# On entropy-like functionals and codes for metrized probability spaces II

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*Abstract.* In Part I, we have proved characterization theorems for entropy-like functionals $\delta$, $\lambda$, $E$, $\Delta$ and $\Lambda$ restricted to the class consisting of all finite spaces $P \in \mathfrak{W}$, the class of all semimetric spaces equipped with a bounded measure. These theorems are now extended to the case of $\delta$, $\lambda$ and $E$ defined on the whole of $\mathfrak{W}$, and of $\Delta$ and $\Lambda$ restricted to a certain fairly wide subclass of $\mathfrak{W}$.

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In Part I of this paper, published in 1990 (see[1]), we have introduced the functionals $\delta$, $\lambda$, $E$, $\Delta$ and $\Lambda$, defined on $\mathfrak{W}$, the class of all semimetrized measure spaces, by means of a suitably extended and modified concept of a code. It has been shown that these functionals restricted to $\mathfrak{W}_F$, the class of all finite $P \in \mathfrak{W}$, can be characterized as the largest ones satisfying certain simple conditions.

In Part II we prove that the corresponding theorems remain valid, with certain modifications, for $\delta$, $\lambda$ and $E$ defined on all semimetrized measure spaces. As for $\Delta$ and $\Lambda$, we also prove characterization theorems, though only for $\Delta$ and $\Lambda$ restricted to a certain subclass of $\mathfrak{W}$.

For reasons not connected with mathematics, this Part II has been written two years later than Part I and appears only now. In view of this fact, it seems necessary to recall a number of definitions from Part I, correcting at the same time several misprints and minor errors, and also adding some further definitions. This is done in Section 5, the first section of this Part II.

Section 6 contains several lemmas and the proof of the fact that the entropies $\widehat{E}$, $E$ and $E^*$, defined in different ways, do coincide on $\mathfrak{W}$. Section 7 contains the characterization theorems for $\delta$, $\lambda$ and $E$ defined on $\mathfrak{W}$. In Section 8, we prove the restricted versions of characterization theorems for $\Delta$ and $\Lambda$.

## 5.

In this section, some definitions and notational conventions (and also two lemmas) from Part I are restated, in particular if the pertinent formulations in Part I contain a misprint or error (there is a number of these; fortunately, they do not affect the subsequent propositions). We also introduce some additional concepts (see 5.2, 5.5–5.9 below). The terminology and notation of Part I is retained with few exceptions explicitly stated (see 5.2, 5.6 and 5.9).

**5.1.** We list some terms and notational conventions from Part I, referring to relevant passages in Part I or to the definitions restated in the present section. — The basic notation is contained in 1.1–1.3. The definitions of semimetric spaces, $W$-spaces and some related concepts are in 1.4.–1.8 (for a concept of a subspace see, however, 5.2 below). Recall that the class of all semimetric spaces (respectively, $W$-spaces) is denoted by $\mathfrak{S}$ (respectively, $\mathfrak{W}$). The diameter of a space was defined in 1.9. — For Hamming spaces, codes, etc. see 5.10 below. — The symbols $\varepsilon * P$ and $\varepsilon \odot P$ are defined in 1.17. Recall that if $P = \langle Q, \varrho \rangle \in \mathfrak{S}$ or $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}$ and $\varepsilon > 0$, then $\varepsilon \odot P$ is $\langle Q, \sigma \rangle$ or $\langle Q, \sigma, \mu \rangle$, respectively, where $\sigma(x, y) = \varrho(x, y)$ if $\varrho(x, y) > \varepsilon$ and $\sigma(x, y) = 0$ if $\varrho(x, y) \leq \varepsilon$.

For $\delta P$, $\lambda P$, etc. see 2.8, for $\delta f$ and $\lambda f$ see 1.20; for $E(\varepsilon, P)$, $\widehat{E}(P)$, etc. see 5.14 below. The definitions of a strictly branching and well-fitting code are restated in 5.15. For dyadic expansions see 2.23, 2.24 and also 5.9 below.

**5.2.** The definition (1.4) of a subspace of $P \in \mathfrak{S}$ is retained. However, for $W$-spaces we will have subspaces in a wide and in a narrow sense; the latter will be called pure.

**Definition.** Let $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}$. If $S \in \mathfrak{W}$, $S = \langle Q, \varrho, \nu \rangle$, $\operatorname{dom} \nu = \operatorname{dom} \mu$ and $\nu \leq \mu$, we will say that $S$ is a <u>subspace</u> of $P$ in the <u>wide sense</u>, abbreviated <u>subspace (w.s.)</u>. If $S = \langle Q, \varrho, \nu \rangle$ is a subspace (w.s.) of $P = \langle Q, \varrho, \mu \rangle$ and there is a $\bar{\mu}$-measurable set $T \subset Q$ such that $\nu(X) = \bar{\mu}(X \cap T)$ for all $X \in \operatorname{dom} \mu$, we will say that $S$ is a <u>pure subspace</u> of $P$. — Thus a subspace (of $P \in \mathfrak{W}$) in the sense of 1.7 is now called a pure subspace.

**5.3. Notation.** Let $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}$. If $f : Q \to R$ is $\bar{\mu}$-measurable, $\bar{\mu}\{q \in Q : fq < 0\} = 0$ and $\int_Q f \, d\mu < \infty$, then $f \cdot \mu$ will denote the measure $X \mapsto \int_X f \, d\mu$ defined on $\operatorname{dom} \mu$, and $f \cdot P$ will denote the $W$-space $\langle Q, \varrho, f \cdot \mu \rangle$. — Clearly $f \cdot P$ is a subspace (w.s.) of $P$ iff $\bar{\mu}\{q \in Q : fq > 1\} = 0$.

If $T \subset Q$ is $\bar{\mu}$-measurable, then $T \cdot \mu$ will denote the measure $X \mapsto \bar{\mu}(T \cap X)$ defined on $\operatorname{dom} \mu$, and $T \cdot P$ will denote the pure subspace $\langle Q, \varrho, T \cdot \mu \rangle$ of $P$.

**5.4. Fact.** If $S$ is a subspace (w.s.) of $P \in \mathfrak{W}$, then there is a function $f$ such that $S = f \cdot P$.

**5.5. Notation.** If $P_t = \langle Q, \varrho, \nu_t \rangle$, $t \in T$, $T$ finite, are subspaces (w.s.) of a $W$-space, then $\Sigma(P_t : t \in T)$ will denote the $W$-space $\langle Q, \varrho, \Sigma(\nu_t : t \in T) \rangle$.

**5.6.** A partition of a semimetric space is defined in the usual way (see 1.8). For $W$-spaces, we introduce partitions in a wide and in a narrow sense; the latter will be called pure.

**Definition.** Let $P \in \mathfrak{W}$. If $U_t$, $t \in T$, $T$ finite, are subspaces (w.s.) of $P$ and $\Sigma(U_t : t \in T) = P$, then $\mathcal{U} = (U_t : t \in T)$ will be called a <u>partition</u> of $P$ in the <u>wide sense</u>, abbreviated <u>partition (w.s.)</u>. If, in addition, $U_t$ are pure subspaces of $P$, then $\mathcal{U}$ will be called a <u>pure partition of $P$</u>. — Thus partitions (of $P \in \mathfrak{W}$) in the sense of 1.8 are now called pure partitions. Observe that, e.g., "$\mathcal{U}$ is a pure partition of $P \in \mathfrak{S} \cup \mathfrak{W}$" means that either $P \in \mathfrak{S}$ and $\mathcal{U}$ is a partition of $P$ or $P \in \mathfrak{W}$ and $\mathcal{U}$ is a pure partition of $P$.

**5.7. Notation.** If $P \in \mathfrak{S} \cup \mathfrak{W}$, $\mathcal{U} = (U_t : t \in T)$, $T$ finite, and $U_t$ are subspaces of $P$ (subspaces (w.s.) if $P \in \mathfrak{W}$), then we put $d(\mathcal{U}) = \max(d(U_t) : t \in T)$.

**5.8.** In Part I, we used the concept of a totally bounded space $P \in \mathfrak{S} \cup \mathfrak{W}$ without giving the definition. It will now be stated explicitly.

**Definition.** A space $P \in \mathfrak{S} \cup \mathfrak{W}$ is called totally bounded, if $d(P) < \infty$ and, for every $\varepsilon > 0$, there is a pure partition $\mathcal{U}$ of $P$ with $d(\mathcal{U}) < \varepsilon$.

**5.9.** Similarly as with subspaces and partitions, we will have, for $W$-spaces, dyadic expansions in a wide and in a narrow sense, whereas the concept of a dyadic expansion of a semimetric space or of a set remains unchanged (see 2.23).

**Definition.** Let $P$ be a $W$-space. Let $D$ satisfy the conditions stated in 2.23 and let $D'$ and $D''$ have the meaning described in 2.23. We will say that $\mathcal{P} = (P_u : u \in D)$ is a dyadic expansion of $P$ in the wide sense, abbreviated dyadic expansion (w.s.) or merely d.e. (w.s.), if $P_u$ are subspaces (w.s.) of $P$, $P_{u0} + P_{u1} = P_u$ for all $u \in D'$ and $P_\emptyset = P$. If, in addition, all $P_u$ are pure subspaces, then $\mathcal{P}$ will be called a pure dyadic expansion, abbreviated pure d.e. — Observe that a dyadic expansion (of $P \in \mathfrak{W}$) in the sense of 2.23 will now be called a pure dyadic expansion.

In 2.24, we have introduced, for any $FW$-space $P$ and any pure d.e. $\mathcal{Z} = (P_u : u \in D)$ of $P$, the symbol $E(P, \mathcal{Z})$ denoting $\Sigma(H(wP_{u0}, wP_{u1}) d(P_u) : u \in D')$. This notation will now be extended: for a dyadic expansion (w.s.) $\mathcal{Z} = (P_u : u \in D)$ of a $W$-space $P$, we put $E(\mathcal{Z}) = E(P, \mathcal{Z}) = \Sigma(H(wP_{u0}, wP_{u1}) d(P_u) : u \in D')$. — Later on (see 7.7), we will also introduce the notation $\lambda\mathcal{Z}$, $\delta\mathcal{Z}$, where $\mathcal{Z}$ is a dyadic expansion.

**5.10.** In the present Part II, only one Hamming space, namely $K_\infty$ (see 1.11 and 1.12) and only $\varepsilon$-codes in $K_\infty$ will be considered. We state the pertinent definitions restricted to this special case.

We put $A = \{0,1\} \times R_+$, $K_\infty = \langle A^*, \pi, \lambda \rangle$, where $A^* = \bigcup(A^n : n \in N)$ and, for every $x \in A$, $x = \langle \pi x, \lambda x \rangle$. We put $|K_\infty| = A^*$. If $x, y \in A^*$, $x = (x_i : i < m)$, $y = (y_j : j < n)$, we put $\tau(x,y) = \Sigma(\lambda x_i \wedge \lambda y_i : i < m \wedge n, x_i \neq y_i)$. Then $K_\infty$ is a Hamming space in the sense of 1.11 and $\tau$ is a semimetric on $|K_\infty|$.

**5.11.** We now restate the definition of an $\varepsilon$-code (in $K_\infty$). If $P \in \mathfrak{S} \cup \mathfrak{W}$, $\varepsilon \geq 0$, then a mapping $f : |P| \to A^*$ will be called an $\varepsilon$-code (or an approximative code) of $P$ (in $K_\infty$), if (1) $fP$ is finite, (2) if $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}$, then all $f^{-1}u$, $u \in fP$, are $\bar{\mu}$-measurable, (3) if $u, v \in fP$, then $d(f^{-1}\{u,v\}) \leq \tau(u,v) \vee \varepsilon$, (4) if $u \cdot (a), u \cdot (b) \in [fP]$, $\pi a = \pi b$, then $a = b$ (recall that, for any ordered set $X$ and any $Y \subset X$, $[Y]$ denotes the set of all $x \in X$ such that $x \prec y$ for some $y \in Y$). — A 0-code will be called an exact code.

**5.12.** To introduce the concept of a regular $\varepsilon$-code, some notation from 2.2 and 2.3 is needed. For the reader's convenience, we restate the pertinent notational conventions, correcting some misprints occurring in 2.2. Let us note that the notation described below concerns an arbitrary $M^*$ and an arbitrary semimetric on $M^*$; however, it will be used only for the case of $M = A$ and $\varrho = \tau$ (see 5.10).

If $M$ is a set, $S \subset M^*$, $x \in [S]$, then

(I) $br(x, S)$ denotes the set of all $b \in M$ such that $x \cdot (b) \in S$;

(II) $Br(x, S)$ denotes the set of all $z \in M^*$ such that
   (1) $x \cdot z \in [S]$, $|z| \geq 1$,
   (2) $br(x \cdot z', S)| = 1$ whenever $z' \prec z$, $\emptyset \neq z' \neq z$,
   (3) $br(x \cdot z, S)| \neq 1$;

(III) if $u \in [S]$, $x \prec u$, $x \neq u$, then $\beta(x, u, S)$ denotes the (unique) $z \in M^*$ such that
   (1) $x \cdot z \prec u$, $|z| \geq 1$,
   (2) $|br(x \cdot z', S)| = 1$ if $z' \prec z$, $\emptyset \neq z' \neq z$,
   (3) $x \cdot z = u$ or $|br(x \cdot z, S)| \neq 1$;

   if $x = u$, we put $\beta(x, u, S) = \emptyset$.

If $M$ is a set, $S \subset M^*$, $\varrho$ is a semimetric on $M^*$, then we put, for $x, y \in [S]$, $\varrho'_S(x, y) = \varrho(x', y')$, where $x' = \beta(x \wedge y, x, S)$, $y' = \beta(x \wedge y, y, S)$. Then $\varrho'_S$ is a semimetric on $[S]$, denoted often simply by $\varrho'$. — If $X \subset [S]$, we put $d'(X) = d'_S(X) = d(\langle X, \varrho' \rangle)$.

**5.13.** An $\varepsilon$-code of a space $P \in \mathfrak{S} \cup \mathfrak{W}$ is called <u>regular</u> (see 2.4), if $d(f^{-1}\{u, v\}) \leq d'(Br(s, fP)) \vee \varepsilon$ whenever $u, v \in fP$, $s \prec u \wedge v$ and $|br(s, fP)| \neq 1$.

**5.14.** The notation from 2.11. 2.12 and 2.13 will now be given.

Let $f$ be an $\varepsilon$-code of $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}$. Then

(1) $B(f) = \{u \in [fP] : |br(u, fP)| = 2\}$;

(2) if $u \in B(f)$, then $E(u, f) = H(\bar{\mu}S, \bar{\mu}T) \cdot \tau'(s, t)$, where$Br(u, fP) = \{s, t\}$, $S = \{x \in P : u \cdot s \prec fx\}$, $T = \{x \in P : u \cdot t \prec fx\}$;

(3) $E(f) = \Sigma(E(u, f) : u \in B(f))$;

(4) for $\varepsilon > 0$, $E(\varepsilon, P) = \inf(E(f) : f$ is a regular $\varepsilon$-code of $P)$;

(5) $\widehat{E}(P) = \sup(E(\varepsilon, P) : \varepsilon > 0)$.

Note that $\widehat{E}(P)$ is called the coding entropy (or simply the entropy) of $P$.

**5.15.** Recall (see 2.17) that an $\eta$-code $f$ of $P \in \mathfrak{S} \cup \mathfrak{W}$ is called (1) strongly branching, if $B(f) = [fP] \setminus fP$, (2) well-fitting, if, for every $u \in B(f)$, $d\{x \in P : u \prec fx\} = d'(Br(u, fP)) = \lambda(s)$, where $s \in Br(u, fP)$.

**5.16.** Since there is an error (not affecting the subsequent assertions) in 2.18, we state it here in the correct form.

**Fact.** Every strongly branching well-fitting $\varepsilon$-code of a space $P \in \mathfrak{S} \cup \mathfrak{W}$ is regular. If, in addition, $d(f^{-1}u) = 0$ for all $u \in fP$, then $f$ is exact.

**5.17.** In 2.20, there are also some misprints and errors in the formulation of the lemma and in its proof. Therefore, we now state the lemma in a modified form and present its proof. Let us note that some technical details of the proof are omitted. Recall that $\lambda f = \int (\lambda \circ f)\, d\mu$ if $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}$.

**Lemma.** *Let $f$ be a regular $\varepsilon$-code of $P \in \mathfrak{S} \cup \mathfrak{W}$. Then there exists a strongly branching, well-fitting regular $\varepsilon$-code $g$ of $P$ such that*

(1) *for every $x \in P$, $\lambda(gx) \leq \lambda(fx)$,*

(2) $\delta g \leq \delta f$,
(3) *if* $P \in \mathfrak{W}$, *then* $\lambda g \leq \lambda f$, $E(g) \leq E(f)$,
(4) *there is a bijection* $\psi : (fP \cup B(f)) \to [gP]$ *such that*
  (a) *for all* $u_1, u_2 \in fP \cup B(f)$, $u_1 \prec u_2$ *iff* $\psi u_1 \prec \psi u_2$, *hence* $\psi(fP) = gP$,
  (b) $g^{-1}(\psi v) = f^{-1} v$ *for all* $v \in fP$.

Proof: I. We are going to construct a strongly branching regular $\varepsilon$-code $h$ of $P$ and a mapping $\psi$ such that (1)–(4) are satisfied (with $h$ instead of $g$).

Let $M$ denote the set of all $\langle u, s\rangle$ such that $u \in B(f)$, $s \in Br(u, fP)$. Choose a mapping $\eta : M \to \{0, 1\}$ such that if $\langle u, s\rangle$, $\langle u, t\rangle \in M$, $s \neq t$, then $\eta\langle u, s\rangle \neq \eta\langle u, t\rangle$.

For every $u \in fP \cup B(f)$, we define $\psi u$ as follows. Let $(z_i : i < k) = (z_i(u) : i < k(u))$ be the strictly increasing sequence of all $z \in fP \cup B(f)$ such that $z \prec u$. For each $i < k$ there is exactly one $s_i = s_i(u)$ such that $z_i s_i = z_{i+1}$. For $i < k$, we put $v_i = \langle \eta\langle z_i, s_i\rangle, d'(Br(z_i, fP))\rangle \in A^*$, and we put $\psi u = (v_i : i < k)$. Finally, we put $hx = \psi(fx)$ for every $x \in P$. It is easy to prove that $h$ is a strongly branching regular $\varepsilon$-code of $P$ in $K_\infty$ and that $h$ and $\psi$ satisfy (1)–(4), with $g$ replaced by $h$.

II. Define $g$ as follows. If $x \in P$, $hx = (v_i : i < k)$, put $gx = (u_i : i < k)$, where $u_i \in A$, $\pi u_i = \pi v_i$, $\lambda u_i = d\{x \in P : v_i \prec fx\}$. It is easy to see that $g$ is a strongly branching well-fitting regular $\varepsilon$-code of $P$ satisfying (1)–(4). □

## 6.

**6.1. Notation.** A) If $\mathcal{P} = (P_u : u \in D)$ is a dyadic expansion of a space $P \in \mathfrak{W} \cup \mathfrak{S}$, we put $\mathcal{P}'' = (P_v : v \in D'')$. — B) In this section, (1) if $\mathcal{U}$ is a partition (w.s.) of $P \in \mathfrak{W}$, then $\eta(\mathcal{U})$ denotes the infimum of all $E(\mathcal{P})$, where $\mathcal{P}$ is a d.e. (w.s.) of $P$ such that $\mathcal{P}''$ refines $\mathcal{U}$, (2) if $\mathcal{U}$ is a pure partition of $P \in \mathfrak{W}$, then $\eta^*(\mathcal{U})$ denotes the infimum of all $E(\mathcal{P})$, where $\mathcal{P}$ is a pure d.e. of $P$ such that $\mathcal{P}''$ refines $\mathcal{U}$.

**6.2. Proposition.** *If* $P \in \mathfrak{W}_F$, *then* $E^*(P) = \sup\{\eta^*(\mathcal{U}) : \mathcal{U}$ *is a pure partition of* $P\} = E(P) = \sup\{\eta(\mathcal{U}) : \mathcal{U}$ *is a partition* (w.s.) *of* $P\}$.

Proof: Since $E(P) = E^*(P)$ for every $P \in \mathfrak{W}_F$, by 2.31, we have only to show that $E^*(P) = \sup(\eta^*(\mathcal{U}) : \dots)$, $E(P) = \sup(\eta(\mathcal{U}) : \dots)$.

Let $P = \langle Q, \varrho, \mu\rangle$. Put $\mathcal{V} = (\{q\} \cdot P : q \in Q)$. By definition (2.24), $E^*(P) = \eta^*(\mathcal{V})$. Evidently, $\mathcal{V}$ refines every pure partition $\mathcal{U}$ of $P$, hence $\eta^*(\mathcal{U}) \leq \eta^*(\mathcal{V})$ and therefore the supremum in question is equal to $\eta^*(\mathcal{V})$, hence to $E^*(P)$.

We are going to show that $E(P) = \sup\{\eta(\mathcal{U}) : \mathcal{U}$ is a partition (w.s.) of $P\}$. Clearly, it is sufficient to prove that $E(P) = \sup\{\eta(\mathcal{U}) : \mathcal{U}$ is a partition (w.s.) of $P, \mathcal{U}$ refines $\mathcal{V}\}$. Let $\mathcal{U} = (U_t : t \in T)$ be a partition (w.s.) of $P$ refining $\mathcal{V}$. We can assume that $\mathcal{U}$ is of the form $(b_{qk}\{q\} \cdot P : q \in Q, k \in K(q))$. We denote by $\psi(\mathcal{U})$ the infimum of all $E(\mathcal{P})$, where $\mathcal{P}$ is a d.e. (w.s.) of $P$ and $\mathcal{P}''$ is equal (up to indexing) to $\mathcal{U}$, and we denote by $P \vartriangle \mathcal{U}$ the $FW$-space $\langle T, \sigma, \nu\rangle$, $T = \{\langle q, k\rangle : q \in Q, k \in K(q)\}$, obtained from $P$ by splitting (see 2.28). It is easy to see that $\psi(\mathcal{U})$ is equal to $E^*(P \vartriangle \mathcal{U})$. Hence $\psi(\mathcal{U})$ is equal to $E^*(P)$; this follows easily from $E^*(P) = E(P)$ for $P \in \mathfrak{W}_F$. Since $\eta(\mathcal{U}) = \inf\{\psi(\mathcal{T}) : \mathcal{T}$ refines $\mathcal{U}\}$, we get $\eta(\mathcal{U}) = E^*(P)$ for every partition (w.s.) $\mathcal{U}$ refining $\mathcal{V}$. □

**6.3.** We are going to consider the functionals $E^*$ and $E$ defined on the class of all $W$-spaces. These functionals have been introduced in [2] (the notation in [2] is $C_E^*$ and $C_E$). In Part I of the present article, the definitions of $E^*(P)$ and $E(P)$ have been given only for $P \in \mathfrak{W}_F$ and in a form different from, though equivalent to that in [2]; see 2.24 and 2.28.

The functionals $E^*$ and $E$ on $\mathfrak{W}$ can be defined in various ways. We choose to define them here by transforming 4.29 in [2] to a definition. The advantage of this procedure lies in the fact that $E^*$ and $E$ introduced in this manner are immediately seen (due to 6.2) to coincide on $\mathfrak{W}_F$ with $E^*$ and $E$ introduced in 2.24 and 2.28.

**6.4. Definition.** If $P$ is a $W$-space, we put $E^*(P) = \sup\{\eta^*(\mathcal{U}) : \mathcal{U}$ is a pure partition of $P\}$, $E(P) = \sup\{\eta(\mathcal{U}) : \mathcal{U}$ is a partition (w.s.) of $P\}$.

**6.5.** By 6.2, $E^*$ and $E$ defined above coincide on $\mathfrak{W}_F$ with $E^*$ and $E$ defined in 2.24 and 2.28, respectively. We are now going to prove that, for every $P \in \mathfrak{W}$, $E^*(P)$ and $E(P)$ coincide and are equal to the coding entropy $\widehat{E}(P)$ introduced in 2.13. To this end, we shall need a number of lemmas.

**6.6. Notation and definition.** A $W$-space possessing a partition (w.s.) $\mathcal{U}$ with $d(\mathcal{U}) = 0$ will be called a $W_0$-space. The class of all $W_0$-spaces will be denoted by $\mathfrak{W}_0$.

**6.7. Proposition.** *Let $P$ be a $W$-space, let $\mathcal{U}$ be a partition* (w.s.) *of $P$ and let $d(\mathcal{U}) = 0$. Then $E(P) = \eta(\mathcal{U}) = \inf\{E(\mathcal{P}) : \mathcal{P}$ is a d.e.* (w.s.) *of $P$, $\mathcal{P}''$ refines $\mathcal{U}\}$, and if $\mathcal{U}$ is pure, then $E^*(P) = \eta^*(\mathcal{U}) = \inf\{E(\mathcal{P}) : \mathcal{P}$ is a pure d.e. of $P$, $\mathcal{P}''$ refines $\mathcal{U}\}$.*

Proof: We prove only the equality for $E^*(P)$ since the proof of the first equality is analogous. Clearly, it is sufficient to show that $\eta^*(\mathcal{V}) \le \eta^*(\mathcal{U})$ for every pure partition $\mathcal{V}$ of $P$. We can assume that $\eta^*(\mathcal{U}) < \infty$. Let $\varepsilon > 0$. Choose a pure d.e. $\mathcal{P} = (P_u : u \in D)$ of $P$ such that $\mathcal{P}''$ refines $\mathcal{U}$, $E(\mathcal{P}) \le \eta^*(\mathcal{U}) + \varepsilon$. It is easy to see that there is a pure d.e. $\widehat{\mathcal{P}}$ of $P$ such that $\widehat{\mathcal{P}} \supset \mathcal{P}$ (i.e., $\widehat{\mathcal{P}} = (\widehat{P}_u : u \in \widehat{D})$, $D \subset \widehat{D}$, $\widehat{P}_u = P_u$ for $u \in D$) and $\widehat{\mathcal{P}}''$ refines $\mathcal{V}$. Since $d(\widehat{P}_u) = 0$ for $u \in \widehat{D} \setminus D'$, we have $E(\widehat{\mathcal{P}}) = E(\mathcal{P})$, and therefore $E(\widehat{\mathcal{P}}) \le \eta^*(\mathcal{U}) + \varepsilon$. This implies $\eta^*(\mathcal{V}) \le \eta^*(\mathcal{U}) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $\eta^*(\mathcal{V}) \le \eta^*(\mathcal{U})$. □

**6.8.** Proposition 6.7 has the following straightforward corollary.

**Proposition.** *If $P$ is a $W$-space, then $E(P) = \inf\{E(\mathcal{P}) : \mathcal{P}$ is a d.e.* (w.s.) *of $P$, $d(\mathcal{P}'') = 0\}$, $E^*(P) = \inf\{(\mathcal{P}) : \mathcal{P}$ is a pure d.e. of $P$, $d(\mathcal{P}'') = 0\}$, $E^*(P) \ge E(P)$.*

**6.9.** For $\delta$ and $\lambda$, there also are propositions analogous to 6.7 and 6.8. First, we restate the relevant definitions (cf. 2.8) in a simplified form.

If $P \in \mathfrak{S} \cup \mathfrak{W}$, then, for every $\varepsilon \ge 0$, $\delta(\varepsilon, P)$ is the infimum of $\delta f$, where $f$ is a regular $\varepsilon$-code of $P$, and $\delta P = \sup(\delta(\varepsilon, P) : \varepsilon > 0)$.

If $P \in \mathfrak{W}$, then, for every $\varepsilon \ge 0$, $\lambda(\varepsilon, P)$ is the infimum of $\lambda f$, where $f$ is a regular $\varepsilon$-code of $P$, and $\lambda P = \sup(\lambda(\varepsilon, P) : \varepsilon > 0)$.

**6.10. Proposition.** *Let $P \in \mathfrak{S} \cup \mathfrak{W}$, let $\mathcal{U}$ be a pure partition of $P$ and let $d(\mathcal{U}) = 0$. Then $\delta P$, respectively $\lambda P$ (if $P \in \mathfrak{W}$), are equal to the infimum of all $\delta f$, respectively $\lambda f$, where $f$ is a regular $\varepsilon$-code of $P$ such that $(f^{-1}v : v \in fP)$ refines $\mathcal{U}$.*

We omit the proof since it is similar to that of 6.7.

**6.11. Proposition.** *If $P \in \mathfrak{W}_0$, then $\delta P$ and $\lambda P$ are equal to the infimum of all $\delta f$ ad $\lambda f$, respectively, where $f$ is a regular $\varepsilon$-code of $P$.*

This follows easily from 6.10.

**6.12. Fact.** Let $\mathcal{P} = (P_u : u \in D)$ be a d.e. (w.s.) of a $W$-space $P$ and let $d(\mathcal{P}'') = a < \infty$. Then there is a d.e. (w.s.) $\widehat{\mathcal{P}} = (\widehat{P}_u : u \in \widehat{D})$ of $P$ such that $\widehat{\mathcal{P}} \subset \mathcal{P}$, $d(\widehat{\mathcal{P}}'') = a$ and $d(P_u) > a$ whenever $u \in \widehat{D}'$.

**6.13. Definition.** A) Let $P$ be a $W$-space, let $d(P) < \infty$ and let $\mathcal{U} = (U_t : t \in T)$ be a partition (w.s.) of $P$. Then $P/\mathcal{U}$ will denote the $W$-space $\langle T, \sigma, \nu \rangle$, where $\nu(\{t\}) = \bar{\mu} U_t$ for all $t \in T$, $\sigma(s,t) = d(U_s + U_t)$ for $t \neq s$. We will say that $P/\mathcal{U}$ is the <u>quotient</u> of $P$ <u>with respect to</u> $\mathcal{U}$. — B) If $P = \langle Q, \varrho \rangle$ is a semimetric space, $d(P) < \infty$ and $\mathcal{U} = (U_t : t \in T)$ is a partition of $P$, then $P/\mathcal{U}$ will denote the semimetric space $\langle T, \sigma \rangle$, where $\sigma(t,s) = d(U_t \cup U_s)$ for $t \neq s$.

**6.14. Fact.** Let $P \in \mathfrak{W}$, $d(P) < \infty$; let $\mathcal{U}$ be a pure partition of $P$ and let $d(\mathcal{U}) = 0$. Then $E^*(P/\mathcal{U}) \geq E^*(P)$.

PROOF: Put $\mathcal{U} = (U_t : t \in T)$, $S = P/\mathcal{U} = \langle T, \sigma, \nu \rangle$. Let $\mathcal{S} = (S_v : v \in D)$ be a pure d.e. of $S$ such that all $S_v$, $v \in D''$, are of the form $\{t\} \cdot S$, $t = t(v) \in T$. For $v \in D''$, put $P_v = U_{t(v)}$; for $k \in D$, let $P_k = \Sigma(P_v : v \in D'', k \prec v)$. Then $\mathcal{P} = (P_k : k \in D)$ is a pure d.e. of $P$. It is easy to see that $E(\mathcal{P}) = E(\mathcal{S})$, $d(\mathcal{P}'') = 0$ and therefore, by 6.7, $E(\mathcal{P}) \geq E^*(P)$, hence $E(\mathcal{S}) \geq S^*(P)$. Since $\mathcal{S}$ was arbitrary, we get $E^*(P/\mathcal{U}) \geq E^*(P)$. □

**6.15. Lemma.** *Let $P$ be a $W_0$-space. If $\mathcal{P}$ is a d.e. (w.s.) of $P$ and $d(\mathcal{P}'') = 0$, then $E(\mathcal{P}) \geq E(P) \vee E^*(P)$.*

PROOF: By 6.8, $E(\mathcal{P}) \geq E(P)$. Thus we have only to prove $E(\mathcal{P}) \geq E^*(P)$. We can assume that $d(P) < \infty$ and $wP_u > 0$ for all $u \in D$. Let $f_v$, $v \in D''$, be functions such that $P_v = f_v \cdot P$. Then there are disjoint $\mu$-measurable sets $Q_t$, $t \in T$, $T$ finite, such that, for every $v \in D''$,

$$(*) \qquad \{q \in Q : f_v q > 0\} = \bigcup (Q_t : t \in T(v)),$$

where $T(v) = \{t \in T : f_v q > 0 \text{ for all } q \in Q_t\}$.

Put $M_t = Q_t \cdot P$, $\mathcal{M} = (M_t : t \in T)$, $S = P/\mathcal{M}$, $S_t = \{t\} \cdot S$ for $t \in T$. For $v \in D''$, $t \in T$, put $b_{vt} = w(Q_t \cdot P_v)$; evidently, $b_{vt} > 0$ iff $t \in T(v)$. For $v \in D''$, put $S(v) = \Sigma(b_{vt} S_t : t \in T(v))$. If $x, y \in D''$, then $s(S(x) + S(y)) = w(P_x + P_y)$ and, by $(*)$, $d(S(x) + S(y)) = d(P_x + P_y)$. Hence $E(\mathcal{P}) = E(\mathcal{S})$, where $\mathcal{S} = (S_u : u \in D)$, $S_u = \Sigma(S(x) : x \in D'', u \prec x)$.

Since $d(P_v) = 0$ for $v \in D''$, we get $d(\mathcal{S}'') = 0$ and therefore, by 6.8, $E(\mathcal{S}) \geq E(S)$. Since $S \in \mathfrak{W}_F$, we have, by 6.2, $E(S) = E^*(S)$. By 6.14, $E^*(S) \geq E^*(P)$, which proves $E(\mathcal{P}) \geq E^*(P)$. □

**6.16. Proposition.** *If $P$ is a $W_0$-space, then $E(P) = E^*(P)$.*

Proof: By 6.15, $E(\mathcal{P}) \geq E(P) \vee E^*(P)$ whenever $\mathcal{P}$ is a d.e. (w.s.) of $P$ and $d(\mathcal{P}'') = 0$. Hence, by 6.7, $E(P) \geq E(P) \vee E^*(P)$. This proves the proposition, since by 6.8, $E(P) \leq E^*(P)$. □

**6.17. Fact.** If $\mathcal{P} = (P_u : u \in D)$ is a d.e. (w.s.) of a $W$-space $P$, $\mathcal{U} = (U_t : t \in T)$ is a partition (w.s.) of $P$, and $|T| \leq 2^m$, then there exists a d.e. (w.s.) $\widehat{\mathcal{P}} = (\widehat{P}_u : u \in \widehat{D}) \supset \mathcal{P}$ of $P$ such that $\widehat{\mathcal{P}}''$ refines $\mathcal{U}$ and, for every $u \in D''$, $E(S_{u,x} : x \in D_u) \leq m \cdot wP \cdot d(P_u)$, where $D_u = \{x : ux \in \widehat{D}\}$, $S_{u,x} = \widehat{P}_{ux}$.

**6.18. Lemma.** *Let $\mathcal{U}$ be a partition (w.s.) of a $W$-space $P$ and let $\varepsilon > 0$. Then, for some $\vartheta > 0$,*

(1) *$E(\mathcal{P}) + \varepsilon \geq \eta(\mathcal{U})$ for every d.e. (w.s.) $\mathcal{P}$ of $P$ satisfying $d(\mathcal{P}'') \leq \vartheta$,*
(2) *if $\mathcal{U}$ is pure, then $E(\mathcal{P}) + \varepsilon \geq \eta^*(\mathcal{U})$ for every pure d.e. $\mathcal{P}$ of $P$ satisfying $d(\mathcal{P}'') \leq \vartheta$.*

Proof: We are going to prove (1); the proof of (2) is similar. Let $\mathcal{U} = (U_t : t \in T)$, $|T| = n$ and let $m$ be the least integer such that $n \leq 2^m$. Put $\vartheta = \varepsilon / m \cdot wP$. Let $\mathcal{P} = (P_u : u \in D)$ be a d.e. (w.s.) of $P$, $d(\mathcal{P}'') \leq \vartheta$. Let $\widehat{\mathcal{P}}$ be a d.e. (w.s.) of $P$ with the properties stated in 6.17. Clearly, $E(\widehat{\mathcal{P}}) \leq E(\mathcal{P}) + \varepsilon$. Since $\widehat{\mathcal{P}}''$ refines $\mathcal{U}$, we have $\eta(\mathcal{U}) \leq E(\widehat{\mathcal{P}}) \leq E(\mathcal{P}) + \varepsilon$. □

**6.19. Proposition.** *For every $W$-space $P$, $E(P) = \sup(E(\vartheta \odot P) : \vartheta > 0)$, $E^*(P) = \sup(E^*(\vartheta \odot P) : \vartheta > 0)$.*

Proof: We are going to prove the first equality; the second equality is proved in an analogous way. By 6.18, the following is true: for every $\varepsilon > 0$, there is a $\vartheta > 0$ such that $E(\vartheta \odot P) + \varepsilon \geq \eta(\mathcal{U})$ for every partition (w.s.) $\mathcal{U}$ of $P$, hence $E(\vartheta \odot P) + \varepsilon \geq E(P)$. This proves that $\sup(E(\vartheta \odot P) : \vartheta > 0) \geq E(P)$. The reverse inequality is evident. □

**6.20. Proposition.** *Let $P \in \mathfrak{W}$ and let $\varepsilon > 0$. Then $E(\varepsilon \odot P) = \inf\{E(\mathcal{P}) : \mathcal{P}$ is a d.e. (w.s.) of $P$, $d(\mathcal{P}'') \leq \varepsilon\}$, $E^*(\varepsilon \odot P) = \inf\{E(\mathcal{P}) : \mathcal{P}$ is a pure d.e. (w.s.) of $P$, $d(\mathcal{P}'') \leq \varepsilon\}$.*

Proof: We prove only the first assertion. For every d.e. (w.s.) $\mathcal{P} = (P_u : u \in D)$ of $P$, we put $\varepsilon \odot \mathcal{P} = (\varepsilon \odot P_u : u \in D)$; clearly, $\mathcal{S} = \varepsilon \odot \mathcal{P}$ is a d.e. (w.s.) of $\varepsilon \odot P$ and $d(\mathcal{S}'') = 0$ iff $d(\mathcal{P}'') \leq \varepsilon$. On the other hand, it is easy to see that every d.e. (w.s.) $\mathcal{S} = (S_v : v \in D)$ of $S = \varepsilon \odot P$ is of the form $\varepsilon \odot \mathcal{P}$, $\mathcal{P}$ being a d.e. (w.s.) of $P$. By 6.8, we have $E(\varepsilon \odot P) = \inf\{E(\mathcal{S}) : \mathcal{S} = (E_v : v \in D)$ is a d.e. (w.s.) of $\varepsilon \odot P$, $d(\mathcal{S}'') = 0\}$. Hence, by 6.12, $E(S)$ is equal to the infimum of all $E(\mathcal{S})$, $\mathcal{S}$ being a d.e. (w.s.) of $S$ such that $d(\mathcal{S}'') = 0$ whereas $d(S_v) > 0$ for $v \in D'$. Evidently, for every $\mathcal{S} = \varepsilon \odot \mathcal{P}$ satisfying the condition just stated, we have $E(\mathcal{S}) = E(\mathcal{P})$. Thus,

$E(S)$ is equal to the infimum of all $E(\mathcal{P})$, where $\mathcal{P} = (P_v : v \in D)$ is a d.e. (w.s.) of $P$, $d(\mathcal{P}'') \le \varepsilon$, $d(P_v) > \varepsilon$ for $v \in D'$. It is easy to see that this infimum is equal to $\inf\{E(\mathcal{P}) : \mathcal{P}$ is a d.e. (w.s.) of $P$, $d(\mathcal{P}'') \le \varepsilon\}$. $\square$

**6.21. Fact.** Let $f$ be a strongly branching well-fitting regular $\varepsilon$-code of a $W$-space $P$ in $K_\infty = \langle A^*, \pi, \lambda\rangle$, $A^* = \{0,1\} \times R_+$. If $u = (u_i : i < k) \in A^*$, put $\pi u = (\pi u_i : i < k)$. For every $u \in [fP]$, put $P_{\pi u} = \{x \in P : u \prec fx\}$. Put $D = \{\pi u : u \in [fP]\}$. Then $\mathcal{P} = (P_v : v \in D)$ is a pure d.e. of $P$, $E(\mathcal{P}) = E(f)$, $d(\mathcal{P}'') \le \varepsilon$.

**6.22. Fact.** Let $P \in \mathfrak{W}$, $\varepsilon > 0$. Then $E(\varepsilon, P) = E^*(\varepsilon \odot P)$.

Proof: By 5.17, $E(\varepsilon, P)$ is equal to the infimum of all $E(f)$, where $f$ is a strongly branching well-fitting regular $\varepsilon$-code of $P$. Hence, by 6.21, $E(\varepsilon, P)$ is equal to the infimum of all $E(\mathcal{P})$, where $\mathcal{P}$ is a pure d.e. of $P$ and $d(\mathcal{P}'') \le \varepsilon$. Hence, by 6.20, $E(\varepsilon, P) = E^*(\varepsilon \odot P)$. $\square$

**6.23. Theorem.** *For every $W$-space $P$, the coding entropy $\widehat{E}(P)$ coincides with $E^*(P)$ and $E(P)$.*

Proof: By 6.16 and 6.19, we have $E^*(P) = E(P)$ for every $P \in \mathfrak{W}$. Due to $\widehat{E}(P) = \sup(E(\varepsilon, P) : \varepsilon > 0)$, we have, by 6.22 and 6.16, $\widehat{E}(P) = E^*(P)$ for every $P \in \mathfrak{W}$. $\square$

**6.24. Convention.** In what follows, we will write, for every $W$-space $P$, $E(P)$ instead of $\widehat{E}(P)$ and $E^*(P)$.

## 7.

This section contains characterization theorems for the functionals $\delta$, $\lambda$ and $E$ defined on $\mathfrak{W}$ (the analogous theorems concerning the restrictions of $\delta$, $\lambda$ and $E$ to $\mathfrak{W}_F$ have been proved in Section 3 of [1]).

**7.1. Proposition.** *Let $\varphi$ be one of the functionals $\delta$, $\lambda$ and $E$. Then $\varphi P = \sup(\varphi(\varepsilon \odot P) : \varepsilon > 0)$ whenever either (1) $P \in \mathfrak{S} \cup \mathfrak{W}$, $\varphi = \delta$, or (2) $P \in \mathfrak{W}$, $\varphi = \lambda$ or $\varphi = E$.*

Proof: If $\varphi = \delta$ or $\varphi = \lambda$, then the assertion follows, by 2.6, from the definitions (see 2.8). For the case $\varphi = E$, see 6.19. $\square$

**7.2. Proposition.** *Let $P$ be a $W$-space and let $(P_0, P_1)$ be a pure partition of $P$. Then*

(1) $\delta P \le d(P) + \delta P_0 \vee \delta P_1$,
(2) $\lambda P \le d(P) \cdot wP + \lambda P_0 + \lambda P_1$,
(3) $E(P) \le d(P) H(wP_0, wP_1) + E(P_0) + E(P_1)$.

*The inequality (1) is also valid if $P$ is a semimetric space.*

Proof: I. Let $P = \langle Q, \varrho, \mu\rangle$ be a $W$-space. Let $\mathcal{U} = (U_t : t \in T)$ be a pure partition of $P$ refining $(P_0, P_1)$ and satisfying $d(\mathcal{U}) = 0$. Let $\varphi$ be one of the functionals $\delta$, $\lambda$ and $E$. Put $T_j = \{t \in T : U_t \le P_j\}$, $j = 0, 1$. Then $\mathcal{U}_j = (U_t :$

$t \in T_j$) is a pure partition of $P_j$. We can assume that $\varphi(P_j) < \infty$, $j = 0, 1$. Let $\varepsilon > 0$. By 6.10 and 5.17, there exist (1) strongly branching well-fitting codes $f_0$ and $f_1$ of $P_0$ and $P_1$ such that $\delta f_j \le \delta P_j + \varepsilon$ and $(f_j^{-1}u : u \in f_j P_j)$ refines $\mathcal{U}_j$, $j = 0, 1$; (2) strongly branching well-fitting codes $g_0$ and $g_1$ of $P_0$ and $P_1$ such that $\lambda g_j \le \lambda P_j + \varepsilon$ and $(g_j^{-1}u : u \in g_j P_j)$ refines $\mathcal{U}_j$; (3) dyadic expansions $\mathcal{P}_0$ and $\mathcal{P}_1$ of $P_0$ and $P_1$ such that $E(\mathcal{P}_j) \le E(P_j) + \varepsilon$ and $\mathcal{P}_j''$ refines $\mathcal{U}_j$, $j = 0, 1$.

In the case (1) and, respectively, (2) define $f$ and $g$ as follows: let $P_j = Q_j \cdot P$, where $Q_0 \cup Q_1 = Q$, $Q_0 \cap Q_1 = \emptyset$; for $x \in Q_j$ put $f(x) = (a_j) \cdot f_j(x)$, $g(x) = (a_j) \cdot g_j(x)$, where $a_j = (j, d(P))$. Then $f$ and $g$ are regular codes of $P$, $\delta f = d(P) + \delta f_0 + \delta f_1$, $\lambda g = d(P) \cdot wP + \lambda g_0 + \lambda g_1$, hence $\delta f \le d(P) + \delta P_0 \vee \delta P_1 + \varepsilon$, $\lambda g \le d(P) \cdot wP + \lambda P_0 + \varepsilon + \lambda P_1 + \varepsilon$. Evidently, $d(f^{-1}u : u \in fP) = 0$, $d(g^{-1}u : u \in gP) = 0$ and therefore, by 6.11, $\delta P \le d(P) + \delta P_0 \vee \delta P_1 + \varepsilon$, $\lambda P \le d(P) \cdot wP + \lambda P_0 + \lambda P_1 + 2\varepsilon$. Since these inequalities hold for every $\varepsilon > 0$, we have shown that the inequalities (1) and (2) stated in the proposition are valid whenever $P \in \mathfrak{W}_0$.

Consider the case (3). Define a dyadic expansion $\mathcal{P}$ of $P$ as follows: if $\mathcal{P}_0 = (P_u^{(0)} : u \in D_0)$, $\mathcal{P}_1 = (P_u^{(1)} : u \in D_1)$, let $D$ consist of $\emptyset$, all $(0) \cdot v$, where $v \in D_0$, and all $(1) \cdot v$, where $v \in D_1$. Put $P_\emptyset = P$, $P_{(0)\cdot v} = P_v^{(0)}$, $P_{(1)\cdot v} = P_v^{(1)}$; put $\mathcal{P} = (P_u : u \in D)$. Then $\mathcal{P}''$ refines $\mathcal{U}$, $E(\mathcal{P}) = d(P)H(wP_0, wP_1) + E(\mathcal{P}_0) + E(P_1)$ and therefore $E(\mathcal{P}) \le d(P)H(wP_0, wP_1) + E(P_0) + \varepsilon + E(\mathcal{P}_1) + \varepsilon$. By 6.8, we have $E(P) \le E(\mathcal{P})$; since $\varepsilon > 0$ is arbitrary, we get $E(P) \le d(P)H(wP_0, wP_1) + E(P_0) + E(P_1)$.

II. Let $P$ be an arbitrary $W$-space. Then, for every $\varepsilon > 0$, $\varepsilon \odot P$ is a $W_0$-space and therefore we obtain the inequalities (1)–(3) with $P$ replaced by $\varepsilon \odot P$, $\varepsilon > 0$ arbitrary. By 7.1, we get the inequalities (1)–(3) for every $P \in \mathfrak{W}$.

III. The proof of (1) for $P \in \mathfrak{S}$ is analogous and can be omitted. □

**7.3. Characterization theorem for $\delta$.** Let $\mathfrak{P} = \mathfrak{S}$ or $\mathfrak{P} = \mathfrak{W}$. The functional $\delta$ (defined on $\mathfrak{P}$) is the largest of all functionals $\varphi$ on $\mathfrak{P}$ satisfying the following conditions for all $P \in \mathfrak{P}$:

(1) $\varphi P = 0$ whenever $d(P) = 0$,
(2) $\varphi P = \sup(\varphi(\varepsilon \odot P) : \varepsilon > 0)$,
(3) $\varphi P \le d(P) + \varphi P_0 \vee \varphi P_1$ for all pure partitions $(P_0, P_1)$ of $P$.

Proof: We consider the case $\mathfrak{P} = \mathfrak{W}$; the other case is analogous.

I. Evidently, $\delta$ satisfies (1). By 7.1 and 7.2, it satisfies (2) and (3).

II. Let $\varphi$ be a functional on $\mathfrak{W}$ satisfying (1)–(3). By 7.1, it is sufficient to prove that $\varphi P \le \delta P$ whenever $P$ is of the form $\varepsilon \odot S$ for some $S$ (hence $P \in \mathfrak{W}_0$). — Let $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}_0$. Let $f$ be an arbitrary strongly branching well-fitting code of $P$ such that $d(f^{-1}u : u \in fP) = 0$ and $w(f^{-1}u) > 0$ for all $u \in fP$. For every $v \in [fP]$ put $P_u = \{x \in Q : v \prec fx\} \cdot P$, and for every $u \in [fP] \setminus fP$ put $a_u = (0, d(P_u))$ if $\varphi P_{u0} \ge \varphi P_{u1}$, $a_u = (1, d(P_{u1}))$ if $\varphi P_{u0} < \varphi P_{u1}$, $s(u) = u \cdot (a_u)$. Then there exists exactly one $v \in fP$ such that $v \upharpoonright (n+1) = s(v \upharpoonright n)$ for $n < |v|$. It is easy to see that, by (3) and (1), we have $\varphi P \le \Sigma(d(P_u) : u \prec v)$. Since $f$ is well-fitting, we get $\varphi P \le \lambda v$, hence $\varphi P \le \delta f$. Since $f$ is arbitrary, we get $\varphi P \le \delta P$. □

**7.4. Characterization theorem for $\lambda$.** *The functional $\lambda$ (defined on $\mathfrak{W}$) is the largest of all functionals $\varphi$ on $\mathfrak{W}$ satisfying the following conditions for all $W$-spaces $P$:*

(1) $\varphi P = 0$ *whenever* $d(P) = 0$,
(2) $\varphi P = \sup(\varphi(\varepsilon \odot P) \,:\, \varepsilon > 0)$,
(3) $\varphi P \leq d(P) \cdot wP + \varphi P_0 + \varphi P_1$ *for all pure partitions* $(P_0, P_1)$ *of* $P$.

Proof: We can proceed in the same way as in the proof of 7.3 except the part concerning the inequality $\varphi P \leq \lambda P$ for $P \in \mathfrak{W}_0$.

Let $\varphi$ satisfy (1)–(3). Let $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}_0$. Let $f$ be an arbitrary strongly branching well-fitting code of $P$ such that $d(f^{-1}u \,:\, u \in fP) = 0$ and $w(f^{-1}u) > 0$ for all $u \in fP$. For every $v \in [fP]$ put $P_v = \{x \in Q \,:\, v \prec fx\} \cdot P$. By (3), we have $\varphi P_u \leq d(P_u) \cdot wP_u + \varphi P_{u0} + \varphi P_{u1}$ for every $u \in [fP] \setminus fP$. This implies $\varphi P \leq \Sigma(d(P_u)wP_v \,:\, v \in [fP] \setminus fP)$, since, by (1), $\varphi P_v = 0$ for $v \in fP$. Due to the fact that $f$ is well-fitting, we have, for any $u \in fP$ and any $x \in f^{-1}v$, $\lambda(fx) = \Sigma(d(P_v) \,:\, v \prec u)$. Consequently, $\lambda f = \Sigma(w(f^{-1}u)\Sigma(d(P_v) \,:\, v \prec u) \,:\, u \in fP) = \Sigma(d(P_v)\Sigma(w(f^{-1}u) \,:\, u \in fP,\, v \prec u) \,:\, v \in [fP] \setminus fP) = \Sigma(d(P_v)wP_v \,:\, v \in [fP] \setminus fP)$ and therefore $\varphi P \leq \lambda f$. Since $f$ is arbitrary, we get $\varphi P \leq \lambda P$. □

**7.5. Characterization theorem for $E$.** *The functional $E$ (defined on $\mathfrak{W}$) is the largest of all functionals $\varphi$ on $\mathfrak{W}$ satisfying the following conditions for all $W$-spaces $P$:*

(1) $\varphi P = 0$ *whenever* $d(P) = 0$,
(2) $\varphi P = \sup(\varphi(\varepsilon \odot P) \,:\, \varepsilon > 0)$,
(3) $\varphi P \leq d(P)H(wP_0, wP_1) + E(P_0) + E(P_1)$ *for all pure partitions* $(P_0, P_1)$ *of* $P$.

Proof: As with 7.4, we prove only that $\varphi P \leq E(P)$ whenever $P \in \mathfrak{W}_0$ and $\varphi$ satisfies (1)–(3).

Let $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}_0$. Let $\mathcal{P} = (P_u \,:\, u \in D)$ be an arbitrary pure dyadic expansion of $P$ such that $d(\mathcal{P}'') = 0$. By (3), we have $\varphi P_u \leq d(P_u)H(wP_{u0}, wP_{u1}) + \varphi P_{u0} + \varphi P_{u1}$ for each $u \in D'$; by (1), $\varphi P_u = 0$ for $u \in D''$. Consequently, $\varphi P \leq \Sigma(d(P_u)H(wP_{u0}, wP_{u1}) \,:\, u \in D') = E(\mathcal{P})$. By 6.8, this proves $\varphi P \leq E(P)$.

**7.6.** It is possible to generalize Proposition 7.2 quite substantially, namely to prove the corresponding assertions for an arbitrary partition (w.s.) instead for a pure binary partition. To this end, we need some auxiliary definitions and some lemmas.

**7.7. Notation.** If $\mathcal{P} = (P_u \,:\, u \in D)$ is a dyadic expansion (w.s.) of a $W$-space $P$, we put (1) for any $v \in D''$, $\lambda(\mathcal{P}, v) = \Sigma(d(P_u) \,:\, u \prec v,\, u \neq v)$;
(2) $\lambda\mathcal{P} = \Sigma(\lambda(\mathcal{P}, v) \cdot wP_w \,:\, v \in D'')$; (3) $\delta\mathcal{P} = \max(\lambda(\mathcal{P}, v) \,:\, v \in D'',\, wP_v > 0)$.

**7.8. Fact.** If $\mathcal{P} = (P_u \,:\, u \in D)$ is a dyadic expansion (w.s.) of a $W$-space $P$, then $\lambda\mathcal{P} = \Sigma(d(P_u) \cdot wP_u \,:\, u \in D')$.

**7.9. Fact.** If $P \in \mathfrak{W}_0$, then $\lambda P$ (respectively, $\delta P$) is equal to the infimum of all $\lambda\mathcal{P}$ (respectively, $\delta\mathcal{P}$), where $\mathcal{P}$ is a pure dyadic expansion of $P$ satisfying $d(\mathcal{P}'') = 0$.

Proof: By 6.11 and 5.17, $\lambda P$ is equal to the infimum of all $\lambda f$, where $f$ is a strongly branching well-fitting exact code of $P$. To every such code, there corresponds (as it is easy to show) a pure dyadic expansion $\mathcal{P}$ of $P$ such that $d(\mathcal{P}'') = 0$, $\lambda\mathcal{P} = \lambda f$. Hence $\lambda P$ is not less than the infimum (of $\lambda\mathcal{P}$) in question. On the other hand, for every pure dyadic expansion $\mathcal{P} = (P_u : u \in D)$ of $P$ satisfying $d(\mathcal{P}'') = 0$ there is a strongly branching well-fitting code $f$ of $P$ such that $\lambda f = \lambda\mathcal{P}$. This proves the equality in question for $\lambda$; for $\delta$, the proof is similar. □

**7.10. Lemma.** *Let $P = \langle Q, \varrho, \mu\rangle$ be a $W$-space and let $(P_0, P_1) = (\langle Q, \varrho, \mu_0\rangle, \langle Q, \varrho, \mu_1\rangle)$ be a partition* (w.s.) *of $P$. Let $P^*$ denote the $W$-space $\langle Q^*, \varrho^*, \mu^*\rangle$, where $Q^* = Q \times \{0,1\}$, $\varrho^*(\langle x, i\rangle, \langle y, j\rangle) = \varrho(x,y)$, $\operatorname{dom}\mu^*$ consists of all sets of the form $(X_0 \times \{0\}) \cup (X_1 \times \{1\})$, $X_0, X_1 \in \operatorname{dom}\mu$, and $\mu^*(X \times \{j\}) = \mu_j(X)$. Then $\delta P^* = \delta P$, $\lambda P^* = \lambda P$, $E(P^*) = E(P)$.*

Proof: We prove only $\lambda P^* = \lambda P$ since the proof of $\delta P^* = \delta P$ is analogous whereas $E(P^*) = E(P)$ is easily proved using 6.7 (for the case $P \in \mathfrak{W}_0$), the equality $E^* = E$, and 6.19. By 6.19, it is sufficient to prove $\lambda P^* = \lambda P$ for the case $P \in \mathfrak{W}_0$.

It is easy to show that $\lambda P^* \le \lambda P$. Hence we have to prove $\lambda P \le \lambda P^*$ only. To this end, it is sufficient, by 7.9, to find, for any given pure dyadic expansion $\mathcal{P} = (P_u : u \in D)$ of $P^*$ satisfying $d(\mathcal{P}'') = 0$, a pure dyadic expansion $\mathcal{Z}$ of $P$ such that $d(\mathcal{Z}) = 0$, $\lambda\mathcal{Z} \le \lambda\mathcal{P}$. It is easy to see that there is a partition $(T_m : m \in M)$ of $Q$ and a pure dyadic expansion $\mathcal{S} = (S_u : u \in \widehat{D})$ of $P^*$ such that $D \subset \widehat{D}$, $S_u = P_u$ for $u \in D$, $\mu T_m > 0$ for all $m \in M$, and the set of all $S_u$, $u \in \widehat{D}''$, coincides with the set of all $V_{mj} \cdot P^*$, where $V_{mj} = T_m \times \{j\}$, $m \in M$, $j = 0,1$. Obviously, $\lambda\mathcal{S} = \lambda\mathcal{P}$.

Let $\psi : M \times \{0,1\} \to \widehat{D}''$ be a bijection. Define $X_u$, $u \in D''$, as follows. If $u = \psi(m,j)$, put (1) $X_u = T_m$ if $\lambda(\mathcal{S}, u) \le \lambda(\mathcal{S}, \psi(m, 1-j))$, (2) $X_u = \emptyset$ if $\lambda(\mathcal{S}, u) > \lambda(\mathcal{S}, \psi(m, 1-j))$. Put $Z_u = X_u \cdot P$ for $u \in \widehat{D}''$, $Z_u = \Sigma(Z_v : v \in \widehat{D}'', u \prec v)$ for $u \in \widehat{D}'$. Clearly, $\mathcal{Z} = (Z_u : u \in \widehat{D})$ is a pure dyadic expansion of $P$. It is not difficult to show that $d(Z_u) \le d(S_u)$ for all $u \in \widehat{D}$, hence $\lambda(\mathcal{Z}, v) \le \lambda(\mathcal{S}, v)$ for all $v \in \widehat{D}''$.

Let $u, v \in \widehat{D}''$, $u = \psi(m, 0)$, $v = \psi(m, 1)$. We shall treat only the case $\lambda(\mathcal{S}, u) \le \lambda(\mathcal{S}, v)$ (the case $\lambda(\mathcal{S}, v) \ge \lambda(\mathcal{S}, v)$ is analogous). We have $wZ_u = wS_u + wS_v$, $wZ_v = 0$, and therefore $wZ_u \cdot \lambda(\mathcal{Z}, u) + wZ_v \cdot \lambda(\mathcal{Z}, v) \le (wS_u + wS_v)\lambda(\mathcal{S}, u) \le wS_u \cdot \lambda(\mathcal{S}, u) + wS_v \cdot \lambda(\mathcal{S}, u) \le wS_u \cdot \lambda(\mathcal{S}, u) + wS_v \cdot \lambda(\mathcal{S}, v)$. This proves that $\Sigma(wZ_v \cdot \lambda(\mathcal{Z}, v) : v \in \widehat{D}'') \le \Sigma(wS_v \cdot \lambda(\mathcal{S}, v) : v \in \widehat{D}'')$. Thus $\lambda\mathcal{Z} \le \lambda\mathcal{S} = \lambda\mathcal{P}$, which proves the lemma. □

**7.11. Proposition.** *Let $P$ be a $W$-space and let $(P_0, P_1)$ be a partition* (w.s.) *of $P$. Then*

(1) $\delta P \le d(P) + \delta P_o \vee \delta P_1$,
(2) $\lambda P \le d(P) \cdot wP + \lambda P_0 + \lambda P_1$,
(3) $E(P) \le d(P) \cdot H(wP_0, wP_1) + E(P_0) + E(P_1)$.

Proof: Let $P = \langle Q, \varrho, \mu\rangle$. Let $P^* = \langle Q^*, \varrho^*, \mu^*\rangle$ denote the space described in 7.10. Put $Q_j = Q \times \{j\}$, $j = 0,1$. Then $Q_j \cdot P^*$ is isomorphic to $P_j$ and $(Q_0 \cdot P^*, Q_1 \cdot P^*)$

is a pure partition of $P^*$. Then, by 7.2, (1) $\delta P^* \le d(P^*) + \delta(Q_0 \cdot P^*) \vee \delta(Q_1 \cdot P^*)$, (2) $\lambda P^* \le d(P^*) \cdot wP^* + \lambda(Q_0 \cdot P^*) + \lambda(Q_1 \cdot P^*)$, (3) $E(P^*) \le d(P^*) \cdot H(w(Q_0 \cdot P^*), w(Q_1 \cdot P^*)) + E(Q_0 \cdot P^*) + E(Q_1 \cdot P^*)$. By 7.10, this proves the proposition since, evidently, $d(P^*) = d(P)$ and $wP^* = wP$. □

**7.12. Proposition.** *Let $P$ be a $W$-space. Let $\mathcal{U} = (U_t : t \in T)$ be a partition (w.s.) of $P$. Then*

(1) $\delta P \le \delta(P/\mathcal{U}) + \max(\delta U_t : t \in T)$,
(2) $\lambda P \le \lambda(P/\mathcal{U}) + \Sigma(\lambda U_t : t \in T)$,
(3) $E(P) \le E(P/\mathcal{U}) + \Sigma(E(U_t) : t \in T)$.

Proof: I. It follows easily from 7.9 that there is a pure dyadic expansion $\mathcal{S} = (T_u \cdot S : u \in D)$ of the space $S = P/\mathcal{U} = \langle T, \sigma, \nu \rangle$ such that $\delta(\mathcal{S}) = \max(\lambda(\mathcal{S}, v) : v \in D'') = \delta S$ and all $T_v$, $v \in D''$, are singletons. For every $u \in D$, put $P_u = \Sigma(U_t : t \in T_u)$. Then $(P_u : u \in D')$ is a dyadic expansion (w.s.) of $P$. By 7.11, we have $\delta P_u \le d(P_u) + \delta P_{u0} \vee \delta P_{u1}$ for all $u \in D'$, which implies $\delta P \le \delta \mathcal{P} + \max(\delta P_v : v \in D'')$. Clearly, for every $u \in D'$, we have $|T_u| \ge 2$ and therefore, by the definition of $P/\mathcal{U}$, $d(T_u \cdot S) = d(P_u)$. Consequently, $\delta\mathcal{P} = \delta\mathcal{S}$ and therefore $\delta P \le \delta\mathcal{S} + \max(\delta P_v : v \in D'')$. Since $\delta\mathcal{S} = \delta S$ and the collection $(P_v : v \in D'')$ coincides, up to the indexing, with $(U_t : t \in T)$, we obtain the inequality (1).

II. For the inequality (2), the proof is similar. We take a pure dyadic expansion of $S$, denoted again by $\mathcal{S} = (T_u \cdot S : u \in D)$, such that $\lambda\mathcal{S} = \lambda S$; this is possible by 7.9. For every $u \in D$, we put $P_u = \Sigma(U_t : t \in T_u)$, and we denote $(P_u : u \in D)$ by $\mathcal{P}$. By 7.11, we have $\lambda P_u \le d(P_u) \cdot wP_u + \lambda P_{u0} + \lambda P_{u1}$ for all $u \in D'$. This implies $\lambda P \le \lambda\mathcal{P} + \Sigma(\lambda P_v : v \in D'')$. The inequality (2) then follows similarly as in I. — As for the inequality (3), the proof is analogous and can be omitted. □

**7.13. Notation.** A) If $f_j$ is a function on $Q_j$, $j = 1, 2$, then $f_1 \times f_2$ denotes the function on $Q_1 \times Q_2$ defined by $(f_1 \times f_2)(q_1, q_2) = f_1(q_1) f_2(q_2)$. — B) Let $P_j = \langle Q_j, \varrho_j, \mu_j \rangle$, $j = 1, 2$, be $W$-spaces. For $j = 1, 2$, let $\mathcal{U}_j = (f_{jm} \cdot P_j : m \in M_j)$ be a partition (w.s.) of $P_j$. Then $\mathcal{U}_1 \times \mathcal{U}_2$ denotes the partition (w.s.) $\mathcal{U}$ of $P = P_1 \times P_2$ defined as follows: $\mathcal{U} = (g_{km} \cdot P : (k, m) \in M_1 \times M_2)$, where $g_{km} = f_{1k} \times f_{2m}$. — If $\mathcal{U}_j = (T_{jm} \cdot P_j : m \in M_j)$ are pure partitions, then $\mathcal{U}_1 \times \mathcal{U}_2 = ((T_{1k} \times T_{2m}) \cdot P : (k, m) \in M_1 \times M_2)$.

**7.14. Fact.** For $j = 1, 2$, let $P_j$ be a $W$-space and let $\mathcal{U}_j = (U_{jm} : m \in M_j)$ be a partition (w.s.) of $P_j$. Put $P = P_1 \times P_2$, $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, $P/\mathcal{U} = \langle M_1 \times M_2, \sigma, \nu \rangle$, $P_j/\mathcal{U}_j = \langle M_j, \sigma_j, \nu_j \rangle$. Then $\nu = \nu_1 \times \nu_2$, $\sigma \le \sigma_1 \times \sigma_2$.

**7.15. Lemma.** *Let $P$ be a $W_0$-space. Then $\lambda P$ is equal to the infimum of all $\lambda(P/\mathcal{U})$, where $\mathcal{U}$ is a pure partition of $P$ and $d(\mathcal{U}) = 0$.*

Proof: I. Let $\varepsilon > 0$. By 7.9, there is a pure d.e. $\mathcal{P} = (P_u : u \in D)$ of $P$ such that $d(\mathcal{P}'') = 0$ and $\lambda\mathcal{P} \le \lambda P + \varepsilon$. Put $S = P/\mathcal{P}''$. Let $\mathcal{S}$ denote the pure d.e. $(S_u : u \in D)$ of $S$ such that $S_v = \{v\} \cdot S$ whenever $v \in D''$. We have $\lambda\mathcal{S} = \lambda\mathcal{P}$ and therefore, by 7.9, $\lambda S \le \lambda\mathcal{S} = \lambda\mathcal{P} \le \lambda P + \varepsilon$. Hence the infimum in question does not exceed $\lambda P$.

II. Let $\mathcal{U} = (U_t : t \in T)$ be a pure partition of $P$, $d(\mathcal{U}) = 0$. We are going to show that $\lambda P \le \lambda(P/\mathcal{U})$. Put $S = P/\mathcal{U}$. Since $S$ is finite, there is a pure d.e. $\mathcal{S} = (S_u : u \in D)$ of $S$ such that $\lambda\mathcal{S} = \lambda S$ and every $S_v$, $v \in D''$, is of the form $\{t(v)\}\cdot S$, $t(v) \in T$. Let $\mathcal{P} = (P_u : u \in D)$ be the pure d.e. of $P$ such that $P_v = U_{t(v)}$ for $v \in D''$. We have $\lambda\mathcal{P} = \lambda\mathcal{S}$ and $d(\mathcal{P}'') = 0$. Hence, by 7.9, $\lambda P \le \lambda\mathcal{S} = \lambda(P/\mathcal{U})$. □

**7.16. Proposition.** *Let $\mathfrak{P} = \mathfrak{S}$ or $\mathfrak{P} = \mathfrak{W}$. Let $P_1, P_2 \in \mathfrak{P}$. Then (1) $\delta(P_1 \times P_2) \le \delta P_1 + \delta P_2$, and if $\mathfrak{P} = \mathfrak{W}$, then (2) $\lambda(P_1 \times P_2) \le \lambda P_1 \cdot wP_2 + \lambda P_2 \cdot wP_1$, (3) $E(P_1 \times P_2) \le E(P_1) \cdot wP_2 + E(P_2) \cdot wP_1$.*

Proof: We will consider only the case $\mathfrak{P} = \mathfrak{W}$. By 7.1, it is sufficient to prove the inequalities for the case $P_1 \in \mathfrak{W}_0$, $P_2 \in \mathfrak{W}_0$. We are going to prove (2); the proof of (1) and (3) is similar. Clearly, we can assume $wP_1 = wP_2 = 1$, $\lambda P_j < \infty$.

Since $P_1, P_2 \in \mathfrak{W}_0$, $\lambda P_j$ is equal, by 7.15, to the infimum of all $\lambda(P_j/\mathcal{U}_j)$, where $\mathcal{U}_j$ is a pure partition of $P_j$ and $d(\mathcal{U}_j) = 0$. Choose an arbitrary $\varepsilon > 0$ and choose a pure partition $\mathcal{V}_j$ of $P_j$ such that $d(\mathcal{V}_j) = 0$, $\lambda(P_j/\mathcal{V}_j) \le \lambda P_j + \varepsilon$. Let $\mathcal{V}_j = (V_{jm} : m \in M_j)$, $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, $P_j/\mathcal{V}_j = S_j = \langle M_j, \sigma_j, \nu_j\rangle$, $(P_1 \times P_2)/\mathcal{V} = S = \langle M_1 \times M_2, \sigma, \nu\rangle$. By 4.1, $\lambda(S_1 \times S_2) \le \lambda S_1 + \lambda S_2$. By 7.14, $\lambda S \le \lambda(S_1 \times S_2)$, hence $\lambda((P_1 \times P_2)/\mathcal{V}) = \lambda S \le \lambda P_1 + \lambda P_2 + 2\varepsilon$. By 7.15, this proves $\lambda(P_1 \times P_2) \le \lambda P_1 + \lambda P_2$. □

**7.17. Remarks.** 1) The connection between the functional $\delta$ on $\mathfrak{S}$ and the Kolgomorov entropy $\mathcal{H}_\varepsilon$ (see, e.g., [3] and [4]) is given by the following almost evident formula: $\mathcal{H}_\varepsilon(P) \le \delta(\varepsilon * P) \le \mathcal{H}_\varepsilon(P) + 1$. (Recall that $\varepsilon * \langle Q, \varrho\rangle = \langle Q, \varepsilon * \varrho\rangle$, where $(\varepsilon * \varrho)(x,y) = 0$ if $\varrho(x,y) \le \varepsilon$, and $(\varepsilon * \varrho)(x,y) = 1$ if $\varrho(x,y) > \varepsilon$). — 2) Let $J^n$, $n = 1, 2, \dots$, denote the cube $[0,1]^n$ equipped with the metric $\varrho((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max(|x_i - y_i| : i = 1, \dots, n)$. It is easy to see that $\delta(J^n) \le 2n$. It can be shown that $\delta(J^1) = 2$; however, I do not know whether $\delta(J^n) = 2n$ for $n = 2, 3, \dots$ .

## 8.

In this section, we generalize the characterization theorems for $\Delta$ and $\Lambda$ proved for the class $\mathfrak{W}_F$ in Part I, to a certain fairly wide subclass of $\mathfrak{W}$.

**8.1.** If $P \in \mathfrak{S} \cup \mathfrak{W}$, then $\inf(\delta(P^n)/n : n \in N, n > 0)$ is denoted by $\Delta(P)$. If $P \in \mathfrak{W}$, then $\inf(\lambda(P^n)/n(wP)^{n-1} : n \in N, n > 0)$ is denoted by $\Lambda(P)$. — See 4.6.

**8.2. Fact.** Let $m, n \in N$, $m > 0$, $n > 0$. If $P \in \mathfrak{S} \cup \mathfrak{W}$, then $\delta(P^{m+n}) \le \delta(P^m) + \delta(P^n)$. If $P \in \mathfrak{W}$, $wP = 1$, then $\lambda(P^{m+n}) \le \lambda(P^m) + \lambda(P^n)$.

This is a consequence of 7.16.

**8.3. Fact.** If $P \in \mathfrak{S} \cup \mathfrak{W}$, then $\Delta(P) = \lim(\delta(P^n)/n)$. If $P \in \mathfrak{W}$, $wP > 0$, then $\Lambda(P) = \lim(\lambda(P^n)/n(wP)^{n-1})$; in particular, $\Lambda(P) = \lim(\lambda(P^n)/n)$ if $wP = 1$.

This is a consequence of 8.2 and 4.5.

**8.4. Proposition.** *If $P \in \mathfrak{S} \cup \mathfrak{W}$, then $\Delta(P^m) = m\Delta(P)$ for every $m \in N$, $m > 0$. If $P, S \in \mathfrak{S}$ or $P, S \in \mathfrak{W}$, then $\Delta(P \times S) \le \Delta(P) + \Delta(S)$.*

Proof: From 4.7, $\Delta(P^m) = m\Delta(P)$ follows at once. By 8.2, $\delta(P^m \times S^n) \le \delta(P^m) + \delta(S^n)$, from which the inequality for $\Delta$ follows easily, by 8.3. $\square$

**8.5. Proposition.** *If $P \in \mathfrak{S} \cup \mathfrak{W}$ and $(P_0, P_1)$ is a pure partition of $P$, then $\Delta(P) \le d(P) + \Delta(P_0) \vee \Delta(P_1)$.*

The proof is the same, word for word, as the one of 4.10, except that instead of 4.7, 8.3 is used.

**8.6. Facts.** I. For every $P \in \mathfrak{W}$ and every $m \in N$, $m > 0$, $\Lambda(P^m) = m(wP)^{m-1}\Lambda(P)$; in particular, $\Lambda(P^m) = m\Lambda(P)$, if $wP = 1$. — II. If $P, S \in \mathfrak{W}$, then $\Lambda(P \times S) \le \Lambda(P) \cdot wS + \Lambda(S) \cdot wP$.

Proof: I. We can assume that $wP = 1$. By 8.3., $\Lambda(P^m) = \lim_{n\to\infty}(\lambda(P^{mn})/n) = m \cdot \lim_{n\to\infty}(\lambda(P^{mn})/mn)$. Hence, again by 8.3, $\Lambda(P^m) = m\Lambda(P)$. — II. We can assume that $wP = wS = 1$. By 8.3, $\Lambda(P \times S) = \lim(\lambda(P^n \times S^n)/n)$. Hence, by 7.16, $\Lambda(P \times S) \le \lim(\lambda(P^n)/n) + \lim \lambda(S^n)/n = \Lambda(P) + \Lambda(S)$. $\square$

**8.7. Proposition.** *For every $W$-space $P$ and every partition* (w.s.) *of $P$, $\Lambda(P) \le d(P) \cdot H(wP_0, wP_1) + \Lambda(P_0) + \Lambda(P_1)$.*

Proof: It follows from 7.11 and 7.16 that $\mathfrak{W}$ satisfies the conditions stated in 4.13. By 4.18, this proves the proposition. $\square$

**8.8.** In the proof of characterization theorems for $\delta$, $\lambda$ and $E$ (see 7.3, 7.4 and 7.5), Proposition 7.1 plays a substantial role. I do not know whether there are analogous propositions on $\Delta$ and $\Lambda$, i.e., whether $\Delta(P) = \sup(\Delta(\varepsilon \odot P) : \varepsilon > 0)$, $\Lambda(P) = \sup(\Lambda(\varepsilon \odot P) : \varepsilon > 0)$ for all $P \in \mathfrak{W}$. Therefore, the characterization theorems for $\Delta$ and $\Lambda$ will be proved only in a restricted form, namely for $\Delta$ and $\Lambda$ restricted to certain subclasses.

**8.9. Notation and definition.** Let $P \in \mathfrak{S} \cup \mathfrak{W}$. For every $t \in R_+$, we put $C_\delta[P](t) = \sup(\delta S : S \le P, d(S) \le t)$. — If $P \in \mathfrak{S} \cup \mathfrak{W}$ and the function $C_\delta[P]$ is continuous at 0 (i.e., $C_\delta[p](t) \to 0$ for $t \to 0$), we will say that $P$ is $\underline{\delta\text{-regular}}$. The class of all $\delta$-regular $P \in \mathfrak{S}$ (respectively, $P \in \mathfrak{W}$) will be denoted by $\mathfrak{S}_\delta$ (by $\mathfrak{W}_\delta$).

**8.10. Facts.** I. If $P \in \mathfrak{S} \cup \mathfrak{W}$ and $S$ is a subspace of $P$, then $C_\delta[S] \le C_\delta[P]$; hence every subspace of a $\delta$-regular space is $\delta$-regular. — II. Let $\mathfrak{P} = \mathfrak{S}$ or $\mathfrak{P} = \mathfrak{W}$. If $P, S \in \mathfrak{P}$, then $C_\delta[P \times S] \le C_\delta[P] + C_\delta[S]$. Hence, if $P$ and $S$ are $\delta$-regular, then so is $P \times S$.

**8.11. Facts.** I. Every $W_0$-space is $\delta$-regular. — II. If $\langle Q, \varrho\rangle$ is a subspace of $R^n$, $\mu$ is the Lebesgue measure restricted to $Q$ and $\mu Q < \infty$, then $\langle Q, \varrho\rangle \in \mathfrak{S}_\delta$, $P = \langle Q, \varrho, \mu\rangle \in \mathfrak{W}_\delta$.

**8.12. Fact.** Let $f$ be a regular $\varepsilon$-code of $P \in \mathfrak{S} \cup \mathfrak{W}$. Put $T = fP$; for $t \in T$, put $U_t = f^{-1}t$; put $\mathcal{U} = (U_t : t \in T)$. For $t \in T$ put $f't = t$. Then (1) $f'$ is a regular 0-code of $P/\mathcal{U}$, $\delta f' = \delta f$, (2) if $P \in \mathfrak{W}$, then $\lambda f' = \lambda f$.

This follows easily from the definition of $P/\mathcal{U}$.

**8.13. Fact.** For every $P \in \mathfrak{S}\cup\mathfrak{W}$ and every $\varepsilon > 0$, $\delta(\varepsilon \odot P)$ is equal to the infimum of all $\delta f$, where $f$ is a regular $\varepsilon$-code of $P$.

Proof: By 5.17, $\delta(\varepsilon \odot P)$ is equal to the infimum of all $\delta f$, where $f$ is a strongly branching well-fitting regular 0-code of $\varepsilon \odot P$. Clearly, every code of this kind is a strongly branching well-fitting regular $\varepsilon$-code of $P$, and vice versa. By 5.17, this proves the equality in question. □

**8.14. Proposition.** *Let $P \in \mathfrak{S}\cup\mathfrak{W}$. Let $\varepsilon > 0$. Let $f$ be a regular $\varepsilon$-code of $P$. If $d(f^{-1}v : v \in f \cdot P) \le \varepsilon$, then (1) $\delta P \le \delta(\varepsilon \odot P) + C_\delta[P](\varepsilon)$, (2) if $P \in \mathfrak{W}$, then $\lambda P \le \lambda(\varepsilon \odot P) + C_\delta[P](\varepsilon) \cdot wP$.*

Proof: Let $\eta > 0$. By 8.13, there is a regular $\varepsilon$-code $f$ of $P$ such that $\delta f \le d(\varepsilon \odot P) + \eta$. For $v \in fP$, put $u_v = (f^{-1}v) \cdot P$; put $\mathcal{U} = (U_v : v \in fP)$. By 8.12, $\delta f \ge \delta(P/\mathcal{U})$. By 7.12, $\delta P \le \delta(P/\mathcal{U}) + \max(\delta U_v : v \in fP)$, hence $\delta P \le \delta(P/\mathcal{U}) + C_\delta[P](\varepsilon)$. It follows that $\delta P \le \delta(\varepsilon \odot P) + \eta + C_\delta[P](\varepsilon)$ for every $\eta > 0$. This proves the inequality (1). The proof of (2) is analogous. □

**8.15. Proposition.** *Let $P \in \mathfrak{S}\cup\mathfrak{W}$ be $\delta$-regular. Then $\Delta(P) = \sup(\Delta(\varepsilon \odot P) : \varepsilon > 0)$, and if $P \in \mathfrak{W}$, then $\Lambda(P) = \sup(\Lambda(\varepsilon \odot P) : \varepsilon > 0)$.*

Proof: We prove only the first assertion, since the proof of the second one is analogous. Let $P \in \mathfrak{W}$ be $\delta$-regular. By 8.14 and 8.10, we have $|\delta(P^n)/n - \delta(\varepsilon \odot P^n)/n| \le C_\delta[P](\varepsilon)$. Since $P$ is $\delta$-regular, $C_\delta[P](\varepsilon) \to 0$ for $\varepsilon \to 0$, which proves the assertion. □

**8.16. Characterization theorem for $\Delta$ restricted to $\delta$-regular spaces.** *Let $\mathfrak{P} = \mathfrak{S}_\delta$ or $\mathfrak{P} = \mathfrak{W}_\delta$. The functional $\Delta$ restricted to $\mathfrak{P}$ is the largest of all functionals $\varphi$ on this class satisfying the following conditions for all $P \in \mathfrak{P}$:*

(1) *$\varphi P = 0$ whenever $d(P) = 0$,*
(2) *$\varphi P = \sup(\varphi(\varepsilon \odot P) : \varepsilon > 0)$,*
(3) *$\varphi P \le d(P) + \varphi P_0 \vee \varphi P_1$ for all pure partitions $(P_0, P_1)$ of $P$,*
(4) *$\varphi(P^n) = n \cdot \varphi P$ for all $P \in \mathfrak{P}$ and all $n \in N$, $n > 0$.*

Proof: I. By 8.15, 8.5 and 8.4, $\Delta$ satisfies the conditions in question. — II. Let $\varphi$ satisfy the conditions. Then, by 7.3, $\varphi S \le \delta S$ for every $S \in \mathfrak{P}$ and therefore $n \cdot \varphi P = \varphi(P^n) \le \delta(P^n)$, $\varphi P \le \delta(P^n)/n$ for all $P \in \mathfrak{P}$ and $n \in N$, $n > 0$. This implies $\varphi \le \Delta$. □

**8.17. Characterization theorem for $\Lambda$ restricted to $\delta$-regular spaces.**
*The functional $\Lambda$ on the class $\mathfrak{W}_\delta$ of all $\delta$-regular spaces is*

A) *the largest of all functionals $\varphi$ on $\mathfrak{W}_\delta$ satisfying the following conditions for all $P \in \mathfrak{W}_\delta$:*

(1) *$\varphi P = 0$ whenever $d(P) = 0$,*
(2) *$\varphi P = \sup(\varphi(\varepsilon \odot P) : \varepsilon > 0)$,*
(3) *$\varphi P \le d(P) \cdot wP + \varphi P_0 + \varphi P_1$ for every pure partition $(P_0, P_1)$ of $P$,*
(4) *$\varphi(P^n) = n \cdot \varphi P$ for $n \in N$, $n > 0$, provided $wP = 1$,*

B) *the largest of all functionals on* $\mathfrak{W}_\delta$ *satisfying* (1), (2), (4) *and* (3') $\varphi P \leq d(P) \cdot H(wP_0, wP_1) + \varphi P_0 + \varphi P_1$ *for every pure partition* $(P_0, P_1)$ *of* $P$.

Proof: I. Clearly, $\Lambda$ satisfies (1). By 8.15, $\Lambda$ satisfies (2). By 8.7, $\Lambda$ satisfies (3') and also (3); by 8.6, it satisfies (4). — II. Let a functional $\varphi$ on $\mathfrak{W}_\delta$ satisfy (1)–(4). Then, by the same argument as in 7.4, we get $\varphi P \leq \lambda P$ for all $P \in \mathfrak{W}_0$. By (4) and 8.10, we have $\varphi P = \varphi(P^n)/n \leq \lambda(P^n)/n$ for all $\mathfrak{W}_0$ such that $wP = 1$ and all $n \in N$, $n > 0$. It is easy to see that 4.7, asserting the convergence $\lambda(P^n)/n \to \Lambda(P)$, does hold for all $P \in \mathfrak{W}$, $wP = 1$. It follows that $\varphi P \leq \Lambda(P)$ whenever $P \in \mathfrak{W}_0$. By (2), we get $\varphi P \leq \Lambda(P)$ for all $P \in \mathfrak{W}_\delta$. — III. Evidently, (3') implies (3). Therefore every functional $\varphi$ on $\mathfrak{W}_\delta$ for which (1), (2), (3') and (4) are true, satisfies, by II, the inequality $\varphi P \leq \Lambda(P)$ for all $P \in \mathfrak{W}_\delta$. □

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