Čech–Stone-like compactifications for general topological spaces

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Abstract. The problem whether every topological space X has a compactification Y such that every continuous mapping f from X into a compact space Z has a continuous extension from Y into Z is answered in the negative. For some spaces X such compactifications exist.

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By a space we always mean a topological space. In this paper, compact spaces are regarded without any separation axiom, so that they are such spaces that every their open cover contains a finite subcover (they are called quasi-compact in [En]). A compactification of a space X is a compact space containing X as a dense subspace. We shall explore compactifications from the point of view of continuous extensions of continuous mappings. Such a point of view is related to the concept *reflection* of a space X in a subclass $\mathcal{C}$ of topological spaces: it is a space rX from $\mathcal{C}$ and a continuous mapping $r : X \to rX$ such that for every continuous mapping $f : X \to Y$, $Y \in \mathcal{C}$, there exists a unique continuous mapping $g : rX \to Y$ such that $g \circ r = f$. If we omit the word "unique" from the previous definition, we get the concept *weak reflection*. The reflectivity is rather strong, the weak reflectivity is rather weak, and there are some modifications of the above definitions, for instance existence of a functor F into $\mathcal{C}$ and a natural transformation $r : 1 \to F$.

If X has a (weak) reflection $r : X \to rX$ in the class of compact spaces, then r is an embedding and rX may be found as a compactification of X. The Čech–Stone compactification βX of a completely regular Hausdorff space X is the reflection of X in the class of compact Hausdorff spaces; composing it with the reflection into completely regular Hausdorff spaces, we get a reflection of any space in compact Hausdorff spaces. If we do not need the uniqueness of the continuous extensions, the Čech–Stone compactification realizes a weak reflection of any space in compact regular spaces. Of course, the reflection mapping r is an embedding only for completely regular Hausdorff spaces in the former case and for the completely regular spaces in the latter case. It is easy to show that there are continuous mappings from X into a compact (nonregular) space which cannot be continuously extended to βX.

The Wallman compactification ωX is usually constructed for T_1-spaces X, but the T_1-axiom is not needed in its definition and the proof of its basic properties; one of the properties asserts that every continuous mapping from X into a compact regular space Y can be continuously extended to ωX (see e.g. [En] for T_1-spaces). Again, there are continuous mappings from X into a compact (nonregular) space which cannot be continuously extended to ωX (see e.g. [Ha]).

Thus, it is quite natural to ask: *Is there a compactification γX of X such that every continuous mapping from X into any compact space Y can be continuously extended to γX?* In other words: *Is the class of compact spaces weakly reflective in the class of topological spaces?* It is probably impossible to find out who first asked that question. I remember that 25 years ago Z. Frolík mentioned it by some occasion; then I did not hear it for a long time and about two years ago, J. Adámek and J. Rosický came with it again (oral communication). Now, the problem is stated explicitly in [AR] and [He].

The occasion when we spoke about the problem with Z. Frolík was an interest in improving some results on extension of various mappings onto compactifications. I used some categorical methods (extension of functors) to get functors from various categories into the category of compactifications (in fact, into a more general category of extensions) — see [Hu_1]. Those categorical methods could solve the problem positively only. In [Hu_2, Example 3], it was proved that the answer to the problem is in the negative if one considers closure spaces in the sense of [Če] instead of topological spaces (even if one requires extensions of mappings into compact Hausdorff closure spaces only), but some spaces (e.g. those having finitely many nonconverging ultrafilters only) have the requested compactification. In the next part of this paper we will show that the answer for topological spaces is similar as for closure spaces: it is in the negative but there exist noncompact spaces X having the requested compactification γX. For normal spaces such situations are fully characterized. At first we shall describe some spaces having a weak reflection in compact spaces (Theorem 1) and then some spaces having no weak reflection in compact spaces (Theorem 2).

As far as I know, most recent constructions suggested for the negative solution of the problem used the topological modification of the closure space constructed in [Hu_2], i.e., adding ultrafilters; maybe, that approach works but it is probably difficult to manage it. The construction we use in this paper, is a modification of the example from [GH] (it was used to produce non-co-well-poweredness of a certain class of spaces): instead of uncountable families it suffices to use countable families (the same modified space was used in [GS] to produce non-co-well-poweredness of another class of spaces).

Now, we shall repeat some basic facts concerning the Wallman compactification ωX of X. As a set, $\omega X = X \cup \{\mathcal{F} : \mathcal{F}$ is a free maximal centered collection of closed sets in $X\}$. An open base of ωX consists of $G \cup \{\mathcal{F} \in \omega X - X : F \subset G$ for some $F \in \mathcal{F}\}$, G open in X. Then $\omega X - X$ is called the *Wallman remainder* and every its point is closed in ωX.

Theorem 1. *If the Wallman remainder of X is finite, then the Wallman compactification of X is the weak reflection of X in compact spaces.*

PROOF: Let $|\omega X - X| < \omega, f : X \to Y$ be continuous, Y be compact. For $x \in \omega X - X$ put $\tilde{f}x$ to be an accumulation point of $\{F : F$ is closed in $Y, f^{-1}(F) \in x\}$, for $x \in X$ define $\tilde{f}x = fx$. We shall prove that $\tilde{f} : \omega X \to Y$ is continuous. Clearly, $\tilde{f}$ is continuous on X since the restriction of $\tilde{f}$ to X coincides with f and X is open in ωX. Take $x \in \omega X - X$ and an open set G in Y containing $\tilde{f}x$. Then $f^{-1}(G)$ is open in X. Since there is some $F \in x$ such that $F \subset f^{-1}(G)$ (otherwise, $X - f^{-1}(G) \in x$, which contradicts the fact that $\tilde{f}x \in G$), we can choose an open subset U of $f^{-1}(G)$ such that $X - U$ belongs to all elements of $\omega X - X$ but not to x. Then $U \cup (x)$ is a neighborhood of x in ωX and $\tilde{f}$ maps this neighborhood into G. □

Theorem 2. *If X contains an infinite family $\{X_n\}$ of closed noncompact subsets such that $X_n \cap X_m$ is compact for $n \neq m$, then X has no weak reflection in compact spaces.*

Proof: For any infinite cardinal κ define $Z_\kappa = X \cup (\kappa \times \omega)$. Let $\{N_n\}$ be a partition of ω with $|N_n| = \omega$ for every $n \in \omega$. The topology on Z_κ will be defined transfinitely on κ such that

$$\alpha < \kappa \Rightarrow X \cup (\alpha \times \omega) \text{ is open in } Z_\kappa$$

(hence X is an open subset of Z_κ):

(1) A neighborhood base of $(0, n)$ is composed of the sets $(0, n) \cup (X - (C \cup \bigcup_K X_i)$ for finite $K \subset \omega$ with $n \notin K$, and for closed compact sets C in X.
(2) A neighborhood base of $(\beta+1, n)$ is composed of the sets $(\beta+1, n) \cup \bigcup\{V_x : x \in (\beta) \times N_n - F\}$ for finite sets F, and for neighborhoods V_x of x.
(3) A neighborhood base of (α, n), for α limit, is composed of the sets $(\alpha, n) \cup \bigcup\{V_{(\beta,n)} : \gamma < \beta < \alpha\}$ for $\gamma < \alpha$, and for neighborhoods V_x of x.

Claim: *Let S be either of the following three subsets of Z_κ for some $k \in \omega, 1 \leq \delta < \kappa : (0, k) \cup X_k, (\delta, k) \cup (\delta - 1) \times N_k$ for isolated δ, $\{(\beta, k) : \beta \leq \delta\}$ for limit δ. Then S is closed compact in Z_κ.*

Proof of Claim: The compactness is clear in all three cases (the first two spaces are one-point compactifications, the last space is homeomorphic to a space of ordinals). The proof of closedness will proceed by transfinite induction (take $z \in Z_\kappa - S$):

(i) If $z \in X$ then either $X - X_k$ or X is a neighborhood of z disjoint with S.
(ii) If $z \in (0) \times \omega$, then either $(z) \cup X - X_k$ or $(z) \cup X$ is a neighborhood of z disjoint with S.
(iii) If $z = (\alpha, n), \alpha > 0$, then the respective sets $(\alpha - 1) \times N_n - F$ or $\{(\beta, n) : \gamma < \beta < \alpha\}$ from (2) and (3) can be chosen disjoint with S (in the latter case, either $\delta < \alpha$ and then the choice $\gamma = \delta$ works, or $\delta \geq \alpha$ and then $k \neq n$ and $\delta = 0$ works). The corresponding neighborhoods V_x can be chosen disjoint with S by the induction hypothesis.

The proof of Claim is finished.

Suppose now that X has a weak reflection rX in compact spaces and take a cardinal κ bigger than the cardinality of rX. The identity mapping of X extends continuously to a mapping from rX into the one-point compactification of Z_κ; denote the image by Y — it is a compact space containing X. We shall prove that Y contains the whole space Z_κ, which is impossible by our cardinality assumption. Indeed, if there is the least $\delta < \kappa$ such that for some $k \in \omega$ we have $(\delta, k) \notin Y$, then by our Claim, $S \cap Y$ (S is the set from Claim corresponding to δ, k) is closed in Y and hence compact, which is a contradiction because $S \cap Y$ is one of the following sets: $X_k, (\delta) \times N_k$ for isolated δ, $\{(\beta, k) : \beta < \delta\}$ for limit δ. $\square$

The following Corollary was deduced from Theorem 2 in a joint discussion at Math. Dept. of Kansas St. Univ. also with colleagues from Univ. of Kansas.

Corollary 1. *If the Wallman remainder of X contains an infinite discrete subspace, then X has no weak reflection in compact spaces.*

Proof: Let $\{x_n\}$ be a countable discrete subspace of $\omega X - X$; take a sequence $\{U_n\}$ of basic open neighborhoods U_n of x_n in ωX such that $x_m \notin U_n$ for $n \neq m$. For every n there is some $A_n \in x_n$ with $A_n \subset U_n$. Put $F_n = A_n - \bigcup_{i<n} U_i$ for $n \in \omega$. Then the sets F_n are disjoint, closed and noncompact. The noncompactness follows from the fact that $F_n \in x_n$ since otherwise there exists $B_n \in x_n$ disjoint with F_n, hence $B_n \cap A_n \in x_n$, $B_n \cap A_n \subset \bigcup_{i<n} U_i$, but $X - U_i \in x_n$ for each $i \neq n$ and $A_n \cap B_n \cap (X - \bigcup_{i<n} U_i) = \emptyset$. $\square$

By the Čech–Stone remainder of a topological space X we mean the Čech–Stone remainder $\beta cX - cX$ of the completely regular T_1-modification cX of X.

Corollary 2. *If the Čech–Stone remainder of X is infinite, then X has no weak reflection in compact spaces.*

Proof: There is a canonical surjection $g : \omega X \to \beta cX$ extending the canonical mapping $X \to \beta cX$. If the Čech–Stone remainder of X is infinite, it contains an infinite discrete subspace (since βcX is Hausdorff) and, hence, also $\omega X - X$ contains an infinite discrete subspace and we may use Corollary 1. $\square$

As the following example and Corollary show, Corollary 2 can be converted for normal spaces only.

Example 1. There is a completely regular T_1-space X with $|\beta X - X| = 1$, having no weak reflection in compact spaces.

Take $X = [0,1]^{\omega_1} - \{0\}$ (by $\{0\}$ we mean the point with all the coordinates equal to 0). Then $\beta X = [0,1]^{\omega_1}$, and the edges X_n (i.e., the subsets of X of those points having all the coordinates 0 except the n-th one which is in $]0,1]$) fulfil the conditions of Theorem 2.

Corollary 3. *A normal T_1-space has a weak reflection in compact spaces iff its Čech–Stone remainder is finite.*

Proof: If X is normal T_1, then the Wallman compactification of X coincides with the Čech–Stone compactification of X. $\square$

In their preprint [DW], A. Dow and S. Watson constructed another compactification of X than ωX having the property of continuous extension for continuous mappings into compact Hausdorff (or regular) spaces; their compactification is generated by a four-point space. The authors also mention a modified problem (by S. Todorćevič): *Does there exist a space U such that every topological space X has a compactification γX embeddable into a power of U such that every continuous* *mapping from X into a compact T_1-space has a continuous extension onto γX?* If we start with a T_1-space X in the proof of Theorem 2, we get a T_1-space Z_κ and its one-point compactification is T_1, too. So, the Todorćevič' problem has the negative answer.

Corollary 4. *There are T_1-spaces having no weak reflection in the class of compact T_1-spaces.*

Theorem 1 implies that every space with finite Wallman remainder has a weak reflection in compact T_1-spaces (it is the Wallman compactification of the T_1-modification of the space). The following trivial example shows that a space may have a weak reflection in compact T_1-spaces but not in compact spaces.

Example 2. Take the half line $[0, \rightarrow[$ endowed with the following topology: the basic neighborhood of p is $[0, p]$. Define X as the sum of countably many copies of such half lines sewed together at the point 0. Then X has no weak reflection in compact spaces by Theorem 2, but it has a weak reflection in compact T_1-spaces, namely the T_1-modification of X, which is a singleton.

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