gamma + 1 \le \alpha$ and γ is not a limit ordinal. Write $\gamma = \beta + 1$. We have $\beta + 1 < \alpha$, $\mathrm{Soc}(N) \subseteq T_\beta$, but $\mathrm{Soc}(N)$ is not contained in $T_{\beta+1}$, consequently there exist σ-cocritical submodules $U \subseteq \mathrm{Soc}(N) \subseteq \mathrm{H}_\beta(M)$ and $S \subseteq \mathrm{H}_{\beta+2}(M)$ but not contained in N. Thus there is $S \subset X \subseteq \mathrm{H}_{\beta+1}(M)$ with $\ell(X) = 2$. By Lemma 3.4, $V = \mathrm{Cl}^M(R(x - v)) \subset X \oplus U$ and since V is not contained in $\mathrm{Soc}(M)$, V is σ-uniserial of σ-length 2. Denoting $v = x - u$, we have $\mathrm{H}_1(V) = S = \mathrm{H}_1(X)$ by Lemma 2.1 (iv). Hence $Rv \cap N = 0$. Moreover $h(u) = h(v) = \beta < \beta + 1 \le h(u + v)$, $\mathrm{e}(v) = \mathrm{e}(x)$ and we finish the proof by applying Theorem 4.4. □

Example 4.6. Let $M = \langle a\rangle \oplus \langle b\rangle$ be a direct sum of cyclic groups, a be of order p^3, b order $q \neq p$ and denote $N = \langle b\rangle$. Then $T_0 = \langle p^2 a\rangle \oplus \langle b\rangle = \mathrm{Soc}(M)$, $T_1 = T_2 = \langle p^2 a\rangle$, $T_3 = 0$, T_2 is not contained in the $\mathrm{Soc}(N)$, $\mathrm{Soc}(N) \subseteq T_0$. So the condition is not satisfied, but N is a center of purity: Let K be any complement, $K \cap N = 0$. If no element of the form $\lambda a + \mu b$, $\mu b \neq 0$ is in K, then $K = \langle a\rangle$. If some element of the form $\lambda a + \mu b$, $\mu b \neq 0$, lies in K, then $p^3(\lambda a + \mu b) = p^3 \mu b \in K$. So $\alpha p^3 \mu + \beta q = 1$ gives $b = \alpha p^3 \mu b \in K$, which is impossible.

We see that the converse in Theorem 4.5 does not hold in general.

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Department of Algebra, Faculty of Mathematics and Physics, Charles University, 186 00 Praha 8, Czechoslovakia

Departamento de Algebra, Facultad de Ciencias de Almeria, Universidad de Granada, 04071 Almeria, Spain

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